As a final remark we note that under (31) the monotone sequence $\langle d[f^n B]\rangle$ for B a bounded subset of X either converges to 0 or is ultimately constant.

References

[1] D. F. Bailey, *Some theorems on contractive mappings*, J. London Math. Soc. 41 (1966), pp. 101–106.

[2] S. Banach, *Sur les opérations dans les ensembles abstraits et leur applications aux intégrales*, Fund. Math. 3 (1922), pp. 133–181.

[3] F. E. Browder, *On the convergence of successive approximations for nonlinear functional equations*, Proc. Kon. Nedrl. Akad. Wetensch. Amsterdam A71 (1968), pp. 27–35.

[4] M. Edelstein, *On fixed and periodic points under contractive mappings*, J. London Math. Soc. 37 (1962), pp. 74–79.

[5] A. Goetz, *On a notion of uniformity for L-spaces of Fréchet*, Colloq. Math. 9 (1962), pp. 223–231.

[6] R. Kannan, *Some results on fixed points*, Bull. Calcutta Math. Soc. 60 (1968), pp. 71–76.

[7] — *Some results on fixed points II*, Amer. Math. Monthly 76 (1969), pp. 405–408.

[8] R. J. Knill, *Fixed points of uniform contractions*, J. Math. Anal. Appl. 12 (1965), pp. 449–455.

[9] A. Meir and E. Keeler, *A theorem on contraction mappings*, J. Math. Anal. Appl. 28 (1969), pp. 326–329.

[10] S. G. Mrówka and W. J. Pervin, *On uniform connectedness*, Proc. Amer. Math. Soc. 15 (1964), pp. 446–449.

[11] S. A. Naimpally and B. D. Warrack, *Proximity Spaces*, Cambridge 1970.

[12] S. Reich, *Some remarks concerning contraction mappings*, Canad. Math. Bull. 14 (1971), pp. 121–124.

[13] — Problem 5775, Amer. Math. Monthly 78 (1971), p. 84.

[14] V. M. Sehgal, *A fixed point theorem for mappings with a contractive iterate*, Proc. Amer. Math. Soc. 23 (1969), pp. 631–634.

[15] E. Wattel and P. P. N. de Groen, *A general fixed point theorem*, Math. Cent. Amsterdam Afd. Zuiv. Wisk. 9 (1969), p. 5.

[16] C. S. Wong, *A fixed point theorem for certain functions on a complete Hausdorff uniform space*, Notices Amer. Math. Soc. 18 (1971), p. 191.

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Quasi-nonexpansive multi-valued maps and selections

by

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Abstract. Two classes of quasi-nonexpansive multi-valued maps are investigated. (1) Let f be a map of a metrically convex complete metric space X into the family of all non-empty compact subsets of X. Then f has a fixed point if there exists a self map φ on $[0, \infty)$ such that $\varphi(t) < t$ for all $t > 0$ and f is φ-contractive. (2) Let X be a weakly compact convex subset of a Banach space B and f be a continuous map of X into the family of all non-empty closed convex subset of X. Then f has a fixed point and has a Kannan selection; the selection so chosen is continuous if further B is strictly convex and X is compact. The relation between selections and fixed points are investigated. As an example, it is proved that every Kannan map of the unit interval into its subintervals has a Kannan selection and therefore a fixed point; all such maps can be explicitly illustrated.

1. Introduction. Let (X, d) be a (non-empty) metric space. Let $bc(X)$ be the family of all non-empty bounded closed subsets of X endowed with the Hausdorff metric D induced by d [9]. Let f be a map of X into $bc(X)$. f is *contractive* (*nonexpansive*) *at a point* x in X if $D(f(x), f(y)) < d(x, y)$ $(\leqslant d(x, y))$ for all y in X other than x. Let φ be a map of $[0, \infty)$ into itself. φ is *contractive* if $\varphi(0) = 0$ and $\varphi(t) < t$ for all $t > 0$. Let φ be a contractive self map on $[0, \infty)$. f is *φ-contractive at a point* x in X if $D(f(x), f(y)) \leqslant \varphi(d(x, y))$ for all y in X. f is *nonexpansive* (*contractive*, *φ-contractive*) on X if f is nonexpansive (resp. contractive, φ-contractive) at each point in X. f is *quasi-nonexpansive* (*quasi-contractive*, *quasi-φ-contractive*) if the fixed point set $F_f = \{x \in X\colon x \in f(x)\}$ of f is non-empty and f is nonexpansive (resp. contractive, φ-contractive) at each point in F_f. For convenience, we shall identify a singleton with the point it contains. Thus if f is single-valued, our notion of quasi-nonexpansiveness coincides with the one introduced by W. G. Dotson [6].

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One reason for our interest in such maps is as follows. A fixed point of f usually corresponds to a solution of certain equation. For solving an equation, one cares only the existence of a solution and how to approximate it; so it is not important whether f is nonexpansive at points outside F_f or not. Of course, the nonexpansiveness of f outside F_f may enforce stronger properties of f on F_f. The two classes of maps which we are dealing with are as follows. (A) Let φ be a self map on $[0, \infty)$. φ is a *distribution function* on $[0, \infty)$ if φ is continuous from the right and is monotonically non-decreasing on $[0, \infty)$. Let f be a map on a complete metric space (X, d) into the family $c(X)$ of all non-empty compact subsets of X. We prove that f has a fixed point in X if there exists a contractive distribution self map φ on $[0, \infty)$ such that f is φ-contractive. Related work in this direction can be found in [16]. (B) Let f be a map on a metric space (X, d) into $(\mathrm{bc}(X), D)$. f is *Kannan* if

$$D(f(x), f(y)) \leqslant \tfrac{1}{2}\big(d(x, f(x)) + d(y, f(y))\big) \quad \text{for all } x, y \text{ in } X$$
$$(d(x, A) = \inf\{d(x, y):\ y \in A\},\ x \in A,\ A \subset X).$$

Single-valued Kannan maps were first considered by R. Kannan [10], [11], [12]. We need one more definition for stating our results. A function g (not necessarily continuous) on X is a *selection* of f if $g(x) \in f(x)$ for each x in X [14]. We shall prove that: (a) If X is a weakly compact convex subset of a separable Banach space and if f is a Kannan map of X into the family $\mathrm{wcc}(X)$ of all non-empty weakly compact convex subsets of X, then f has a fixed point if and only if f has a Kannan selection. (b) If X is a compact convex subset of a strictly convex Banach space and if f is a continuous Kannan map of X into $\mathrm{wcc}(X)$, then f has a fixed point and a continuous selection. A simple characterization of Kannan maps with fixed points is given. As an example, it is shown that every Kannan map of the unit interval into the family of its subintervals has a fixed point; the family of all such maps is explicitly illustrated.

2. φ-contractive multi-valued maps. Let (X, d) be a metric space. For any non-empty subset A of X, $\delta(A)$ will denote the diameter of A, i.e. $\delta(A) = \sup\{d(x, y):\ x, y \in A\}$.

THEOREM 1. *Let (X, d) be a complete metric space and f be a map of X into $\mathrm{bc}(X)$. Suppose that there exists a contractive distribution function φ on $[0, \infty)$ such that f is φ-contractive. Then*

(a) *There exists a unique non-empty bounded closed subset X_0 of X such that*

$$X_0 = \mathrm{cl} \bigcup \{f(x):\ x \in X_0\}.$$

(b) *If $f(x)$ is compact for each x in X, then f has a fixed point.*

Proof. (a) Consider the lifting map F on $\mathrm{bc}(X)$ defined by

$$F(A) = \mathrm{cl} \bigcup \{f(x):\ x \in A\}, \qquad A \in \mathrm{bc}(X).$$

It suffices to prove that F has a unique fixed point. We shall first prove that F is a self map on $\mathrm{bc}(X)$. Let $A \in \mathrm{bc}(X)$, $x, x_0 \in A$, $y \in f(x)$, $y_0 \in f(x_0)$. Then

$$D(f(x), f(x_0)) \leqslant \varphi(d(x, x_0)) \leqslant d(x, x_0) \leqslant \delta(A).$$

Now use the definition of Hausdorff metric, one can prove that

$$d(y, y_0) \leqslant \delta(A) + \delta(f(x_0)).$$

So

$$\delta(F(A)) \leqslant 2\big(\delta(A) + \delta(f(x_0))\big)$$

and therefore $F(A)$ is bounded. By repeating the argument in the proof of Theorem 1 in [16], we conclude that F is φ-contractive. It can be proved that $(\mathrm{bc}(X), D)$ is complete. So by a result of D. W. Boyd and J. S. W. Wong [2, Theorem 1], F has a unique fixed point X_0.

(b) Let G be the restriction of F in (a) to $c(X)$. Let $A \in c(X)$. Since f is continuous and $f(x)$ is compact for each x in A, by a result of E. Michael [13, Theorem 4.2], $\bigcup \{f(x):\ x \in X\}$ is compact. So G is a self map on $c(X)$. Since G is φ-contractive and $(c(X), D)$ is complete, G has a fixed point X_0. So f/X_0 (f restricted to X_0) is a φ-contractive and therefore a contractive self map on X_0. By a result of R. B. Fraser and Sam B. Nadler Jr. [8, Theorem 4], f/X_0 and therefore f has a fixed point.

We need the following definition for our next result. Let (X, d) be a metric space. X is *metrically convex* [1, p. 41] if for any distinct x, y in X, there exists z in X different from x and y such that $d(x, y) = d(x, z) + d(z, y)$. Note that every normed linear space is metrically convex. In Theorem 1, with metric convexity on X, we can drop the condition that φ is a distribution function.

THEOREM 2. *Let (X, d) be a metrically convex complete metric space and f be a map of X into $\mathrm{bc}(X)$. Suppose that there exists a contractive self map on $[0, \infty)$ such that f is φ-contractive. Then*

(a) *There exists a unique non-empty bounded closed subset X_0 of X such that*

$$X_0 = \mathrm{cl} \cup \{f(x):\ x \in X_0\}.$$

(b) *If $f(x)$ is compact for each x in X, then f has a fixed point.*

Proof. For each $t \geqslant 0$, let

$$\alpha(t) = \sup\{d(f(x), f(y)):\ x, y \in X,\ d(x, y) \leqslant t\}.$$

By Lemma 1 in [16], α is monotonically non-decreasing and is continuous from the right. Since $\alpha \leqslant \varphi$, $\alpha(t) < t$ for all $t > 0$. So (a), (b) follow from Theorem 1.

(b) in Theorem 2 generalizes a result of D. W. Boyd and J. S. W. Wong [2, Theorem 2] for single-valued maps.

We conclude this section with the following open problem:

PROBLEM 1. Let (X, d) be a complete metric space. Let f be a map of X into $c(X)$. Suppose that there exists a contractive self map φ on $[0, \infty]$ such that φ is upper semicontinuous from the right and f is φ-contractive. Does f have a fixed point?

f in Problem 1 has a fixed point if f is single-valued [2, Theorem 1].

3. Kannan maps. Before investigating Kannan maps, we first note that every Kannan map is nonexpansive at its fixed points. In fact, we can prove the following more general result. Let (X, d) be a metric space and f be a map of X into $\mathrm{bc}(X)$. Suppose that for any x, y in X, $D(f(x), f(y))$ lies in the convex hull of

$$\{d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x)), d(x, y)\}.$$

Then f is nonexpansive at its fixed points. Now suppose that f is a Kannan map. Then f is quasi-nonexpansive if and only if f has a fixed point. Since a Kannan map may not even be contractive, it is too much to hope that f has a fixed point. Thus we assume that X is a weakly compact convex subset of a Banach space B. Then f may not have a fixed point. However, if f is single-valued and continuous on X, then f has a fixed point [17, Theorem 1]. In this section, we shall prove that f has a fixed point if f is continuous. In [18, Theorem 3], it is proved that f has a fixed point if f is single-valued and X has close-to-normal structure, i.e. for any closed convex subset H of X with $\delta(H) > 0$, there exists x in H for which $\|x-y\| < \delta(H)$ for all y in H. Thus f has a fixed point if X has close-to-normal structure and f has a Kannan selection (a selection which is a Kannan map). This leads to the following related problem which is of interest in itself.

PROBLEM 2. Let X be a weakly compact convex subset of a Banach space and f be a Kannan map of X into $\mathrm{cc}(X)$. (a) Does f have a Kannan selection? (b) When f is continuous, does f have a continuous Kannan selection? ($\mathrm{cc}(X) = \{A \in \mathrm{wcc}(X)\colon A \text{ is compact}\}$).

In this section, we shall solve the above problem for the case when f is continuous, B is strictly convex and X is compact convex.

Let B be a Banach space. We shall denote by d the metric for B induced by the norm $\|\ \|$ on B. For any subset X of B, $\mathrm{wc}(X)$ denotes the family of all non-empty weakly compact subsets of X.

THEOREM 3. *Let X be a non-empty subset of a Banach space B. Let f be a Kannan map of X into $\mathrm{wc}(X)$ which has a fixed point a. Then f has a Kannan selection with unique fixed point a.*

Proof. Let $x \in X$. Since $f(x)$ is weakly compact, there exists a point $f_a(x)$ in $f(x)$ such that $d(a, f_a(x)) = d(a, f(x))$. Thus f_a is a selection of f. For any x, y in X,

$$\begin{aligned} d(f_a(x), f_a(y)) &\leqslant d(a, f_a(x)) + d(a, f_a(y)) = d(a, f(x)) + d(a, f(y)) \\ &\leqslant D(f(a), f(x)) + D(f(a), f(y)) \\ &\leqslant \tfrac{1}{2} d(x, f(x)) + \tfrac{1}{2} d(y, f(y)) \\ &\leqslant \tfrac{1}{2} d(x, f_a(x)) + \tfrac{1}{2} d(y, f_a(y)). \end{aligned}$$

Hence f_a is a Kannan selection of f. Clearly a is the unique fixed point of f_a.

Let X be a weakly compact convex subset of a Banach space B. For our next result, we need to emphasize that to assume that X has a close-to-normal structure is a very weak condition. It is proved in [19] that X has a close-to-normal structure if any one of the following conditions holds: (a) X has normal structure [3], (b) B is strictly convex, (c) B is separable, (d) B has property (A), i.e. for any sequence $\{x_n\}$ in B, $\{x_n\}$ converges to a point x in B if it converges weakly to x and if $\{\|x_n\|\}$ converges to $\|x\|$ [14]. It is proved in [18] that every Kannan self map on a non-empty weakly compact convex subset of B has a fixed point if and only if every weakly compact convex subset of B has a close-to-normal structure. So we have, from Theorem 3, the following result.

THEOREM 4. *Let X be a non-empty weakly compact convex subset of a Banach space B which has a close-to-normal structure and f be a Kannan map of X into $\mathrm{wc}(X)$. Then f has a Kannan selection if and only if f has a fixed point.*

THEOREM 5. *Let K be a non-empty weakly compact convex subset of a Banach space B and T be a Kannan map of K into $\mathrm{wcc}(K)$. Then*

(a) *There exists x_0 in K such that $d(x_0, T(x_0)) \leqslant d(x, T(x))$ for all x in K, i.e. the map $x \to d(x, T(x))$ on K attains its infimum r_0.*

(b) *The set $A = \{x \in K\colon d(x, T(x)) = r_0\}$ is T-invariant, i.e. $T(x) \subseteq A$ for each x in A.*

(c) *A contains a non-empty T-invariant closed convex subset of K.*

Proof. (a) For each $r \geqslant 0$, let $K_r = \{x \in K\colon d(x, T(x)) \leqslant r\}$. Since K is bounded, the set $I = \{r \geqslant 0\colon K_r \neq \emptyset\}$ is non-empty. It is sufficient to show that $\bigcap_{r \in I} K_r \neq \emptyset$.

For each $r \in I$, let H_r be the closed convex hull $\operatorname{cl}\operatorname{co}(T(K_r))$ of $T(K_r)$ $\big(= \bigcup_{x \in K} T(x)\big)$. Then $\{H_r: r \in I\}$ is a family of weakly compact convex subsets of K which has the finite intersection property and therefore has a non-empty intersection. Thus it suffices to prove that $H_r \subset K_r$ for each r in I. Let $r \in I$, $y \in H_r$ and $\varepsilon > 0$. Then there exist t_i in $[0, 1]$, y_i in K_r and $\bar{y}_i$ in $T(y_i)$, $i = 1, 2, \ldots, n$, such that

$$\sum_{i=1}^{n} t_i = 1$$

and

$$d\Big(y, \sum_{i=1}^{n} t_i \bar{y}_i\Big) < \varepsilon .$$

Thus

$$d\big(y, T(y)\big) \leqslant d\Big(y, \sum_{i=1}^{n} t_i \bar{y}_i\Big) + d\Big(\sum_{i=1}^{n} t_i \bar{y}_i, T(y)\Big). \tag{1}$$

Since $T(y)$ is weakly compact, for each $i = 1, \ldots, n$, there exists z_i in $T(y)$ such that $d(\bar{y}_i, z_i) = d(\bar{y}_i, T(y))$. Since

$$\sum_{i=1}^{n} t_i z_i \in T(y),$$

we have

$$\begin{aligned} d\Big(\sum_{i=1}^{n} t_i \bar{y}_i, T(y)\Big) &\leqslant d\Big(\sum_{i=1}^{n} t_i \bar{y}_i, \sum_{i=1}^{n} t_i z_i\Big) \leqslant \sum_{i=1}^{n} t_i d(\bar{y}_i, z_i) \\ &\leqslant \sum_{i=1}^{n} t_i d\big(\bar{y}_i, T(y)\big) \leqslant \sum_{i=1}^{n} t_i D\big(T(y_i), T(y)\big) \\ &\leqslant \sum_{i=1}^{n} t_i \Big(\tfrac{1}{2} d\big(y_i, T(y_i)\big) + \tfrac{1}{2} d\big(y, T(y)\big)\Big) \\ &\leqslant \sum_{i=1}^{n} t_i \Big(\tfrac{1}{2} r + \tfrac{1}{2} d\big(y, T(y)\big)\Big) \\ &= \tfrac{1}{2}\Big(r + d\big(y, T(y)\big)\Big). \end{aligned} \tag{2}$$

From (1) and (2), $d(y, T(y)) \leqslant 2\varepsilon + r$. Since $\varepsilon > 0$ is arbitrarily chosen, $d(y, T(y)) \leqslant r$. So $y \in K_r$. Hence $H_r \subseteq K_r$.

(b) $T(K_{r_0}) \subseteq H_{r_0} \subseteq K_{r_0}$.

(c) $T(H_{r_0}) \subseteq T(K_{r_0}) \subseteq \operatorname{cl}\operatorname{co}(T(K_{r_0})) = H_{r_0}$ and H_{r_0} is non-empty closed and convex.

For our next result, we need the following definition. Let X be a convex subset of a Banach space and f be a map of X into a metric space. Then f is *continuous along line segments* if for any x, y in X, $\{f(tx + (1-t)y)\}$ converges to $f(x)$ when t tends to 1 from below.

THEOREM 6. *Let K be a non-empty weakly compact convex subset of a Banach space and T be a Kannan map of K into* $\operatorname{wcc}(K)$. *Suppose that T is continuous along line segments. Then T has a fixed point.*

Proof. By Theorem 5(c), we may assume that $d(x, T(x))$ is constant, say r, on K. It remains to show that $r = 0$. Let $x \in K$, $y \in T(x)$ and $z \in T(y)$. Let $t \in (0, 1)$. Then

$$\begin{aligned} d\big(z, T(ty + (1-t)z)\big) &\leqslant D\big(T(y), T(ty + (1-t)z)\big) \\ &\leqslant \tfrac{1}{2}\Big(d\big(y, T(y)\big) + d\big(ty + (1-t)z, T(ty + (1-t)z)\big)\Big) = r. \end{aligned}$$

Similarly, $d\big(y, T(ty + (1-t)z)\big) \leqslant r$. Since $T(ty + (1-t)z)$ is weakly compact, there are $\bar{y}$, $\bar{z}$ in $T(ty + (1-t)z)$ such that

$$d(y, \bar{y}) = d\big(y, T(ty + (1-t)z)\big), \quad d(z, \bar{z}) = d\big(z, T(ty + (1-t)z)\big).$$

Since

$$\begin{aligned} r &= d\big(ty + (1-t)z,\ T(ty + (1-t)z)\big) \\ &\leqslant d\big(ty + (1-t)z,\ t\bar{y} + (1-t)\bar{z}\big) \\ &\leqslant t d(y, \bar{y}) + (1-t) d(z, \bar{z}) \\ &= t d\big(y, T(ty + (1-t)z)\big) + (1-t) d\big(z, T(ty + (1-t)z)\big) \\ &\leqslant tr + (1-t) r = r, \end{aligned}$$

all of the above inequalities are equalities. So we have, in particular, $d\big(z, T(ty + (1-t)z)\big) = r$. Since T is continuous along line segments, $\{T(ty + (1-t)z)\}$ converges to $T(y)$ as t tends to 1 from below. Since $|d(z, A) - d(z, B)| \leqslant D(A, B)$ for any non-empty subsets A, B of K, $\{d\big(z, T(ty + (1-t)z)\big)\}$ converges to $d(z, T(y))$. Therefore $r = d(z, T(y))$. Since $z \in T(y)$, $r = 0$.

The proofs of the above two results are respectively refinements of the proofs of Theorem 1 in [17] and [18]. Earlier contribution of P. Soardi [15] should also be recalled here. We would like to emphasize here that even if f is single-valued, the above result is far general than the corresponding result of R. Kannan in [10] and [12].

THEOREM 7. *Let X be a non-empty compact convex subset of a strictly convex Banach space B and f be a continuous Kannan map of X into* $\operatorname{cc}(X)$. *Then f has a continuous Kannan selection.*

Proof. By Theorem 6, f has a fixed point a. Define f_a as in Theorem 3. Then f_a is a Kannan selection of f. We shall prove that f_a is continuous.

Let $\{x_n\}$ be any sequence in X which converges to a point x in X. It suffices to show that a subsequence of $\{f_a(x_n)\}$ converges to $f_a(x)$. By compactness of X, $\{f_a(x_n)\}$ has a subsequence $\{f_a(x_{h(n)})\}$ which converges to some point y_0 in X. For each n, there exists y_n in $f(x)$ such that

$$d\big(f_a(x_{h(n)}), y_n\big) < D\big(f(x_{h(n)}), f(x)\big) + 1/n .$$

Since f is continuous, $\{d\big(f_a(x_{h(n)}), y_n\big)\}$ converges to 0. Thus $\{y_n\}$ converges to y_0. Since $f(x)$ is closed, $y_0 \in f(x)$. Therefore $\{d\big(a, f_a(x_{h(n)})\big)\}$ converges to $d(a, y_0)$. Since f is continuous and $\{x_{h(n)}\}$ converges to x, $\{d\big(a, f(x_{h(n)})\big)\}$ converges to $d\big(a, f(x)\big)$. Since $d\big(a, f_a(x_{h(n)})\big) = d\big(a, f(x_{h(n)})\big)$ for each n, $d(a, y_0) = d\big(a, f(x)\big)$. Since B is strictly convex and $f(x)$ is compact convex, $y_0 = f_a(x)$. Thus $\{f_a(x_{h(n)})\}$ converges to $f_a(x)$.

We conclude this section with the following remarks. A fixed point of f in Theorem 7 can also be obtained as follows. Let U be the uniformity on X induced by the metric d on X. Since X is compact, by results of E. Michael [13, Proposition 3.6 and Theorem 3.3], the topology on $c(X)$ induced by the Hausdorff metric D coincides with the uniform topology $|2^U|$ and the finite topology $|2^U|$ on $c(X)$. Since f is a continuous map of X into $\big(c(X), D\big)$, by another result of E. Michael [13, Corollary 9.3], f is lower semicontinuous. Since X is paracompact, by still another result of E. Michael [14, Theorem 3.2''], f has a continuous selection, say g. By Schauder-Tychonoff fixed point theorem, g and therefore f has a fixed point. We note here that since f is upper semicontinuous, one can also obtain a fixed point of f from a result of Ky Fan (see e.g. [7, (4) on p. 13]).

4. Examples. We need some further observations before presenting examples.

THEOREM 8. *Let (X, d) be a metric space and f be a Kannan map of X into* $\mathrm{bc}(X)$. *Then f has a fixed point a if and only if*

$$D\big(f(x), f(a)\big) \leqslant \tfrac{1}{2} d\big(x, f(x)\big), \qquad x \in X .$$

THEOREM 9. *Let K be a non-empty weakly compact convex subset of a Banach space. Let f be a Kannan map of K into* $\mathrm{wcc}(K)$ *with a fixed point a. Then the set F_f of all fixed points of f is $f(a)$. Moreover, F_f is minimal f-invariant.*

Proof. Let $x \in F_f$. Then $D\big(f(x), f(a)\big) = 0$ and therefore $x \in f(x) = f(a)$. Thus $F_f \subset f(a)$. By (a) of Theorem 5, F_f is f-invariant. So from $a \in F_f$, we have $f(a) \subset F_f$. Hence $f(a) = F_f$. Clearly F_f is minimal f-invariant.

In the following result, T is not necessarily continuous.

THEOREM 10. *Let T be a Kannan map of the unit interval $K = [0, 1]$ into* $\mathrm{cc}(K)$. *Then T has a fixed point.*

Proof. For each x in K, let $f(x)$ be the mid-point of the interval $T(x) = [x_1, x_2]$ ($\{f(x)\} = T(x)$ if $T(x)$ is a singleton). Then f is a selection of T. Note also

$$D\big(T(x), T(y)\big) = \max\{|x_1 - y_1|, |x_2 - y_2|\} .$$

So

$$\begin{aligned}|f(x) - f(y)| &= |\tfrac{1}{2}(x_1 + x_2) - \tfrac{1}{2}(y_1 + y_2)| \leqslant \tfrac{1}{2}(|x_1 - y_1| + |x_2 - y_2|)\\ &\leqslant D\big(T(x), T(y)\big) \leqslant \tfrac{1}{2}\Big(d\big(x, T(x)\big) + d\big(y, T(y)\big)\Big)\\ &\leqslant \tfrac{1}{2}\Big(d\big(x, f(x)\big) + d\big(y, f(y)\big)\Big) .\end{aligned}$$

Thus f is a Kannan selection of T. Since K has close-to-normal structure, f and therefore T has a fixed point.

We remark that the selection f of T in Theorem 10 is nonexpansive if T is nonexpansive (instead of Kannan). It thus gives a simple proof that every nonexpansive map of the unit interval K into $\mathrm{cc}(K)$ has a fixed point.

With Theorems 9 and 10, we are able to give all Kannan maps of the unit interval to the family of its subintervals.

THEOREM 11. *Let f be a map of the unit interval $K = [0, 1]$ into* $\mathrm{cc}(K)$. *For each x in K, let $[x_1, x_2]$ be the interval $f(x)$. Then f is a Kannan map if and only if there exists a in K such that*

(a) $x_1 = a_1$, $x_2 = a_2$ *if* $x \in [a_1, a_2]$;

(b) $x_1 \in [\tfrac{1}{3}(2a_1 + x), 2a_1 - x] \cap [0, 1]$,
$x_2 \in [a_2 - \tfrac{1}{2}(x_1 - x), a_2 + \tfrac{1}{2}(x_1 - x)] \cap [x_1, 1]$ *if* $x \in [0, a_1)$;

(c) $x_2 \in [2a_2 - x, \tfrac{1}{3}(2a_2 + x)] \cap [0, 1]$,
$x_1 \in [a_1 - \tfrac{1}{2}(x - x_2), a_1 + \tfrac{1}{2}(x - x_2)] \cap [0, x_2]$ *if* $x \in (a_2, 1]$.

Proof. Suppose first that f is a Kannan map. By Theorem 10, f has a fixed point, say a.

(a) follows from Theorem 9.

(b) Suppose $x \in [0, a_1)$. Then $x < x_1$: If not, then $x \geqslant x_1$. By Theorem 9, $x > x_2$ as $x \notin f(x)$. Then

$$d\big(x, f(x)\big) = x - x_2 < a_1 - x_1 \leqslant D\big(f(x), f(a_1)\big) \leqslant \tfrac{1}{2} d\big(x, f(x)\big) ,$$

a contradiction. Thus from $D\big(f(x), f(a)\big) \leqslant \tfrac{1}{2}(x_1 - x)$, we have

$$\max\{|x_1 - a_1|, |x_2 - a_2|\} \leqslant \tfrac{1}{2}(x_1 - x) .$$

1° Consider $|x_1 - a_1| \leqslant \tfrac{1}{2}(x_1 - x)$. If $x_1 \geqslant a_1$, then $x_1 - a_1 \leqslant \tfrac{1}{2}(x_1 - x)$, i.e. $x_1 \leqslant 2a_1 - x$. So $x_1 \in [a_1, 2a_1 - x]$. If $x_1 < a_1$, then $a_1 - x_1 \leqslant \tfrac{1}{2}(x_1 - x)$, i.e. $x_1 \geqslant \tfrac{1}{3}(2a_1 + x)$. So $x_1 \in \big[\tfrac{1}{3}(2a_1 + x), a_1\big)$. Thus

$$x_1 \in \Big(\big[\tfrac{1}{3}(2a_1 + x), a_1\big) \cup [a_1, 2a_1 + x]\Big) .$$

Hence $x_1 \in [\tfrac{1}{3}(2a_1 + x), 2a_1 + x] \cap [0, 1]$.

2° Consider $|x_2 - a_2| \leqslant \tfrac{1}{2}(x_1 - x)$. If $x_2 \geqslant a_2$, then $x_2 - a_2 \leqslant \tfrac{1}{2}(x_1 - x)$, i.e. $x_2 \leqslant a_2 + \tfrac{1}{2}(x_1 - x)$. Therefore $x_2 \in [a_2, a_2 + \tfrac{1}{2}(x_1 - x)]$. If $x_2 < a_2$, then

$a_2 - x_2 \leqslant \frac{1}{2}(x_1 - x)$, i.e. $x_2 \geqslant a_2 - \frac{1}{2}(x_1 - x)$. Therefore $x_2 \in \left[a_2 - \frac{1}{2}(x_1 - x),\, a_2\right)$. Thus

$$x_2 \in \left[a_2 - \tfrac{1}{2}(x_1 - x),\, a_2\right) \cup [a_2,\, a_2 + \tfrac{1}{2}(x_1 - x)]\,.$$

Hence $x_2 \in [a_2 - \frac{1}{2}(x_1 - x),\, a_2 + \frac{1}{2}(x_1 - x)] \cap [x_1, 1]$.

We leave the proof of (c) to the reader.

Now suppose that (a), (b), (c) hold. Let $x \in K$. By Theorem 8, if suffices to prove that $D\big(f(x), f(y)\big) \leqslant \frac{1}{2} d\big(x, f(x)\big)$, i.e.

$$\max\{|x_1 - a_1|,\, |x_2 - a_2|\} \leqslant \tfrac{1}{2} d\big(x, f(x)\big)\,.$$

From (a), we may assume that $x < a_1$ or $x > a_2$. Suppose that $x > a_2$. Since $x_2 \in [2a_2 - x,\, \frac{1}{3}(2a_2 + x)]$, $x_2 < x$. So $d\big(x, f(x)\big) = x - x_2$.

1° If $x_2 \geqslant a_2$, then from $x_2 \leqslant \frac{1}{3}(2a_2 + x)$, $x_2 - a_2 \leqslant \frac{1}{2}(x - x_2)$.

2° If $x_2 < a_2$, then from $x_2 \geqslant 2a_2 - x$, $\frac{1}{2}(x - x_2) \geqslant a_2 - x$. From 1° and 2°, $|x_2 - a_2| \leqslant \frac{1}{2}(x - x_2)$. Since $x_1 \in [a_1 - \frac{1}{2}(x - x_2),\, a_1 + \frac{1}{2}(x - x_2)] \cap [0, x_2]$, we have $x_1 - a_1 \leqslant \frac{1}{2}(x - x_2)$ and $x_1 \leqslant x_2$. Thus $x_1 \leqslant x_2$ and

$$\max\{|x_2 - a_2|,\, |x_1 - a_1|\} \leqslant \tfrac{1}{2}(x - x_2) = \tfrac{1}{2} d\big(x, f(x)\big)\,.$$

We leave the proof of the case $x < a_1$ to the reader.

From the above result, we know that for a given Kannan map T of $K = [0, 1]$ into $\mathrm{cc}(K)$, there may be many (possibly continuous) Kannan selections of T and in case T is continuous, there may be many continuous and discontinuous Kannan selections, and there may be many selections of T which are not Kannan. To crown all, Theorem 11 gives us a basis for asking further reasonable questions. For those who would like to see a detail example of multiple-valued continuous Kannan maps, we provide the following one.

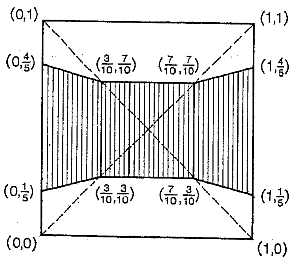
Example. Let T be a map on $K = [0, 1]$ such that for each x in K, $T(x) = [x_1, x_2]$, where

(a) $x_1 = \frac{3}{10}$, $x_2 = \frac{7}{10}$, if $x \in [\frac{3}{10}, \frac{7}{10}]$;

(b) $x_1 = \frac{1}{5} + \frac{1}{3}x$, $x_2 = \frac{4}{5} - \frac{1}{3}x$ if $x \in [0, \frac{3}{10})$;

(c) $x_2 = \frac{7}{15} + \frac{1}{3}x$, $x_1 = \frac{8}{15} - \frac{1}{3}x$ if $x \in (\frac{7}{10}, 1]$.

Then by Theorem 11, T is a Kannan map. $\bigcup_{x \in K} T(x)$ is the shaded part of the following figure.



References

[1] L. E. Blumenthal, *Theory and Applications of Distance Geometry*, Oxford 1953.
[2] D. W. Boyd and J. S. W. Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc. 20 (1969), pp. 458–464.
[3] M. S. Brodskii and D. P. Milman, *On center of a convex set*, Dokl. Akad. Nauk SSSR, 59 (1948), pp. 837–840.
[4] D. F. Cudia, *Rotundity*, Proc. Symposia in Pure Math. VII (Convexity), Amer. Math. Soc. (1963), pp. 73–97.
[5] M. M. Day, *Normed linear spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete n.F. 21 (1962).
[6] W. G. Dotson, Jr., *Fixed points of quasi-nonexpansive mappings*, J. Austral. Math. Soc. 13 (1972), pp. 167–170.
[7] Ky Fan, *Extensions of two fixed point theorems*, Lecture Notes in Mathematics (Set-Valued Mappings, Selections and Topological Properties of 2^X) 171, (1970), pp. 12–16.
[8] R. B. Fraser, Jr. and S. B. Nadler, Jr., *Sequences of contractive maps and fixed points*, Pacific J. Math. 31 (1969), pp. 659–667.
[9] F. Hausdorff, *Set Theory*, New York 1957.
[10] R. Kannan, *Some results on fixed points* — III, Fund. Math. 70 (1971), pp. 169–177.
[11] — *Some results on fixed points* — IV, Fund. Math. 74 (1972), pp. 181–187.
[12] — *Fixed point theorems in reflexive Banach spaces*, Proc. Amer. Math. Soc. 38 (1973), pp. 111–118.
[13] E. Michael, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc. 71 (1951), pp. 152–182.
[14] — *Continuous selections I*, Ann. Math. 63 (1956), pp. 361–382.
[15] P. Soardi, *Su un problema di punto urito di S. Reich*, Boll. Un. Math. Ital. 4 (1971), pp. 841–845.
[16] Chi Song Wong, *Fixed point theorems for point-to-set mappings*, Canad. Math. Bull. (to appear).
[17] — *Fixed points and characterizations of certain maps*, Pacific J. Math. 54 (1974), pp. 305–312.
[18] — *On Kannan maps*, Proc. Amer. Math. Soc. 47 (1975), pp. 105–111.
[19] — *Close-to-normal structure and its applications*, J. Functional Anal. 16 (1974), pp. 353–358.

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