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## A bound for the first k-1 consecutive k-th power non-residues (mod p)

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1. Introduction and summary. Throughout  $\chi$  will be a non-principal character  $\pmod{p}$  of order k, where p is an odd prime with  $p \equiv 1 \pmod{k}$  and k is an integer  $\geq 2$ .

Let r(k, m, p) denote the smallest positive integer r such that

(1.1) 
$$\chi(r) = \chi(r+1) = \dots = \chi(r+m-1) = 1$$

and let  $a_j(k, p)$  denote the smallest positive integer  $\alpha$  such that

(1.2) 
$$\chi(\alpha) \neq 1$$
,  $\chi(\alpha+1) \neq 1$ , ...,  $\chi(\alpha+j-1) \neq 1$ .

Although knowledge of the exact number of triples of consecutive quadratic residues of primes p is classical and approximate formulae are well-known for the number of occurrences of m consecutive residues (with a markedly improved error term (see [4], p. 198) due to the work of A. Weil [7]), the first non-trivial upper bound for r(k, m, p) when k = 2 and m = 3 was obtained by the author; see, in particular [5] and [6]. Namely, for k = 2 and m = 3, I have shown that

(1.3) 
$$r(k, m, p) < 270 \cdot 5p^{1/4} \log p + 62$$
 for all  $p > 17$ .

On the other hand, P. D. T. A. Elliott showed in [2], Lemma 12, that

$$(1.4) \alpha_2(k, p) = O_s(p^{1/4+s})$$

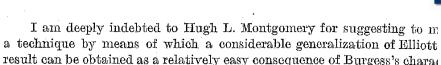
for k=2,  $\varepsilon>0$ , and  $p\geqslant 5$ .

In [3], Elliott gave a slight improvement of (1.4), namely

(1.5) 
$$a_{2}(k, p) = O_{s}(p^{\frac{1}{4}\left(1 - \frac{e^{-10}}{2}\right) + s})$$

for k = 2,  $\varepsilon > 0$ , and  $p \ge 5$ . Unfortunately, Elliott's results applied only to real characters (mod p), and nothing was obtained for  $\alpha_j(k, p)$  when j is > 2.

ter sum estimates.



Specifically, for  $\varepsilon > 0$ ,  $k \ge 2$ , and  $p > p_0(\varepsilon, k)$ , I will show the

$$a_{k-1}(k, p) = O_{\epsilon, k}(p^{1/4+\epsilon}).$$

There are, for example, at least six consecutive seventh power nor residues less than  $p^{1/4+e}$ ! Note also that an immediate consequence of (1.4) and (1.6) is that the second smallest *prime* character non-residue of characters of order  $\geq 2$ ,  $g_2(k, p)$ , satisfies the implied inequality in (1.6) since  $g_2(k, p) \leq a_2(k, p) + 1$ .

## 2. Consecutive character non-residues.

THEOREM. For every  $\varepsilon > 0$  and each  $k \ge 2$ ,

$$(2.1) a_{k-1}(k, p) = O_{s,k}(p^{1/4+s})$$

for all primes  $p > p_0(\varepsilon, k)$  satisfying  $p \equiv 1 \pmod{k}$ .

Proof. Let X denote the set of complex valued characters (mod p of order k defined on the positive integers, and order the elements o X so that  $\chi = \chi_1$  is the character with the property that  $\chi(n) = e^{2\pi i/k}$  i n is a character non-residue of order k.

Then, for (n, p) = 1 we have

(2.2) 
$$\sum_{r=1}^{k-1} \chi^r(n) = \begin{cases} k-1 & \text{for } \chi(n) = 1, \\ -1 & \text{for } \chi(n) \neq 1, \end{cases}$$

since  $e^{2\pi i/k} + e^{4\pi i/k} + \dots + e^{(2\pi i/k)(k-1)} = -1$ . Assume that

$$(2.3) p^{1/4+s} < (k-1)N < a_{k-1}(k,p).$$

Then, obviously, at least N of the integers n with  $n \leq (k-1)N$  are character residues of order k. Consequently, no more than (k-2)N can be character non-residues of order k.

Thus

(2.4) 
$$\sum_{n \leq (k-1)N} \left( \sum_{r=1}^{k-1} \chi^r(n) \right) = \sum_{\substack{n \leq (k-1)N \\ \chi(n)=1}} (k-1) - (k-2) \sum_{\substack{n \leq (k-1)N \\ \chi(n) \neq 1}} 1$$
$$\geq N(k-1) - N(k-2) = N.$$

But, from Burgess [1], if  $a_{k-1}(k, p) > p^{1/4+\epsilon}$  we must have

(2.5) 
$$\left| \sum_{n \leq (k-1)N} \left( \sum_{r=1}^{k-1} \chi^{r}(n) \right) \right| = \left| \sum_{r=1}^{k-1} \left( \sum_{n \leq (k-1)N} \chi^{r}(n) \right) \right| \\ \leq \sum_{r=1}^{k-1} \left| \sum_{n \leq (k-1)N} \chi^{r}(n) \right| \\ < \sum_{r=1}^{k-1} \varepsilon(k-1)N = \varepsilon N(k-1)^{2}.$$

But  $N > \varepsilon N(k-1)^2$  for  $\varepsilon < \frac{1}{(k-1)^2}$  contradicting the assumption (2.3) and, consequently, the theorem is proved.

## References

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