On fundamental deformation retracts and on some related notions

by

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Abstract. The main result of this note is the following

THEOREM. *If* $Y_1 \supset Y_2 \supset \ldots$ *is a sequence of fundamental deformation retracts of a space* $X \in \mathrm{FANR}$ *then the set* $Y = \bigcap_{m=1}^{\infty} Y_m$ *is a fundamental deformation retract of* X.

Several notions, as the notion of the F-stability, of the FR-stability and of the S-stability of a compactum are introduced and also several problems formulated.

§ 1. Introduction. Among notions of the classical homotopy theory, the notion of the stable space, due to H. Hopf and E. Pannwitz ([4], p. 433) plays an essential role. A space X is said to be *stable* if for every map $f\colon X \to X$ homotopic to the identity map $i_X\colon X \to X$, the relation $f(X) = X$ holds true. In particular all closed, compact manifolds are stable.

A closed set $Y \subset X$ is said to be a *homotopy support* (or an H-*support*) of X if there exists a map $f\colon X \to X$ homotopic to i_X and such that $f(X) \subset Y$. Thus stable spaces are the same as spaces X without any H-support different from X. In particular, every deformation retract of X is an H-support of X, but not conversely. The question if every set being both a retract and an H-support of X is a deformation retract of X remains open. It is clear that

(1.1) *If* Y *is an* H-*support of* X *and* Z *is an* H-*support of* Y *then* Z *is an* H-*support of* X.

If X does not contain any deformation retract of X different from X, then X is said to be R-*stable*. Notice that

(1.2) *Every stable space is* R-*stable.*

But the converse is not true, even in the case of polyhedra, as it follows by the following well-known example due to H. Hopf and E. Pannwitz ([4], p. 435):

(1.3) EXAMPLE. Let L be an arc lying on a 2-sphere S and let a, b denote its endpoints. Denote by Z the curvilinear polyhedron which one obtains from S by identifying the points a and b. By this identification the arc L passes onto a simple closed curve C lying in Z. Let A be a disk with the boundary C and with the interior disjoint to Z. Setting

$$P = A \cup Z\,,$$

one gets a curvilinear polyhedron and it is known ([4], p. 447) that Z is an H-support of P, hence P is not stable. However P is R-stable, because none proper subset of P is a deformation retract of P.

The concepts of the stable and of R-stable spaces have in the case of spaces with a rather regular local structure (as polyhedra, or as ANR-spaces) a clear intuitive sense. However another situation is when one considers spaces with a more complicated local structure. The aim of this note is to apply the notions of the theory of shape in order to obtain concepts which, for arbitrary compacta, seem to be reasonable substitutes for the classical concepts of the stable and of R-stable spaces.

§ 2. F-stable and FR-stable spaces. Let X be a compactum lying in a space $M \in \mathrm{AR}(\mathfrak{M})$. A compactum $Y \subset X$ is said to be a *fundamental support* (or an F-*support*) of X (in M) if there exists a fundamental sequence

$$\underline{f} = \{f_k, X, Y\}_{M,M} \quad \text{such that} \quad \{f_k, X, X\}_{M,M} \simeq \underline{i}_{X,M}\,,$$

where $\underline{i}_{X,M}$ denotes the fundamental identity sequence $\{i, X, X\}_{M,M}$.

Let us observe that the choice of the space $M \in \mathrm{AR}(\mathfrak{M})$ containing X is immaterial, that is,

(2.1) *If Y is an* F-*support of X in M and if $X \subset N \in \mathrm{AR}(\mathfrak{M})$, then Y is an* F-*support of X in N.*

In order to see it, consider two maps

$$\alpha\colon M \to N \quad \text{and} \quad \beta\colon N \to M$$

such that $\alpha(x) = \beta(x) = x$ for every point $x \in X$. Setting $\alpha_k = \alpha$ and $\beta_k = \beta$ for $k = 1, 2, \ldots$, we get two fundamental sequences

$$\underline{\alpha} = \{\alpha_k, X, X\}_{M,N} \quad \text{and} \quad \underline{\beta} = \{\beta_k, X, X\}_{N,M}\,.$$

If Y is an F-support of X in M, then there is a fundamental sequence $\underline{f} = \{f_k, X, Y\}_{M,M}$ such that $\{f_k, X, X\}_{M,M} \simeq \underline{i}_{X,M}$. Setting

$$g_k(y) = \alpha f_k \beta(y) \quad \text{for every point } y \in N\,,$$

one gets a fundamental sequence $\underline{g} = \{g_k, X, Y\}_{N,N}$. It remains to show that $\{g_k, X, X\}_{N,N}$ is homotopic to $\underline{i}_{X,N}$. In order to do it, consider a neighborhood W of X in N. Since $N \in \mathrm{AR}(\mathfrak{M})$ and since $\alpha\beta/X = i/X$, there exists a neighborhood $W_0 \subset W$ of X in N such that

(2.2) $$\alpha\beta/W_0 \simeq i/W_0 \quad \text{in} \quad W.$$

Now we can select a neighborhood U of X in M such that

(2.3) $$\alpha(U) \subset W_0\,.$$

Since $\{f_k, X, X\}_{M,M} \simeq \underline{i}_{X,M}$, there exists a neighborhood $U_0 \subset U$ of X in M such that

(2.4) $$f_k/U_0 \simeq i/U_0 \text{ in } U \quad \text{for almost all } k\,.$$

We can assign to U_0 a neighborhood $V_0 \subset W_0$ of X in N such that $\beta(V_0) \subset U_0$. One infers by (2.2), (2.3) and (2.4) that

$$\alpha f_k \beta/V_0 \simeq \alpha\beta/V_0 \simeq i/V_0 \text{ in } W \quad \text{for almost all } k\,,$$

hence $\{g_k, X, X\}_{N,N} = \{\alpha f_k \beta, X, X\}_{N,N} \simeq \underline{i}_{X,N}$. Thus the proof of proposition (2.1) is finished.

It follows that in the definition of the F-support of X the words "in M" are superfluous. We can speak shortly on F-*supports* of X.

If there does not exist any F-support $Y \neq X$ of X, then X is said to be *fundamentally stable* (or F-*stable*). Let us notice that

(2.5) *Every* H-*support of X is an* F-*support of X.*

In fact, if $f\colon X \to X$ and $f(X) \subset Y \subset X$ then there exists a map $\hat{f}\colon M \to M$ such that $\hat{f}(x) = f(x)$ for every point $x \in X$. Setting $f_k = \hat{f}$ for $k = 1, 2, \ldots$, one gets two fundamental sequences

$$\underline{f} = \{f_k, X, Y\}_{M,M} \quad \text{and} \quad \{f_k, X, X\}_{M,M}\,.$$

If $f \simeq i_X$ then $\{f_k, X, X\}_{M,M} \simeq \underline{i}_{X,M}$ and consequently Y is an F-support of X. It follows that

(2.6) *If X is* F-*stable, then X is stable.*

Let us observe that

(2.7) *If $X \in \mathrm{ANR}$ and $Y \neq X$ is an* F-*support of X, then there is a compactum $Y' \neq X$ being an* H-*support of X.*

In fact, if Y is an F-support of X then there exists a fundamental sequence $\underline{f} = \{f_k, X, Y\}_{M,M}$ such that $\{f_k, X, X\}_{M,M} \simeq \underline{i}_{X,M}$. Since $X \in \mathrm{ANR}$, there exists an open neighborhood U of X (in M) and a retraction $r\colon U \to X$. We infer by $Y \neq X$ that there exists a neighborhood $V \subset U$ of Y in M such that $r(V) \neq X$. Moreover, there exists an index k_0 such that

(2.8) $$f_{k_0}/X \simeq i/X \text{ in } U \quad \text{and} \quad f_{k_0}(X) \subset V.$$

Setting $f(x) = rf_{k_0}(x)$ for every point $x \in X$, we get a map $f\colon X \to X$. It follows by (2.8) that $f = rf_{k_0}/X \simeq r/X = i/X$ in the set $r(U) = X$.

Moreover, $f(X) = rf_{k_0}(X) \subset r(V) \neq X$. Hence the set $Y' = f(X)$ is a required H-support of X.

It follows by (2.6) and (2.7) that

(2.9) *An* ANR-*space is stable if and only if it is* F-*stable.*

A simple example of a continuum X which is stable but not F-stable is the closure of the diagram of the function

$$y = \cos\frac{1}{x} + \cos\frac{1}{1-x} \quad \text{for} \quad 0 < x < 1.$$

Let us recall that a compactum $Y \subset X$ is said to be a *fundamental deformation retract* of X if there exists a fundamental sequence $\underline{r} = \{r_k, X, Y\}_{M,M}$ such that $r_k/Y = i/Y$ and that $\{r_k, X, X\}_{M,M} \simeq \underline{i}_{X,M}$. Let us add that using the same standard argument as by the proof of proposition (2.1), one shows that in the definition of the fundamental deformation retracts of X the choice of the space $M \in \mathrm{AR}(\mathfrak{M})$ containing X is immaterial.

It is well known that

(2.10) *If Y is a fundamental deformation retract of X then* $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$.

Moreover

(2.11) *Every fundamental deformation retract of X is an* F-*support of X, but not conversely.*

The question if every set being both a fundamental retract and an F-support of X is a fundamental deformation retract of X remains open.

If there does not exist any fundamental deformation retract $Y \neq X$ of X, then X is said to be *fundamentally* R-*stable* (or FR-*stable*). It is clear that

(2.12) *If X is* FR-*stable, then X is* R-*stable.*

Moreover, (2.11) implies that

(2.13) *If X is* F-*stable, then X is* FR-*stable.*

The following problem remains open:

(2.14) PROBLEM. *Does there exist an* R-*stable* ANR-*space which is not* FR-*stable?*

§ 3. Some properties of F-supports.

A preliminary information on shape properties of F-supports gives the following

(3.1) THEOREM. *If Y is an* F-*support of X then* $\mathrm{Sh}(X) \leqslant \mathrm{Sh}(Y)$.

Proof. Let $\underline{f} = \{f_k, X, Y\}_{M,M}$ be a fundamental sequence such that $\{f_k, X, X\}_{M,M} \simeq \underline{i}_{X,M}$. Setting $\underline{g} = \{i, Y, X\}_{M,M}$ we get a fundamental sequence $\underline{g}\colon Y \to X$ satisfying the condition

$$\underline{gf} = \{if_k, X, X\}_{M,M} = \{f_k, X, X\}_{M,M} \simeq \underline{i}_{X,M}.$$

Hence $\mathrm{Sh}(X) \leqslant \mathrm{Sh}(Y)$.

(3.2) THEOREM. *If Z is an* F-*support of an* F-*support Y of X, then Z is an* F-*support of X.*

Proof. Let $\underline{f} = \{f_k, X, Y\}_{M,M}$ and $\underline{g} = \{g_k, Y, Z\}_{M,M}$ be fundamental sequences such that $\{f_k, X, X\}_{M,M} \simeq \underline{i}_{X,M}$ and $\{g_k, Y, Y\}_{M,M} \simeq \underline{i}_{Y,M}$. If U is a neighborhood of X in M then there exists a neighborhood $\hat{U} \subset U$ of X in M such that

(3.3) $$f_k/\hat{U} \simeq i/\hat{U} \ \text{in} \ U \quad \text{for almost all } k.$$

But $\hat{U}$ is also a neighborhood of Y in M. Consequently, there exists a neighborhood $V \subset \hat{U}$ of Y in M such that

(3.4) $$g_k/V \simeq i/V \ \text{in} \ \hat{U} \quad \text{for almost all } k.$$

Moreover there exists a neighborhood $U' \subset \hat{U}$ of X in M such that

(3.5) $$f_k(U') \subset V \quad \text{for almost all } k.$$

It follows by (3.4) and (3.5) that

(3.6) $$g_k f_k/U' \simeq f_k/U' \ \text{in} \ \hat{U} \quad \text{for almost all } k.$$

Since $U' \subset \hat{U}$, one infers by (3.3) that $f_k/U' \simeq i/U'$ in U for almost all k, which together with the inclusion $\hat{U} \subset U$ and with the relation (3.6) gives

$$g_k f_k/U' \simeq i/U' \ \text{in} \ U \quad \text{for almost all } k.$$

Hence $\{g_k f_k, X, X\}_{M,M}$ is a fundamental sequence homotopic to $\underline{i}_{X,M}$. Since $\{g_k f_k, X, Z\}_{M,M} = \underline{gf}$ is a fundamental sequence, we conclude that Z is an F-support of X.

By a slight modification of the last proof, one gets the following proposition:

(3.7) *If Z is a fundamental deformation retract of Y and Y is a fundamental deformation retract of X, then Z is a fundamental deformation retract of X.*

§ 4. Decreasing sequences of fundamental deformation retracts.

Let us recall that a compactum X is said to be an FANR-*space* if for every compact space M containing X there exists a compact neighborhood U of X in M such that X is a fundamental retract of U (compare [1], p. 65). The class of all FANR-spaces is a shape invariant. It contains, in particular, all ANR-spaces and many other compacta, for instance all plane compacta with finite Betti numbers.

Let us prove the following

(4.1) THEOREM. *If $Y_1 \supset Y_2 \supset \ldots$ is a sequence of fundamental deformation retracts of a space $X \in$ FANR, then the set $Y = \bigcap_{m=1}^{\infty} Y_m$ is a fundamental deformation retract of X.*

Proof. Assume that X lies in the Hilbert cube Q. Then X has arbitrary small neighborhoods in Q being ANR-sets. Consequently, there exists a neighborhood $U \in$ ANR of X in Q and a fundamental retraction

(4.2) $$\underline{s} = \{s_k, U, X\}_{Q,Q}.$$

Let $V_1 \supset V_2 \supset \ldots$ be a sequence of neighborhoods of Y in Q, lying in U and shrinking to Y. Replacing the sequence $Y_1, Y_2, \ldots$ by a suitably selected subsequence, we may assume that

(4.3) V_m is a neighborhood of Y_m in Q for every $m = 1, 2, \ldots$

Let

(4.4) $\underline{r}^m = \{r_k^m, X, Y_m\}_{Q,Q}$ be a fundamental deformation retraction;

hence

(4.5) $$\{r_k^m, X, X\}_{Q,Q} \simeq \underline{i}_{X,Q} \quad \text{for every } m = 1, 2, \ldots$$

By virtue of (4.4), one sees readily that there exists a decreasing sequence $U_1 \supset U_2 \supset \ldots$ of neighborhoods of X in Q, lying in U and shrinking to X, such that

(4.6) For every $m = 1, 2, \ldots$ the inclusion $r_k^m(U_m) \subset V_m$ holds true for almost all k.

Moreover, for every $m = 1, 2, \ldots$ there is a neighborhood $V'_m \subset V_m$ of Y in Q such that

(4.7) $$r_k^m/V'_m \simeq r_{k+1}^m/V'_m \text{ in } V_m \quad \text{for almost all } k.$$

It follows by (4.6) and (4.7) that there is an increasing sequence of indices $\alpha(1) < \alpha(2) < \ldots$ such that

(4.8) $$r_k^m(U_m) \subset V_m \quad \text{and} \quad r_k^m/V'_m \simeq r_{\alpha(m)}^m/V'_m \text{ in } V_m \quad \text{for every } k \geqslant \alpha(m).$$

Since $r_{\alpha(m)}^m/Y = i/Y$, there exists a neighborhood $\hat{V}_m \subset V'_m$ of Y in Q such that $\hat{V}_{m+1} \subset \hat{V}_m$ and that

(4.9) $$r_{\alpha(m)}^m/\hat{V} \simeq i/\hat{V}_m \text{ in } V_m \quad \text{for every } m = 1, 2, \ldots$$

It follows by (4.8) and (4.9) (because $\hat{V}_m \subset V'_m$) that

(4.10) $$r_k^m/\hat{V}_m \simeq i/\hat{V}_m \text{ in } V_m \quad \text{for every } k \geqslant \alpha(m).$$

By virtue of (4.4) and (4.5) there exists for every $m = 1, 2, \ldots$ a neighborhood $\hat{U}_m \subset U_m$ of X in Q such that $\hat{U}_{m+1} \subset \hat{U}_m$ and a sequence of indices $\beta(1) < \beta(2) < \ldots$ such that $\beta(m) \geqslant \alpha(m)$ for $m = 1, 2, \ldots$ and that

(4.11) $$r_k^m(\hat{U}_m) \subset \hat{V}_m \quad \text{and} \quad r_k^m/\hat{U}_m \simeq i/\hat{U}_m \text{ in } U_m \quad \text{for every } k \geqslant \beta(m).$$

Moreover, we infer by (4.2) that there exists a sequence of indices $\gamma(1) < \gamma(2) < \ldots$ such that $\gamma(m) \geqslant \beta(m)$ for every $m = 1, 2, \ldots$ and that

(4.12) $$s_k/U \simeq s_{\gamma(m)}/U \text{ in } \hat{U}_m \quad \text{for every } k \geqslant \gamma(m).$$

Since $s_k/X = i/X$ for every $k = 1, 2, \ldots$, we infer that

(4.13) $$\{s_{\gamma(m)}, X, X\}_{Q,Q} \simeq \{s_k, X, X\}_{Q,Q} \simeq \underline{i}_{X,Q}.$$

Setting

(4.14) $$r_m = r_{\gamma(m)}^m s_{\gamma(m)} \quad \text{for every } m = 1, 2, \ldots,$$

we get a sequence of maps $r_m\colon Q \to Q$ and we infer by (4.11), (4.12) and by the inequality $\gamma(m) \geqslant \beta(m)$ that

(4.15) $$r_m/U = r_{\gamma(m)}^m s_{\gamma(m)}/U \simeq s_{\gamma(m)}/U \text{ in } U_m \quad \text{for every } m = 1, 2, \ldots$$

It follows by (4.13) that

(4.16) $$\{r_m, X, X\}_{Q,Q} \simeq \{s_{\gamma(m)}, X, X\}_{Q,Q} \simeq \underline{i}_{X,Q}.$$

From (4.11) and (4.12), and since $\hat{V}_{m+1} \subset \hat{V}_m$, we infer that

(4.17) $$r_m(U), r_{m+1}(U) \subset \hat{V}_m \quad \text{for every } m = 1, 2, \ldots$$

Now let us set

(4.18) $$\omega_m = r_{\alpha(m)}^m \quad \text{for every } m = 1, 2, \ldots$$

It follows by (4.10) and (4.8) that

(4.19) $$\omega_m/\hat{V}_m \simeq i/\hat{V}_m \text{ in } V_m \quad \text{for every } m = 1, 2, \ldots,$$

and

(4.20) $$\omega_m(U_m) \subset V_m \quad \text{for every } m = 1, 2, \ldots$$

By virtue of (4.15) and (4.20) one infers that

(4.21) $$\begin{gathered} \omega_m r_m/U \simeq \omega_m s_{\gamma(m)}/U \quad \text{in } \omega_m(U_m) \subset V_m, \\ \omega_m r_{m+1}/U \simeq \omega_m s_{\gamma(m+1)}/U \text{ in } \omega_m(U_{m+1}) \subset V_m. \end{gathered}$$

Using (4.17) and (4.19), we infer that

(4.22) $$\omega_m r_m/U \simeq r_m/U \text{ in } V_m \quad \text{and} \quad \omega_m r_{m+1}/U \simeq r_{m+1}/U \text{ in } V_m$$

for every $m = 1, 2, \ldots$

Moreover, (4.12), (4.20) and the inclusion $\hat{U}_m \subset U_m$ imply that

(4.23) $$\omega_m s_{\gamma(m)}/U \simeq \omega_m s_{\gamma(m+1)}/U \text{ in } V_m .$$

It follows by (4.21), (4.22) and (4.23) that

$$r_m/U \simeq r_{m+1}/U \text{ in } V_m \quad \text{for every } m = 1, 2, \ldots,$$

hence $\underline{r} = \{r_m, X, Y\}_{Q,Q}$ is a fundamental sequence. Moreover, (4.14) implies that $r_m/Y = i/Y$ and we conclude that $\underline{r}$ is a fundamental retraction satisfying (4.16). Hence Y is a fundamental deformation retract of X and the proof of Theorem (4.1) is finished.

(4.24) Problem. *Does Theorem* (4.1) *remain true if we replace the hypothesis that $X \in$ FANR by a weaker one that X is a movable compactum?*

(4.25) Problem. *Is it true that if $Y_1 \supset Y_2 \supset \ldots$ is a sequence of fundamental retracts of a space $X \in$ FANR, then the set $Y = \bigcap_{m=1}^{\infty} Y_m$ is a fundamental retract of X?*

By an example due to C. Cox ([3], p. 175) if we replace in this problem the hypothesis that $X \in$ FANR by a weaker one that X is movable, then the answer would be negative. In fact, consider in the Euclidean 3-space E^3 a sequence $T_1, T_2, \ldots$ of topological tori such that T_{m+1} lies in the interior of T_m for every $m = 1, 2, \ldots$ and that the set $Y = \bigcap_{m=1}^{\infty} E_m$ is a solenoid of Van Dantzig. Let $\dot{T}_m$ denote the boundary of T_m. Then the set

$$X = Y \cup \bigcup_{m=1}^{\infty} \dot{T}_m$$

is a movable compactum (see [2], p. 140) and

$$Y_k = Y \cup \bigcup_{m=k}^{\infty} \dot{T}_m$$

is a retract (hence also a fundamental retract) of X and $Y_{k+1} \subset Y_k$ for every $k = 1, 2, \ldots$ However the set $Y = \bigcap_{k=1}^{\infty} Y_k$, being a non-movable compactum, is not a fundamental retract of X.

Using Theorems (3.7) and (4.1), one gets the following

(4.26) Corollary. *If $Y_1 \supset Y_2 \supset \ldots$ are FANR-spaces and if Y_{m+1} is a fundamental deformation retract of Y_m for $m = 1, 2, \ldots$, then the set $Y = \bigcap_{m=1}^{\infty} Y_m$ is an FANR-space.*

Using the well-known Brouwer Reduction Theorem ([5], p. 161), we obtain from Theorem (4.1) the following

(4.27) Corollary. *For every space $X \in$ FANR there exists an FR-stable compactum $Y \subset X$ being a fundamental deformation retract of X.*

It follows, in particular

(4.28) Corollary. *Every space $X \in$ FANR contains an FR-stable compactum $Y \in \mathrm{Sh}(X)$.*

(4.29) Problem. *Is it true that for every sequence $Y_1 \supset Y_2 \supset \ldots$ of F-supports of a compactum X the set $Y = \bigcap_{m=1}^{\infty} Y_m$ is an F-support of X?*

(4.30) Problem. *Do Corollaries* (4.27) *and* (4.28) *remain true if we assume only that X is a compactum?*

(4.31) Problem. *Is it true that for every compactum X there exists an F-stable compactum $Y \in \mathrm{Sh}(X)$?*

§ 5. S-stable compacta. Let us say that a compactum X is *shape-stable* (or is S-*stable*) if $\mathrm{Sh}(X) \neq \mathrm{Sh}(Y)$ for every compactum $Y \subsetneq X$. For instance, each closed manifold is S-stable. Also every continuum $X \subset E^n$ decomposing the space E^n and being the common boundary of each component of the set $E^n \setminus X$ is S-stable. Also the polyhedron P of H. Hopf and E. Pannwitz, mentioned in § 1, is S-stable (though it is not stable), because one sees easily that the shape of every compactum $Y \subsetneq P$ differs from $\mathrm{Sh}(P)$.

It is clear that every S-stable compactum is FR-stable, but the converse is not true. For instance, if X is the union of all circles C_k, $k = 1, 2, \ldots$, given in the plane E^2 by the equations

$$(x_1 - k^{-1})^2 + x_2^2 = k^{-2},$$

then X is FR-stable (it is even F-stable), but it is not S-stable. Moreover one sees readily that X does not contain any S-stable compactum $X_0 \in \mathrm{Sh}(X)$. However there exists an S-stable plane continuum X_0 such that $\mathrm{Sh}(X) = \mathrm{Sh}(X_0)$. In fact, this property has each continuum X_0 decomposing E^2 into $\aleph_0$ regions and being the common boundary of each of these regions. Let us also observe that for Cantor discontinuum D there does not exist any S-stable compactum $D_0 \in \mathrm{Sh}(D)$.

The following problems remain open:

(5.1) PROBLEM. *Does there exist for every continuum X an* S-*stable continuum* $X_0 \in \mathrm{Sh}(X)$?

(5.2) PROBLEM. *Is it true that* S-*stable* FANR-*spaces are the same as* FR-*stable* FANR-*spaces?*

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An open-perfect mapping of a hereditarily disconnected space onto a connected space

by

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Abstract. In this paper the authors construct an example of open-perfect mapping of a hereditarily disconnected space onto a connected space. Both of the spaces are metrizable and separable.

The aim of this paper is to construct the hereditarily disconnected space X and the open-perfect mapping f of X onto the connected space Y. Both of the spaces will be separable metrizable. We should mention here that 0-dimensionality and totally disconnectedness are invariants under open-closed mappings in the class of metrizable spaces. The first of these facts is obvious, the second was proved by Kombarov [3]. The basis of our construction is the well-known Knaster–Kuratowski Broom, the subspace of the plane with dispersion point (see [4], § 46, II). At the end of this paper we shall prove the theorem related to this object.

Terminology and notation are as in [2] and [4]. In particular, the word "mapping" and the symbol $g\colon A \to B$ mean continuous mapping, the symbol $g\colon A \twoheadrightarrow B$ means that $g(A) = B$. By connected space we always understand non-one-point space. For $x \in X$ and $A \subset X$ we write $x \times A$ instead of $\{x\} \times A$.

1. We start from some auxiliary constructions in the Euclidean plane R^2 with the standard metric ϱ. The symbol R denotes the set of all real numbers, N denotes the set of non-negative integers, I is the interval $[-1, 1]$ of reals, P and Q irrationals and rationals of I respectively. For $t \in R$ let $\bar{t} = (t, 0) \in R^2$.

Divide the set P into two disjoint, dense in P sets P^* and Q^* such that $\overline{\overline{Q^*}} = \aleph_0$. Let

$$M = P^* \times P \cup Q^* \times Q .$$

For $x = (x_1, x_2) \in R^2$ and real number $\alpha > 0$ we set

$$U(x, \alpha) = \{y \in R^2 | \varrho(y, x') < \alpha , \quad \text{where} \quad x' = (x_1, x_2 \mp \alpha)\} \cup \{x\} ,$$

and

$$K(x, \alpha) = \{y \in R^2 | \varrho(y, x) < \alpha\} .$$