Some finitely generated subsemigroups of $S(X)$

by

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Abstract. $S(X)$ is the semigroup, under composition, of all continuous selfmaps of the topological space X and the topology on $S(X)$ is the compact-open topology. It is shown that there are a large number of spaces with the property that the corresponding semigroup contains a dense semigroup generated by either two or three functions. Monothetic subsemigroups of $S(X)$ are also investigated. However, these cannot coincide with $S(X)$ if X has more than one point.

1. Introduction. The symbol $S(X)$ denotes the semigroup, under composition, of all continuous selfmaps of a topological space X. We equip $S(X)$ with the compact-open topology. Our primary interest is in finding the least number of functions in $S(X)$ which will generate a dense subsemigroup of $S(X)$. Sierpiński [8] has shown that there exist four functions in $S(I)$ which generate a dense subsemigroup of $S(I)$, where I is the closed unit interval. Jarník and Knichal [4] produced two functions which generate a dense subsemigroup of $S(I)$. Young [10] showed by a completely different approach, that there exist dense two-generator subsemigroups of the family of all continuous onto selfmaps of I.

We first study cyclic subsemigroups of $S(X)$ and their closures to see how large these can be when X is discrete, the reals or the interval I. When X is discrete, $S(X)$ is just the full transformation semigroup on X and for this, we use the notation $\mathcal{C}_X$. We then prove that for a large class of spaces X, there exist dense two-generator or three-generator subsemigroups of $S(X)$.

As a corollary to one of our theorems we obtain the well-known result of Evans [3] that any countable semigroup can be embedded in a semigroup with two generators.

2. Monothetic subsemigroups of $\mathcal{C}_X$. We first recall some notation from the literature on general topological semigroups. Let S be a topological semigroup and let $a \in S$. Then $\mathrm{Cl}(\{a^n\}_{n=1}^{\infty})$ is called the *monothetic subsemigroup* of S generated by a and it is denoted by $\Gamma(a)$. We denote the set of cluster points of $\{a^n\}_{n=1}^{\infty}$ by $K(a)$. It is clear that $\Gamma(a)$ is commutative.

When we topologize $S(X)$, it will be with the compact-open topology. We recall that a typical subbasic open set is of the form $\langle K, G\rangle$

$= \{f \in S(X): f(K) \subset G\}$ where K is compact and G is open. For $\mathscr{C}_X$ this coincides with the topology of pointwise convergence and results in a topological semigroup. In this section we consider $\Gamma(f)$ and $K(f)$ for $f \in \mathscr{C}_X$.

Definition (2.1). For $f \in \mathscr{C}_X$ and $x \in X$, we define the *orbit of x under f* to be the set $\{f^n(x): n \in N\}$ and denote it by $O_f(x)$.

Theorem (2.2). *$\Gamma(f)$ is compact if and only if the orbits under f are all finite.*

Proof. Since X is discrete, we see that if $g \in \Gamma(f)$ then for any given $x \in X$ there is an $n \in N$ such that $g(x) = f^n(x)$. Hence $\Gamma(f)(x) = O_f(x)$ for each $x \in X$. If $\Gamma(f)$ is compact then so is $\Gamma(f)(x)$ because the evaluation map at x is a continuous map from $\mathscr{C}_X$ into X; so $O_f(x) = \Gamma(f)(x)$ is finite for each $x \in X$. On the other hand, if $O_f(x)$ is finite for each $x \in X$, then $\Gamma(f)$ is a closed subset of the compact space $\Pi\{O_f(x): x \in X\}$.

Theorem (2.3). *$\Gamma(f)$ is an infinite compact semigroup if and only if $O_f(x)$ is finite for each $x \in X$ and $\{\operatorname{Card} O_f(x): x \in X\}$ is unbounded.*

Proof. We know by the previous theorem that $\Gamma(f)$ is compact if and only if each $O_f(x)$ is finite. Now suppose $\{\operatorname{Card} O_f(x): x \in X\}$ is bounded. Let $\operatorname{Card} O_f(x) \leqslant k$ for all $x \in X$. Then we have $f^{k!+k} = f^k$, so that $\{f^n\}_{n=1}^\infty$ is a repeating sequence and hence $\Gamma(f) = \{f^n\}_{n=1}^\infty$ is a finite set.

On the other hand, if $\{\operatorname{Card} O_f(x): x \in X\}$ is unbounded, then given any $n \in N$, we can find $x \in X$ such that $\operatorname{Card} O_f(x) > n$, so that $f(x), f^2(x), \ldots$ $\ldots, f^n(x)$ are all different. Hence $\{f^n\}_{n=1}^\infty$ is a non-repeating sequence, so $\Gamma(f)$ is infinite. Thus if each $O_f(x)$ is finite and $\{\operatorname{Card} O_f(x): x \in X\}$ is unbounded we see that $\Gamma(f)$ is an infinite compact semigroup.

Theorem (2.4). *If $O_f(x)$ is infinite for some $x \in X$, then $K(f)$ is empty and $\Gamma(f)$ is an infinite cyclic semigroup.*

Proof. If $O_f(x)$ is infinite for some $x \in X$, then $f^n(x) \neq f^m(x)$ for $n \neq m$. Suppose $K(f)$ is nonempty. Let k be in $K(f)$. Then $\langle\{x\}, \{k(x)\}\rangle$ is a subbasic open set containing k, so we have $f^n \in \langle\{x\}, \{k(x)\}\rangle$ for infinitely many n. That is, $f^n(x) = k(x)$ for infinitely many n which is impossible because $f^n(x) \neq f^m(x)$ for $n \neq m$. Hence $K(f)$ is empty and $\Gamma(f) = \{f^n\}_{n=1}^\infty$ an infinite cyclic semigroup.

We see that $K(f)$ is either empty or a compact monothetic group. If all the $O_f(x)$ are finite then $\Gamma(f)$ is compact and so $K(f)$ is a compact group [7, p. 109]; in fact $K(f) = \Gamma(fe)$ where e is the identity of $K(f)$. So $K(f)$ is also monothetic. If some $O_f(x)$ is infinite, then by the above theorem $K(f)$ is empty.

We can say something more about $K(f)$. The set

$$\{\operatorname{Card} O_f(x): x \in O_f(x)\} \cup \{1\}$$

completely characterizes $K(f)$. Before we prove this, we first establish a lemma.

Lemma (2.5). *Let Y be a discrete space and let $\{A_j: j \in J\}$ be a partition of Y into finite subsets such that $\operatorname{Card} A_k \neq \operatorname{Card} A_j$ for $k \neq j$. Let $p \in \mathscr{C}_Y$ be such that p is a cyclic permutation of each A_j, $j \in J$. Then $\Gamma(p) = K(p)$ and is infinite if and only if $\{\operatorname{Card} A_j: j \in J\}$ is unbounded.*

Proof. Now the identity map i is in $K(p)$ and hence is the identity of the group $K(p)$. We have $K(p) = \Gamma(pi) = \Gamma(p)$ and by Theorem (2.3), it is infinite if and only if $\{\operatorname{Card} A_j: j \in J\}$ is unbounded because the A_j's are precisely the orbits under p.

Theorem (2.6). *Let f and g be any elements of $\mathscr{C}_X$ such that all their orbits are finite. Then $K(f)$ is isomorphic to $K(g)$ if and only if*

$$\{1\} \cup \{\operatorname{Card} O_f(x): x \in O_f(x)\} = \{1\} \cup \{\operatorname{Card} O_g(x): x \in O_g(x)\}.$$

Proof. We first consider $K(f)$. If e is the identity of $K(f)$, we observe that for each $x \in X$ there is an $n \in N$ such that $e(x) = f^n(x)$ since we have the topology of pointwise convergence and X is discrete. Let us denote the range of e by V. Now $x \in V$ if and only if $e(x) = x$ since e is an idempotent. But $e(x) = x$ if and only if $f^n(x) = e(x) = x$, that is, if and only if $x \in O_f(x)$. Hence $V = \{x: x \in O_f(x)\}$.

Also, for $a, b \in V$ either $O_f(a) = O_f(b)$ or $O_f(a) \cap O_f(b) = \emptyset$. Hence there exists a partition $\{D_i: i \in I\}$ of V such that each D_i is the orbit of some element of V. Now let Y be a discrete space with a partition $\{A_j: j \in J\}$ such that $\operatorname{Card} A_j \neq \operatorname{Card} A_k$ for $k \neq j$, and such that

$$\{\operatorname{Card} A_j: j \in J\} = \{\operatorname{Card} D_i: i \in I\} \cup \{1\}.$$

As in Lemma (2.5), we define $p \in \mathscr{C}_Y$ such that p is a cyclic permutation of each A_j. We shall show that $K(p)$ is isomorphic to $K(f)$.

Let q be any element of $K(p)$. Then there exists a subsequence $\{p^{n_i}\}_{i=1}^\infty$ of $\{p^n\}_{n=1}^\infty$ which converges to q. This implies that for each $j \in J$ there exists $l_j \in N$ such that $p^{n_i} = q$ on A_j for all $i \geqslant l_j$, which means $n_t \equiv n_i \operatorname{mod}(\operatorname{Card} A_j)$ for all $i, t \geqslant l_j$. On the other hand, if $\{n_i\}_{i=1}^\infty$ is a strictly increasing sequence of positive integers such that for each $j \in J$ there exists $l_j \in N$ with the property $n_i \equiv n_t \operatorname{mod}(\operatorname{Card} A_j)$ for all $i, t \geqslant l_j$, then $\{p^{n_i}\}_{i=1}^\infty$ converges to the function $q \in K(p)$ given by $q(x) = p^{n_i}(x)$ for all $x \in A_j$ where i is any integer such that $i \geqslant l_j$.

We define a map Φ: $K(p) \to K(f)$ as follows. Let $q \in K(p)$ so that there is a sequence $\{p^{n_i}\}_{i=1}^\infty$ converging to q. We shall show that $\{f^{n_i} \circ e\}_{i=1}^\infty$ converges and then define $\Phi(q) = \lim(f^{n_i} \circ e)$. Given any $x \in X$ we have $e(x) \in D_s$ for some D_s in the partition of V. Now $\operatorname{Card} D_s = \operatorname{Card} A_j$ for some $j \in J$, so there exists l_j in N such that $n_i \equiv n_t \operatorname{mod}(\operatorname{Card} D_s)$ for all $i, t \geqslant l_j$. Hence $f^{n_i}(e(x)) = f^{n_t}(e(x))$ for all $i, t \geqslant l_j$. That is, $\{f^{n_i}(e(x))\}_{i=1}^\infty$

becomes stationary from a certain point on and so is convergent. Since our topology is that of pointwise convergence, this shows that $\{f^{n_i} \circ e\}_{i=1}^{\infty}$ converges. Also

$$K(f) = \mathrm{Cl}\big(\{(f \circ e)^n\}_{n=1}^{\infty}\big) = \mathrm{Cl}(\{f^n \circ e\}_{n=1}^{\infty}) .$$

so $\lim(f^{n_i} \circ e) \in K(f)$.

Next we show that Φ is one-to-one. If q, h are in $K(p)$ and $q \neq h$ then there is a $y \in Y$ such that $q(y) \neq h(y)$. Suppose $y \in A_j$ and let x be any point in some D_s such that $\mathrm{Card}\, D_s = \mathrm{Card}\, A_j$. Let $\{p^{n_i}\}_{i=1}^{\infty}$ and $\{p^{m_i}\}_{i=1}^{\infty}$ be any two subsequences of $\{p^n\}_{n=1}^{\infty}$ converging to q and h, respectively. Then there exists an $l_j \in N$ such that $q(y) = p^{n_i}(y)$ and $h(y) = p^{m_i}(y)$ for any $i \in N$, $i \geqslant l_j$. This means $n_i \not\equiv m_i \bmod(\mathrm{Card}\, A_j)$ for any $i \in N$, $i \geqslant l_j$ because $q(y) \neq h(y)$. But this implies that $\Phi(q)(x) \neq \Phi(h)(x)$ because $\Phi(q)(x) = f^{n_i}(x)$ and $\Phi(h)(x) = f^{m_i}(x)$ for any $i \in N$, $i \geqslant l_j$ and $n_i \not\equiv m_i \bmod(\mathrm{Card}\, D_s)$. Hence $\Phi(q) \neq \Phi(h)$, so Φ is one-to-one.

To see that it is onto, we observe that if $\{f^{n_i} e\}_{i=1}^{\infty}$ is a convergent subsequence of $\{f^n e\}_{n=1}^{\infty}$, then $\{n_i\}_{i=1}^{\infty}$ is a strictly increasing sequence of positive integers such that for each D_s we have $n_i \equiv n_t \bmod(\mathrm{Card}\, D_s)$ for all sufficiently large i and t, so that for each $j \in J$ there exists $l_j \in N$ such that $n_i \equiv n_t \bmod(\mathrm{Card}\, A_j)$ for all $i, t \geqslant l_j$. This is precisely what we need for $\{p^{n_i}\}_{i=1}^{\infty}$ to converge. If $\lim p^{n_i} = q$ it is clear that $\Phi(q) = \lim(f^{n_i} e)$. Hence Φ is onto $K(f)$.

Finally, if $\lim p^{r_i} = q$ and $\lim p^{m_i} = h$ then $\lim p^{(n_i + m_i)} = q \circ h$ and so

$$\begin{aligned}\Phi(q \circ h) &= \lim(f^{n_i + m_i} \circ e) = \lim[(f^{n_i} \circ e) \circ (f^{m_i} \circ e)] \\ &= [\lim(f^{n_i} \circ e)] \circ [\lim(f^{m_i} \circ e)] = \Phi(q) \circ \Phi(h)\end{aligned}$$

and so Φ is a homomorphism.

We have shown that $K(f)$ is isomorphic to $K(p)$. Similarly $K(g)$ is isomorphic to $K(p)$. Hence $K(f) \simeq K(g)$.

3. Monothetic subsemigroups of $S(I)$ and $S(R)$. The behaviour of monothetic subsemigroups of $S(X)$ when X is an arbitrary topological space is not so easy to determine. We consider only the special cases when X is the unit interval I or the reals R. Now Boyce [1, p. 96] has completely determined when $\Gamma(f)$ is compact, for $f \in S(I)$ or $S(R)$. We first state some of the results he has obtained. A map f is *precompact* if it has fixed points, $f(x) > x$ for all x smaller than the smallest fixed point of f and $f(x) < x$ for all x larger than the largest fixed point of f. All the functions in $S(I)$ are precompact because I is compact. A map f is called Γ-compact if $\Gamma(f)$ is compact.

THEOREM (3.1) (Boyce) [1, p. 96]. *Let f be an element of either $S(I)$ or $S(R)$. Then f is Γ-compact if and only if $f \circ f$ is precompact and the fixed point set of $f \circ f$ is connected.*

Our aim is to show that if $\Gamma(f)$ is not compact then it has to be discrete.

THEOREM (3.2). *Let f be an element of either $S(I)$ or $S(R)$. If the fixed point set of f is not connected then $K(f)$ is empty and $\Gamma(f)$ is an infinite cyclic semigroup.*

Proof. Suppose the fixed point set of f is not connected. Then there exist $a, b \in R$ such that $f(a) = a$, $f(b) = b$ and $f(x) \neq x$ for all x in (a, b). It is clear that either $f(x) > x$ for all x in (a, b) or $f(x) < x$ for all x in (a, b). We shall assume $f(x) < x$ for all x in (a, b). (The proof for the other case is similar to this.)

If $f(x) > a$ for all x in (a, b), we see that $\{f^n(x)\}_{n=1}^{\infty}$ is monotonic strictly decreasing, bounded below by a and so converges to some y. But then $f(y) = y$ and so y has to be a. Hence $\lim f^n(x) = a$ for all x in $[a, b)$ and $\lim f^n(b) = b$. Hence $\{f^n\}_{n=1}^{\infty}$ has no cluster points; that is, $K(f)$ is empty and $\Gamma(f) = \{f^n\}_{n=1}^{\infty}$, an infinite cyclic semigroup.

If $f(x) \leqslant a$ for some x in (a, b), then $f(x) = a$ for some x in (a, b). We let $c_1 = \mathrm{Sup}\{x\colon f(x) = a,\ x \in (a, b)\}$. We have $a < c_1 < b$ and f maps $[c_1, b]$ onto $[a, b]$. We see that there exists an x in (c_1, b) such that $f(x) = c_1$; that is, $f^2(x) = a$. We let $c_2 = \mathrm{Sup}\{x\colon f^2(x) = a,\ x \in (a, b)\}$. Proceeding in this manner we get a strictly increasing sequence $\{c_n\}$ bounded above by b, such that $c_n = \mathrm{Sup}\{x\colon f^n(x) = a,\ x \in (a, b)\}$. Let $\lim c_n = c$.

Suppose $\{f^n\}_{n=1}^{\infty}$ has a cluster point k. Then we have $k(c) = a$, so $c \neq b$ since $k(b) = b$. We have $a < c_1 < c_2 < \ldots < c_n \ldots < c < b$. Choose any $\varepsilon < c_1 - a$. Then there is an n such that $|f^n(c) - k(c)| < \varepsilon$; that is, $f^n(c) < a + \varepsilon$ because $k(c) = a$. But $a + \varepsilon < c_1$, so $f^n(c) < c_1$. Hence there is an x in (c, b) such that $f^n(x) = c_1$ because $f^n(b) = b$. This implies that $f^{n+1}(x) = f(c_1) = a$ which is a contradiction because $x > c_{n+1}$. Hence $\{f^n\}_{n=1}^{\infty}$ has no cluster points; that is, $K(f)$ is empty and $\Gamma(f) = \{f^n\}_{n=1}^{\infty}$, and infinite cyclic semigroup.

Remark. The converse of Theorem (3.2) is not true. If f is any homeomorphism from I onto I such that $f(0) = 1$, $f(1) = 0$ and $f \circ f$ is not the identity map then the fixed point set of f is connected but the fixed point set of $f \circ f$ is not, hence $K(f)$ is empty and $\Gamma(f)$ is an infinite cyclic semigroup.

COROLLARY (3.3). *If the fixed point set of $f \circ f$ is not connected then $K(f)$ is empty and $\Gamma(f)$ is an infinite cyclic semigroup.*

Proof. If k is a cluster point of f then $k \circ k$ is a cluster point of $\{(f \circ f)^n\}_{n=1}^{\infty}$. But $\{(f \circ f)^n\}_{n=1}^{\infty}$ has no cluster points by Theorem (3.2).

COROLLARY (3.4). *Let f be an element of either $S(I)$ or $S(R)$. Then $\Gamma(f)$ is either compact or is an infinite cyclic semigroup.*

Proof. If $f \in S(I)$ and $\Gamma(f)$ is not compact then by Theorem (3.1) the fixed point set of $f \circ f$ is not connected and so by Corollary (3.3) $\Gamma(f)$ is an infinite cyclic semigroup.

If $f \in S(R)$ and $\Gamma(f)$ is not compact then either the fixed point set of $f \circ f$ is not connected, in which case again by Corollary (3.3) $\Gamma(f)$ is an infinite cyclic semigroup, or f is not precompact. If the fixed point set J of $f \circ f$ is connected but f is not precompact then either J is empty or $f(x) < x$ for all x less than the least element of J or $f(x) > x$ for all x greater than the greatest element of J, and in all three cases $\{f^n\}_{n=1}^{\infty}$ diverges. So in any case $\Gamma(f)$ is an infinite cyclic semigroup since $K(f)$ is empty.

We see that in $S(I)$ and $S(R)$, $K(f)$ is either empty or is a group of one or two elements. This follows from Theorem 3 of Boyce [1, p. 91] which states that if $\Gamma(f)$ is a compact semigroup then $K(f)$ is either $\{e\}$ or $\{k, e\}$ where e is the identity on the fixed point set J of $f \circ f$ and k is a sense-reversing self-inverse homeomorphism of J onto J.

4. Dense subsemigroups of $S(X)$. Now $S(X)$ is monothetic if and only if X has only one element. This follows from the fact that monothetic semigroups are commutative. In the special case of the unit interval I, we know that there exists a countable family $\{f_n\}_{n=1}^{\infty}$ dense in $S(I)$, because $S(I)$ is separable. Sierpiński [8] showed that, given any countable family $\{f_n\}_{n=1}^{\infty}$ of $S(I)$, there exist four functions in $S(I)$ which actually generate $\{f_n\}_{n=1}^{\infty}$ under composition; Jarník and Knichal [4] produced two functions which generate Sierpiński's four functions. Hence there do exist two functions in $S(I)$ which generate a dense subsemigroup of $S(I)$ and we know this result cannot be improved upon.

We obtain similar results for two classes of spaces which, between them, include all Euclidean spaces, all closed unit cubes in Euclidean spaces, the countable discrete space and the Cantor discontinuum.

Definition (4.1). A space X is said to have the *internal extension property* if any continuous function from a closed subset F of X into X can be extended continuously to all of X.

Theorem (4.2). *Let X be a space with the internal extension property. Suppose there exists a countable family $\{A_n\colon n \in N\}$ of mutually disjoint closed subsets of X such that each A_n is homeomorphic to X and each A_n is open in $\bigcup\{A_n\colon n \in N\}$. Suppose further that*

(1) *there exists Φ_1 in $S(X)$ such that $\Phi_1|_{A_n} = h_{n-1}^{-1} \circ h_n$, for $n > 1$ where, for each n, h_n is the homeomorphism mapping A_n onto X;*

(2) *there exists a homeomorphism $\Phi_2 \in S(X)$ mapping X onto a closed subset of X such that $\Phi_2|_{A_n} = h_{n+1}^{-1} \circ h_n$ and $\Phi_2(X) \cap A_1 = \emptyset$;*

(3) *any map Ψ which maps each A_n into A_n and is continuous on each A_n can be continuously extended to all of X.*

Then given any countable family $\{f_n\}_{n=1}^{\infty}$ of $S(X)$, we can find two functions Ψ_1, Ψ_2 such that $\{f_n\}_{n=1}^{\infty}$ is contained in the semigroup generated by Ψ_1 and Ψ_2.

Proof. First we produce five functions which will generate all the f_n. We take Φ_3 to be any continuous extension of h_1 to all of X. We take $\Phi_4 = h_1^{-1}$, so that Φ_4 maps X homeomorphically onto A_1. We take Φ_5 to be the function in $S(X)$ whose restriction to each A_n is $h_n^{-1} \circ f_n \circ h_n$. Such a function exists by (3).

We observe that $\Phi_1^{n-1}, \Phi_2^{n-1}$ can be expressed in terms of the functions h_n. We have

$$\Phi_1^{n-1} = h_1^{-1} \circ h_n \text{ on } A_n \quad \text{for} \quad n > 1,$$

and

$$\Phi_2^{n-1} = h_n^{-1} \circ h_1 \text{ on } A_1 .$$

It is easily verified that

$$f_n = \Phi_3 \circ \Phi_1^{n-1} \circ \Phi_5 \circ \Phi_2^{n-1} \circ \Phi_4 .$$

We next produce two functions which generate these five functions. We take our first function Ψ_1 to be Φ_4. Before we define our second function we need to define some sets. We let

$$B_i = \Phi_4^i(\Phi_2(X)) \quad \text{for} \quad i = 1, 2, 3, 4, 5 ,$$

and

$$B_6 = \Phi_4^6(X) .$$

Since Φ_4, Φ_2 are homeomorphisms and $\Phi_2(X) \subset X - A_1$, we have $B_i \subset \Phi_4^i(X) - \Phi_4^i(A_1)$; that is,

$$B_i \subset \Phi_4^{i-1}(A_1) - \Phi_4^i(A_1) \quad \text{for} \quad i = 1, \ldots, 5$$

and

$$B_6 = \Phi_4^5(A_1) .$$

The B_i's are clearly disjoint. They are also closed because $\Phi_2(X)$ is closed and Φ_4 is a homeomorphism of X onto the closed set A_1.

Now we are ready to define Ψ_2. We let Ψ_2 be the function in $S(X)$ whose restrictions to the B's are as follows:

$$\Psi_2 = \begin{cases} \Phi_2 \circ \Phi_4^{-6} & \text{on } B_6 , \\ \Phi_i \circ \Phi_2^{-1} \circ \Phi_4^{-i} & \text{on } B_i,\ i = 1, 2, 3, 4, 5 . \end{cases}$$

Such a function exists because X has the internal extension property. It is again verified that

$$\Phi_i = \Psi_2 \circ \Psi_1^i \circ \Psi_2 \circ \Psi_1^6 \quad \text{for} \quad i = 1, 2, 3, 4, 5 .$$

So we see that $\{f_n\}_{n=1}^{\infty}$ is contained in the semigroup generated by the two elements Ψ_1 and Ψ_2.

THEOREM (4.3). *Let X be the m-dimensional closed unit cube I^m. Then, given any countable family $\{f_n\}_{n=1}^{\infty}$ of $S(X)$ there exist two functions in $S(X)$ which generate a semigroup containing $\{f_n\}_{n=1}^{\infty}$.*

Proof. For each $n \in N$, let A_n denote the product of m copies of the closed interval $\left[\frac{1}{2^{2n-1}}, \frac{1}{2^{2n-2}}\right]$ for $n = 1, 2, \ldots$ and let h_n be given by

$$(\pi_i \circ h_n)(x) = 2^{2n-1}x_i - 1 \quad \text{for} \quad i = 1, \ldots, m.$$

Let Φ_1 be any function in $S(X)$ such that

$$(\pi_i \circ \Phi_1)(x) = 4x_i \quad \text{whenever} \quad x_i \,\varepsilon\, [0, \tfrac{1}{4}] \quad \text{for} \quad i = 1, \ldots, m.$$

With some calculation, one shows that $\Phi_1|_{A_n} = h_{n-1}^{-1} \circ h_n$ for all $n \in N$. We let Φ_2 be the function given by $(\pi_i \circ \Phi_2)(x) = \frac{1}{4}x_i$ for all $x \in X$. The space X has the internal extension property and the sets A_n and the functions Φ_1, Φ_2 satisfy the conditions of Theorem (4.2). Hence there exist two functions in $S(X)$ which generate a semigroup containing $\{f_n\}_{n=1}^{\infty}$.

THEOREM (4.4). *Let X be the Cantor set K. Then given any countable family $\{f_n\}_{n=1}^{\infty}$ of $S(X)$, there exist two functions in $S(X)$ which generate a semigroup containing $\{f_n\}_{n=1}^{\infty}$.*

Proof. Here again X has the internal extension property [5, p. 281]. We let $A_n = \left[\frac{2}{3^n}, \frac{1}{3^{n-1}}\right] \cap K$ for $n \in N$ and see that all the conditions of Theorem (4.1) are satisfied. The conclusion follows.

LEMMA (4.5). *Let X be any 0-dimensional separable metric space which is the countable union of clopen sets each homeomorphic to X. Then given any countable family $\{f_n\}_{n=1}^{\infty}$ of $S(X)$, there exist two functions in $S(X)$ which generate a subsemigroup containing $\{f_n\}_{n=1}^{\infty}$.*

Proof. The space X has the internal extension property [5, p. 281]. Conditions (2) and (3) of Theorem (4.2) are certainly satisfied because $X = \bigcup_{n=1}^{\infty} A_n$ and so there are no extensions involved. Condition (1) of Theorem (4.2) is also satisfied because $\bigcup_{n=2}^{\infty} A_n$ is a clopen subset of X and so any function continuous on $\bigcup_{n=2}^{\infty} A_n$ can be extended to all of X. Hence the conclusion of Theorem (4.2) follows.

THEOREM (4.6). *Let X be the rationals, the irrationals or the countable discrete space. Then, given any countable family $\{f_n\}_{n=1}^{\infty}$ of $S(X)$ there exist two functions in $S(X)$ which generate a semigroup containing $\{f_n\}_{n=1}^{\infty}$.*

Proof. Each of these spaces is 0-dimensional and is the countable union of clopen sets each homeomorphic to the whole space. Hence the conclusion follows by Lemma (4.5).

Since every countable semigroup can be embedded in the semigroup of all selfmaps of a countably infinite set, we immediately get a well-known theorem of T. Evans as a corollary to our previous theorem.

THEOREM (4.7) [Evans [3]]. *Every countable semigroup can be embedded in a semigroup with two generators.*

If one is interested only in getting Evans' theorem this is not the most efficient way to do it. A very short proof based on similar ideas is given in [9].

THEOREM (4.8). *Let X be the m-dimensional closed unit cube I^m, the irrationals, the rationals, the Cantor set or the countable discrete space and let $S(X)$ have the compact-open topology. Then there is a subsemigroup of $S(X)$, generated by two functions, which is dense in $S(X)$.*

Proof. We know that $S(X)$ is separable [6, p. 921]. If $\{f_n\}_{n=1}^{\infty}$ is the family that is dense in $S(X)$, then by Theorems (4.3), (4.4) and (4.6) there exist two functions which generate a subsemigroup of $S(X)$ containing $\{f_n\}_{n=1}^{\infty}$. Clearly this subsemigroup is dense in $S(X)$.

We now look at another class of spaces such that for any space X in the class $S(X)$ has a dense subsemigroup which is generated by three functions.

LEMMA (4.9). *Let X be any space containing a countable collection of mutually disjoint sets A_n, each homeomorphic to X, such that each A_n is clopen in $\bigcup\{A_n : n \in N\}$. Suppose further that there exists a function Φ_1 in $S(X)$ such that $\Phi_1|_{A_n} = h_{n+1}^{-1} \circ h_n$ for each n, where h_n is the homeomorphism mapping A_n onto X. Finally, let $\{f_n\}_{n=1}^{\infty}$ be any subfamily of $S(X)$ such that $f|_{A_n} = f_n \circ h_n$ for each n and some $f \in S(X)$. Then there are three functions Ψ_1, Ψ_2 and Ψ_3 such that $\{f_n\}_{n=1}^{\infty}$ is contained in the semigroup generated by Ψ_1, Ψ_2 and Ψ_3.*

Proof. We let $\Psi_1 = \Phi_1$, $\Psi_2 = h_1^{-1}$ and $\Psi_3 = f$. Then $\Psi_1^{n-1}|_{A_n} = h_1^{-1} \circ h_1$ and we obtain

$$f_n = \Psi_3 \circ \Psi_1^{n-1} \circ \Psi_2.$$

We shall denote by $C(X, Y)$ the space of all continuous functions from X into Y where $C(X, Y)$ has the compact-open topology.

LEMMA (4.10). *Let J^m be the product of m copies of the open unit interval and let p be any point of J^m. Then there exists a countable collection $\mathcal{F}$ of functions in $S(J^m)$ and a countable collection $\{B_n\}_{n=1}^{\infty}$ of compact subsets of J^m such that*

(1) *each compact subset of J^m is contained in some B_n,*

(2) *for each B_n the family $\mathcal{F}^*$ of restrictions of members of $\mathcal{F}$ to B_n is dense in $C(B_n, J^m)$, and*

(3) *for each $f \in \mathcal{F}$, there is a compact subset K_f of J^m such that $f(x) = p$ for $x \in J^m - K_f$.*

Proof. For each positive integer n take B_n to be the product of m copies of the closed interval $\left[\frac{1}{n+1}, 1-\frac{1}{n+1}\right]$. Since $C(B_n, J^m)$ is separable, there exists a countable collection $\{g_{h,k}\}_{k=1}^{\infty}$ of functions which is dense in $C(B_n, J^m)$. Extend $g_{n,k}$ continuously over $A_n \cup B_n$ to a function $\hat{g}_{n,k}$ by defining $\hat{g}_{n,k}(x) = p$ for x in A_n where $A_n = J_m - B_{n+1}$. Since $A_n \cup B_n$ is closed and J^m is an absolute retract, each $\hat{g}_{n,k}$ can be continuously extended to a function $f_{n,k}$ in $S(J^m)$. Take $\mathcal{F} = \{f_{n,k}\colon\ n, k \in N\}$ and the proof is complete.

THEOREM (4.11). *Let X be any Euclidean m-space and let $S(X)$ have the compact-open topology. Then there exist three functions in $S(X)$ which generate a dense subsemigroup.*

Proof. It will be convenient for us to take X to be one of the spaces J^m. Take A_n to be the product of m copies of the open interval $\left(\frac{1}{2^{2n-1}}, \frac{1}{2^{2n-2}}\right)$ and let h_n be the homeomorphism from A_n onto J^m which is given by

$$(\pi_i \circ h_n)(x) = 2^{2n-1}x_i - 1 \quad \text{for} \quad i = 1, \ldots, m.$$

Let Φ_1 be the function in $S(J^m)$ which is given by

$$(\pi_i \circ \Phi_1)(x) = \tfrac{1}{4}x_i \quad \text{for} \quad i = 1, \ldots, m.$$

One easily checks that Φ_1 satisfies the condition required in Lemma (4.8). That is, $\Phi_1|_{A_n} = h_{n+1}^{-1} \circ h_n$ for each n. Let $\{f_n\}_{n=1}^{\infty}$ be the family $\mathcal{F}$ of Lemma (4.9) and define a function f in $S(J^m)$ as follows: for x in A_n let $f(x) = f_n(h_n(x))$ and for x in $J^m - \bigcup_{n=1}^{\infty} A_n$, let $f(x) = p$. Because of condition (3) of Lemma (4.10) the function f is continuous. Moreover it satisfies the condition of Lemma (4.9) that $f|_{A_n} = f_n \circ h_n$ for each n. Thus Lemma (4.9) assures us that there are three functions in $S(J^m)$ which generate a subsemigroup that contains the family $\mathcal{F}$. The proof will be complete when we show this family is dense in $S(J^m)$.

Let $H = \langle K_1, G_1\rangle \cap \ldots \cap \langle K_t, G_t\rangle$ be a nonempty basic open subset of $S(J^m)$. According to condition (1) of Lemma (4.10) some B_n contains $\bigcup \{K_i\}_{i=1,2,\ldots,t}$. Then H is a nonempty basic open subset of $C(B_n, J^m)$ and hence contains one of the functions $g_{n,k}$. The corresponding function $f_{n,k}$ of $\mathcal{F}$ then belongs to H and the theorem is proved.

We cannot apply Theorem (4.2) to $S(E^m)$ in order to get a dense subsemigroup of $S(E^m)$ generated by two functions because E^m is not homeomorphic to a closed subset of E^m. We have obtained a three generator dense subsemigroup of $S(E^m)$ but it is still an open question whether this result can be improved upon.

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