[2] C. H. Dowker, *Mapping theorems for non-compact spaces*, Amer. J. Math. 69 (1947), pp. 200–242.
[3] J. Dugundji, *Topology*, Boston 1966.
[4] R. Engelking, *Outline of General Topology*, Amsterdam 1968.
[5] T. Goto, *Homotopical cohomology groups of paracompact spaces*, Sci. Rep. Tokyo Kyoiku Daigaku, Sect. A. 9 (1967), pp. 163–169.
[6] S. T. Hu, *Theory of Retracts*, Detroit 1965.
[7] P. J. Huber, *Homotopical cohomology and Čech cohomology*, Math. Ann. 144 (1961), pp. 73–76.
[8] M. Katětov, *A theorem on the Lebesgue dimension*, Casopis Pest Mat. Fys. 75 (1950), pp. 79–87.
[9] K. Morita, *On the dimension of product spaces*, Amer. J. Math. 75 (1953), pp. 205–223.
[10] — *Cohomotopy groups for fully normal spaces*, Sci. Rep. Tokyo Bunrika Daigaku, Sect. A, 4 (1952), pp. 251–261.
[11] K. Morita, *Paracompactness and product spaces*, Fund. Math. 50 (1962), pp. 223–236.
[12] — *Note on paracompactness*, Proc. Japan Acad. 37 (1961), pp. 1–3.
[13] — *Products of normal spaces with metric spaces*, Math. Ann. 154 (1964), pp. 365–382.
[14] — *Topological completion and M-spaces*, Sci. Rep. Tokyo Kyoiku Daigaku, Sect. A, 10 (1970), pp. 271–288.
[15] — *On the dimension of the product of Tychonoff spaces*, Gen. Topology and Appl. 3 (1973), pp. 125–133.
[16] J. C. Moore, *Semi-simplicial complexes and Postnikov systems*, Sympos. Internat. Topology Algeb. Mexico City, 1958, pp. 232–247.
[17] B. A. Pasynkov, *On universal bicompacta of given weight and dimension*, Dokl. Akad. Nauk SSSR 154 (1964), pp. 1042–1043; Soviet Math. Dokl. 5 (1964), pp. 245–246.
[18] — *On the spectral decomposition of topological spaces*, Amer. Math. Soc. Transl., Ser. 2, 73 (1966), pp. 83–134.
[19] R. Puppier, *La completion universelle d'un produit d'espaces*, Publ. Dept. Math. Lyon 6 (2) (1969), pp. 75–84.
[20] Yu. M. Smirnov, *On the dimension of proximity spaces*, Mat. Sbornik 38 (1956), pp. 283–302.
[21] E. H. Spanier, *Algebraic Topology*, New York 1966.

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Measurable relations

by

C. J. Himmelberg (Lawrence, Kan.)

Abstract. The measurability properties of relations (= set valued functions) are developed. First the logical relations among the various definitions of measurability are worked out and used to determine sufficient conditions for the intersection of measurable relations to be measurable. These results are then used to generalize the selection theorems of Kuratowski and Ryll-Nardzewski, Castaing, and Aumann, to generalize Filippov's implicit function theorem, and to prove the existence of a measurable selector extending a given measurable partial selector. The paper concludes with some applications to relations with values in a locally convex space.

1. Introduction. Measurable relations, i.e., set valued functions which assign to each element t of a measurable space T a subset of a topological space X in a manner satisfying any one of several possible definitions of measurability, have been studied extensively in recent years by numerous authors (Aumann [A-1, 2], Castaing [C], Debreu [D], Jacobs [J], Kuratowski and Ryll–Nardzewski [KR], McShane and Warfield [MW], Rockafellar [R], Van Vleck and the author [HV-1, 2, 3] and many others.) Much of this work either assumes that the measurable structure on T is that of a Radon measure on a locally compact space or that X is a very special kind of space, say compact metric or Euclidean. The purpose of this paper is to develop the properties of measurable relations in the general situation where T is an abstract measurable space and X is separable metric. It turns out that to work with T this general we must usually (but not always) introduce compactness somewhere, either in X or in the values of a multifunction with values in X. Alternatively, we obtain a similar body of results assuming that X is a Souslin space and that a σ-finite measure is defined on the measurable subsets of T.

In the main, we will confine our attention to the general properties of measurable relations, and to selection, extension, and implicit function theorems.

In Section 2 we give most of the necessary definitions and terminology, and state without proof some trivial but often used properties of measurable relations. Section 3 is an account of the logical relationships among the various definitions of measurability. Section 4 is concerned with measurability of the intersection, complement, and boundary of measurable relations. Section 5 contains partial generalizations of the selection theorems of Kuratowski and Ryll–Nardzewski [KR] and Castaing [C], and of Aumann's extension [A-1] of Von Neumann's measurable choice theorem. The first generalization characterizes weak measurability by the existence of dense families of measurable selectors. In section 6 we consider the measurability of relations defined in various ways from a function $f\colon T\times X\to Y$ measurable in t and continuous in x. Section 7 contains several implicit function theorems, i.e., theorems asserting the existence of a measurable function γ such that $\gamma(t)\in\Gamma(t)$ and $g(t)=f(t,\gamma(t))$ for all $t\in T$, given that Γ is a multifunction and g is a function such that $g(t)\in f(\{t\}\times\Gamma(t))$ for all t. Section 8 is concerned with the extension of a partial selector $f\colon S\to X$ of a measurable multifunction $F\colon T\to X$ to a measurable selector defined on all of T. (Here S is an arbitrary (not necessarily measurable) subset of T.) Section 9 contains a few results for relations with values in a linear space. No attempt is made to be complete in this section. For recent work on relations with values in linear space we refer the reader to Valadier [V].

2. Definitions and some elementary properties. Throughout the paper T will denote a measurable space with σ-algebra $\mathcal{A}$. In the absence of any other statement about T, that is all we assume about it. In case there is a σ-finite measure defined on $\mathcal{A}$ we say that T is *σ-finite*, and if there is a complete σ-finite measure defined on $\mathcal{A}$ we call T *complete*.

X will almost always be a separable metrizable space, and, following Bourbaki, we will call X: *Polish*, if X is separable and metrizable by a complete metric; *Lusin*, if X is metrizable and the bijective continuous image of a Polish space; *Souslin*, if X is metrizable and the continuous image of a Polish space.

A relation $F\colon T\to X$ is a subset of $T\times X$. Alternatively, F may be regarded as a function from T to the set of all subsets of X. However, for the sake of conceptual clarity, if $F\colon T\to X$ is a relation, we denote by $\breve{F}$ the corresponding function into the set of subsets of X. Also, when we want to emphasize the properties of F as a subset $T\times X$, we will refer to its graph $\mathrm{Gr}(F)$, even though this is redundant terminology, since F is a subset of $T\times X$. The set $\{t\in T\mid F(t)\neq\emptyset\}$ is called the *domain* of $F\colon T\to X$. If domain $F=T$, then F is called a *multifunction* (or *correspondence*) from T to X. If $B\subset X$, then $F^{-1}(B)=\{t\in T\mid F(t)\cap B\neq\emptyset\}$. Relations are composed in the customary way.

A relation $F\colon T\to X$ is measurable (weakly measurable, $\mathcal{B}$-measurable, $\mathcal{C}$-measurable) iff $F^{-1}(B)$ is measurable for each closed (resp., open, Borel, compact) subset B of X. If $F\colon Y\to X$ where Y is a topological space, then the assertion that F is measurable (weakly measurable, etc.) means that F is measurable (weakly measurable, etc.) when Y is assigned the σ-algebra $\mathcal{B}$ of Borel subsets of Y. Likewise, if $F\colon T\times Y\to X$, then the various kinds of measurability of F are always defined in terms of the product σ-algebra $\mathcal{A}\times\mathcal{B}$ on $T\times Y$ generated by the sets $A\times B$, where $A\in\mathcal{A}$ and $B\in\mathcal{B}$.

The following propositions are trivial exercises. We state them here without proof, and will frequently use them without specific reference.

PROPOSITION 2.1. *$\mathcal{B}$-measurability implies measurability, and, if X is perfectly normal, measurability implies weak measurability.*

Because of Proposition 2.1, it follows that the measurability of $F\colon T\to X$ is equivalent to the measurability of the function $\breve{F}\colon T\to 2^X$, if X is compact metric, and 2^X is the space of compact subsets of X with the exponential topology (see [K-1, p. 160].) Recall that in this case the exponential topology is the topology on 2^X defined by the Hausdorff metric [K-1, p. 215].

PROPOSITION 2.2. *If $F\colon T\to X$ is measurable or weakly measurable, then domain F is measurable.*

PROPOSITION 2.3. *Let J be an at most countable set and let $F_n\colon T\to X$ be a relation for each $n\in J$. Then*

(i) *If each F_n is measurable (weakly measurable, etc.), so is the relation $\bigcup_n F_n\colon T\to X$ defined by $(\bigcup_n F_n)(t)=\bigcup_n F_n(t)$; and*

(ii) *if X is second countable and each F_n is weakly measurable, then so is the relation $\prod_n F_n\colon T\to X^J$ defined by $(\prod_n F_n)(t)=\prod_n F_n(t)$.*

The measurability of the at most countable intersection of measurable (weakly measurable, etc.) relations is more difficult to obtain, and will be discussed in section 4.

PROPOSITION 2.4. *Let $F\colon T\to X$ be a measurable (weakly measurable, $\mathcal{B}$-measurable, $\mathcal{C}$-measurable) relation, and let Z be a closed (resp., open, Borel, closed) subset of X. Then the relation $F_Z\colon T\to X$, defined by $F_Z(t)=F(t)\cap Z$, is measurable (resp. weakly measurable, $\mathcal{B}$-measurable, $\mathcal{C}$-measurable).*

PROPOSITION 2.5. *If X is a subspace of Y, then $F\colon T\to X$ is (weakly) measurable as a relation into X iff F is (weakly) measurable as a relation into Y. I.e., if $i\colon X\subset Y$, then $F\colon T\to X$ is (weakly) measurable iff $i\circ F\colon T\to Y$ is.*

PROPOSITION 2.6. *$F\colon T\to X$ is weakly measurable iff the relation $\bar{F}\colon T\to X$, defined by $\bar{F}(t)=\overline{F(t)}$, is weakly measurable.*

3. Logical implications among the various definitions of measurability.

THEOREM 3.1. *Let X be a metric space, and $F\colon T\to X$ a relation with compact values. Then F is measurable iff F is weakly measurable.*

Proof. Measurability implies weak measurability by Proposition 2.1. On the other hand suppose F is weakly measurable and let B be a closed subset of X. We have $X-B=\bigcup_n A_n$, where $A_n=\{x|\ d(x,B)\geqslant 1/n\}$. By hypothesis, $F^{-1}(X-A_n)$ is measurable. So also $T-F^{-1}(X-A_n)=\{t|\ F(t)\subset A_n\}$ is measurable. Applying the compactness of each $F(t)$, we obtain

$$\begin{aligned} F^{-1}(B) &= T-\{t|\ F(t)\subset X-B\} \\ &= T-\{t|\ F(t)\subset \bigcup_n A_n\} \\ &= T-\bigcup_n \{t|\ F(t)\subset A_n\}. \end{aligned}$$

Hence $F^{-1}(B)$ is measurable.

THEOREM 3.2. (i) *Let X be a separable metrizable space, and $F\colon T\to X$ a relation with closed values. Then measurability of $F\Rightarrow$ weak measurability $\Rightarrow$ $\mathcal{C}$-measurability.*

(ii) *If, in (i), X is also σ-compact (i.e., $X=\bigcup_n X_n$, where each X_n is compact), then all three measurability concepts are equivalent.*

Proof. (i) The first implication follows from Proposition 2.1. So suppose F is weakly measurable. Let Y be a compact metric space which contains X as a dense subspace, and define $G\colon T\to Y$ by $G(t)=\overline{F(t)}$ (closure with respect to Y). Then G is also weakly measurable. By Theorem 3.1, G is measurable. Now let K be a compact subset of X. Then

$$\begin{aligned} F^{-1}(K) &= \{t|\ F(t)\cap K\neq\emptyset\} \\ &= \{t|\ \overline{F(t)}\cap X\cap K\neq\emptyset\} \\ &= \{t|\ \overline{F(t)}\cap K\neq\emptyset\}=G^{-1}(K). \end{aligned}$$

So $F^{-1}(K)$ is measurable.

(ii) Assuming X as in (ii) above, there remains only to prove $\mathcal{C}$-measurability implies measurability. So let F be $\mathcal{C}$-measurable and let B be a closed subset of X. Then

$$F^{-1}(B)=\bigcup_n F^{-1}(B\cap X_n)\in\mathcal{A},$$

since each $B\cap X_n$ is compact.

In many instances it is useful to characterize the measurability of $F\colon T\to X$ in terms of the $\mathcal{A}\times\mathcal{B}$-measurability of $\mathrm{Gr}(F)$ (as, for instance, in the work of Aumann [A-1] and Debreu [D]) or in terms of the measurability of the function $t\to d(x,F(t))$ (as in Castaing [C]). In the latter situation we define $d(x,F(t))=+\infty$ if $F(t)=\emptyset$.

THEOREM 3.3. *Let X be separable metric and let $F\colon T\to X$ be a relation. Consider the statements:*

a) *F is weakly measurable;*

b) *$t\to d(x,F(t))$ is a measurable function of t for each x;*

c) *$\mathrm{Gr}(\overline{F})$ is $\mathcal{A}\times\mathcal{B}$-measurable ($\overline{F}$ is defined by $\overline{F}(t)=\overline{F(t)}$). Then $a\Leftrightarrow b\Rightarrow c$.*

Proof. $a\Leftrightarrow b$. F is weakly measurable iff $F^{-1}(B(x,\varepsilon))$ is measurable for each open ball $B(x,\varepsilon)$ in X. On the other hand, $t\to d(x,F(t))$ is measurable in t iff $\{t|\ d(x,F(t))<\varepsilon\}$ is measurable for each $0<\varepsilon\leqslant+\infty$. But

$$\begin{aligned} F^{-1}(B(x,\varepsilon)) &= \{t|\ F(t)\cap B(x,\varepsilon)\neq\emptyset\} \\ &= \{t|\ d(x,F(t))<\varepsilon\}. \end{aligned}$$

It follows that a $\Leftrightarrow$ b.

b $\Leftrightarrow$ c. This is essentially proved in [D, Theorem 4.3], although Debreu assumes F has compact values and uses a stronger definition of measurability. The implication also follows directly from the fact that

$$\mathrm{Gr}(\overline{F})=\{(t,x)|\ d(x,F(t))=0\}=f^{-1}(0),$$

where $f\colon T\times X\to[0,+\infty]$ is defined by $f(t,x)=d(x,F(t))$. The function f is easily shown to be measurable in t and continuous in x, and hence (by Theorem 6.1 to follow) is measurable in both variables jointly. Thus $f^{-1}(0)$ is measurable.

THEOREM 3.4. *Let T be complete, X be Souslin, and $F\colon T\to X$ a relation such that $\mathrm{Gr}(F)$ is $\mathcal{A}\times\mathcal{B}$-measurable. Then F is $\mathcal{B}$-measurable.*

Proof. If X is Polish, see [D, Theorem 4.4] or [A-1, Projection Theorem]. In the present case, let $\varphi\colon P\to X$ be a continuous function from a Polish space P onto X. Define $i\colon T\to T$ to be the identity function, and let $(i,\varphi)\colon T\times P\to T\times X$ be defined by $(i,\varphi)(t,p)=(t,\varphi(p))$. Then (i,φ) is a measurable function in the sense that $(i,\varphi)^{-1}(\mathcal{A}\times\mathcal{B}_X)\subset\mathcal{A}\times\mathcal{B}_Y$, where $\mathcal{B}_X,\mathcal{B}_Y$ are the families of Borel subsets of X, Y, respectively. Now consider the relation $\varphi^{-1}\circ F\colon T\to P$. It has measurable graph, since $\mathrm{Gr}(\varphi^{-1}\circ F)=(i,\varphi)^{-1}(\mathrm{Gr}(F))$. Thus $\varphi^{-1}\circ F$ is a measurable relation by the Polish case. It follows that $F=\varphi\circ\varphi^{-1}\circ F$ is measurable.

The situation for relations with closed values can now be summarized in the following theorem.

THEOREM 3.5. *Let X be separable metric, and let $F\colon T\to X$ have closed values. Consider the following statements:*

a) *F is $\mathcal{B}$-measurable;*

b) *F is measurable;*

c) *F is weakly measurable;*

d) *F is $\mathcal{C}$-measurable;*

e) *$t\to d(x,F(t))$ is a measurable function of t for each $x\in X$;*

f) *$\mathrm{Gr}(F)$ is $\mathcal{A}\times\mathcal{B}$-measurable.*

We have (i) a $\Rightarrow$ b $\Rightarrow$ c $\Leftrightarrow$ e $\Rightarrow$ d *and* c $\Rightarrow$ f;

(ii) *If* X *is* σ*-compact, then* a $\Rightarrow$ b $\Leftrightarrow$ c $\Leftrightarrow$ d $\Leftrightarrow$ e $\Rightarrow$ f;

(iii) *If* T *is complete and* X *is Souslin, then* a $\Leftrightarrow$ b $\Leftrightarrow$ c $\Leftrightarrow$ e $\Leftrightarrow$ f $\Rightarrow$ d.

In (ii) above we do not have b $\Rightarrow$ a, even if X is the unit interval I. For let T be the collection of non-empty members of the space 2^I, and define a multifunction $F\colon T\to I$ by $F(A) = A$ for each $A \in T$. Then F is clearly continuous (since it corresponds to the identity function from 2^I to 2^I) and has closed values. However, F is not $\mathcal{B}$-measurable when T is assigned the σ-algebra $\mathcal{A}$ of Borel subsets of T. To see this, let Q be the set of rational numbers in I. Then $I-Q$ is a Borel set in I, but $F^{-1}(I-Q) = 2^I - \{A \mid F(A) \subset Q\}$ is not a Borel set in 2^I. (See [K-2, p. 72].)

We do not have an example for f $\not\Rightarrow$ b in (ii), unless the σ-compactness condition is removed. Nor do we have an example for d $\not\Rightarrow$ b in (iii) unless either the completeness condition is removed from T or the Souslin condition is dropped from X.

It would be very interesting to know whether b $\Leftrightarrow$ c if T is just a measurable space and X is Souslin (or even Polish).

4. Intersection, complement, and boundary of measurable relations. For each n in an at most countable set J, let $F_n\colon T\to X$ be a measurable (or perhaps just weakly measurable) relation with closed values, and define $F\colon T\to X$ by $F(t) = \bigcap_n F_n(t)$. If T has a complete σ-finite measure and X is Souslin, then the measurability of F follows immediately from (iii) of Theorem 3.5. Even if T has no complete measure and X is only assumed to be separable metric, it follows from Theorem 3.3 that F has $\mathcal{A}\times\mathcal{B}$-measurable graph. For the measurability of F in this case we have the following.

THEOREM 4.1. *Let X be separable metrizable, and let $F_n\colon T\to X$ be a weakly measurable relation with closed values for each $n\in J$. Also assume that for each $t\in T$, $F_n(t)$ is compact for some $n\in J$. Then $F = \bigcap_n F_n$ is measurable.*

Proof. We first consider the case where $F_n(t)$ is compact for all t and n. Define a relation G from T to the product X^J of J copies of X by $G(t) = \prod_{n\in J} F_n(t)$. By Proposition 2.3, G is weakly measurable. Since G has compact values, it is also measurable, by Theorem 3.1. Now let B be a closed subset of X, and let Δ be the diagonal of X^J. Then

$$\begin{aligned} F^{-1}(B) &= \{t \mid \bigcap_{n\in J} F_n(t) \cap B \neq \emptyset\} \\ &= \{t \mid \prod_{n\in J} F_n(t) \cap \Delta \cap B^J \neq \emptyset\} \\ &= G^{-1}(\Delta \cap B^J)\,. \end{aligned}$$

This last set is measurable, since $\Delta\cap B^J$ is closed in X^J and G is measurable. Thus F is measurable.

Now suppose that only one $F_n(t)$ is assumed to be compact for each $t\in T$. Let Y be a metrizable compactification of X, and define $\bar{F}_n\colon T\to Y$ by $\bar{F}_n(t) = \overline{F_n(t)}$ (closure with respect to Y). Then each $\bar{F}_n$ is weakly measurable, and hence measurable, by Theorem 3.1. By the first part of the proof $\bigcap_n \bar{F}_n$ is measurable. But, as is easily seen, $\bigcap_{n\in J}\bar{F}_n(t) = \bigcap_{n\in J} F_n(t)$, for each t, since $\overline{F_n(t)} = F_n(t)$ for some n. Thus $\bigcap_{n\in J} F_n$ is measurable by Proposition 2.5.

COROLLARY 4.2. *Let X be a σ-compact metrizable space and let each F_n be (weakly) measurable with closed values. Then $\bigcap_n F_n$ is measurable.*

Proof. The F_n's are measurable by Theorem 3.5. Let $X = \bigcup_m X_m$, where each X_m is compact. By Theorem 4.1, $t\to X_m\cap\bigcap_n F_n(t)$ defines a measurable relation for each m. Hence $\bigcap_n F_n$ is the countable union of measurable relations.

COROLLARY 4.3. *Let X be separable metrizable, and let each F_n be $\mathcal{C}$-measurable with closed values. Then $F = \bigcap_n F_n$ is $\mathcal{C}$-measurable.*

Proof. Let K be a compact subset of X, define $G_n\colon T\to X$ by $G_n(t) = F_n(t)\cap K$ for each n, and let $G = \bigcap_n G_n$. Each G_n is measurable with compact values. Hence G is measurable by Theorem 4.1. But $F^{-1}(K) = G^{-1}(K)$. So F is $\mathcal{C}$-measurable.

THEOREM 4.4. *Let X be a separable metric space, and let $F\colon T\to X$ be a measurable relation with closed values. Then the relation $G\colon T\to X$, defined by $G(t) = X - F(t)$, is measurable.*

Remark. In fact, instead of measurability of F, we use only the weaker fact that $F^{-1}(\{x\})$ is measurable for all $x\in X$, and we prove that $G^{-1}(B)\in\mathcal{A}$ for every subset B of X. This happens because G has open values.

Proof. Let $B\subset X$, and let A be a countable dense subset of B. Then

$$\begin{aligned} G^{-1}(B) &= \{t \mid (X - F(t))\cap B \neq \emptyset\} \\ &= T - \{t \mid B\subset F(t)\} \\ &= T - \{t \mid A\subset F(t)\} \\ &= T - \bigcap_{a\in A}\{t \mid a\in F(t)\} \\ &= T - \bigcap_{a\in A} F^{-1}(\{a\})\,. \end{aligned}$$

So $G^{-1}(B)$ is measurable.

If T is complete and X Souslin, the situation is simpler (by Theorem 3.5), and the reader can easily check the following:

THEOREM 4.5. *Let T be complete and X Souslin, and let F, H: $T\to X$ be measurable relations which closed values, then $t\to H(t)-F(t)$ defines a measurable relation.*

THEOREM 4.6. *Let F: $T\to X$ be a relation. Then the relation* $\mathrm{Bd}F$: $T\to X$, *defined by $t\to \mathrm{Bd}F(t)$, is measurable if any of the following conditions is satisfied:*

(i) *X is separable metric and F is measurable with compact values.*
(ii) *X is σ-compact metric and F is measurable with closed values.*
(iii) *T is complete, X is Souslin, and F is measurable with closed values.*
(iv) *T is complete, X is Souslin, and* $\mathrm{Gr}(F)$ *is* $\mathcal{A}\times\mathcal{B}$*-measurable.*

Proof. (i) By Theorem 4.4, $t\to \overline{X-F(t)}$ defines a measurable relation. Hence $T\to \overline{X-F(t)}$ is weakly measurable. It follows from Theorem 4.1 that $t\to \mathrm{Bd}F(t) = F(t)\cap \overline{X-F(t)}$ is measurable.

(ii) Both $t\to F(t)$ and $t\to \overline{X-F(t)}$ are weakly measurable. So $\mathrm{Bd}F$ is measurable by Corollary 4.2.

(iii) Follows from (iv) by Theorem 3.3.

(iv) Both F and $t\to X-F(t)$ have $\mathcal{A}\times\mathcal{B}$-measurable graph, and hence are measurable, by Theorem 3.4. Thus both $t\to \overline{F(t)}$ and $t\to \overline{X-F(t)}$ are weakly measurable, and, by Theorem 3.5, have $\mathcal{A}\times\mathcal{B}$-measurable graph. Hence, so does $\mathrm{Bd}F$, and it follows that $\mathrm{Bd}F$ is measurable by another application of Theorem 3.5.

THEOREM 4.7. *If X is separable metric, and F: $T\to X$ is $\mathcal{C}$-measurable with closed values, then* $\mathrm{Bd}F$ *is* $\mathcal{C}$*-measurable.*

Proof. By the remark after Theorem 4.4 and by Theorem 3.2, $t\to \overline{X-F(t)}$ is $\mathcal{C}$-measurable. Hence $\mathrm{Bd}F$ is $\mathcal{C}$-measurable by Corollary 4.3.

5. Selection theorems. A function f: $T\to X$ is a *selector* for a multifunction F: $T\to X$ iff $f(t)\in F(t)$ for all $t\in T$. The basic theorems giving the existence of measurable selectors are the selection theorems of Kuratowski and Ryll–Nardzewski [KR] and Aumann's extension [A-1] of Von Neumann's measurable choice theorem. We state them here for reference.

THEOREM 5.1 (Kuratowski, Ryll–Nardzewski). *If X is separable metric and F: $T\to X$ is a weakly measurable multifunction with complete values, then F has a measurable selector.*

THEOREM 5.2 (Aumann). *If T is σ-finite, X is a Borel subset of a Polish space, and F: $T\to X$ is a multifunction with measurable graph, then there is a measurable function f: $T\to X$ such that $f(t)\in F(t)$ for all t except those in some set of measure* 0.

Remarks. Theorem 5.1 is a trivial modification of the Kuratowski, Ryll–Nardzewski theorem; they assume that X is Polish and F has closed values. Theorem 5.1 follows from that theorem by embedding X in a complete space. Theorem 5.2 is less abstract then Aumann's Theorem. His hypotheses on X are purely measure theoretic, but specialize to the theorem quoted here.

It is perhaps of interest to observe that Theorem 5.2 can be deduced from Theorem 5.1. For recall that Aumann proves 5.2 essentially by reduction to the case where $T = [0,1] = X$, $\mathcal{A}$ = Borel subsets of T, and the measure μ on $\mathcal{A}$ is Lebesgue measure. This more special theorem follows readily from 5.1. In fact the following argument from [HV-3] deduces 5.2 from 5.1 with the assumption that $T=[0,1]$, $\mathcal{A}$ = Borel subsets of T, μ = Lebesgue measure, X is separable metric, and F: $T\to X$ is a multifunction with Souslin graph: Let φ: $P\to \mathrm{Gr}(F)$ be a map of a Polish space P onto $\mathrm{Gr}(F)$, and define a multifunction G: $T\to P$ by $G(t) = \varphi^{-1}(\{t\}\times F(t))$, if $t\in T$. Clearly, G has closed values. Moreover, G is measurable if $\mathcal{A}$ is replaced by the completion $\mathcal{A}^*$ of $\mathcal{A}$. To see this let B be a closed subset of P. Then

$$\begin{aligned} G^{-1}(B) &= \{t \mid \varphi^{-1}(\{t\}\times F(t))\cap B \neq \emptyset\} \\ &= \{t \mid (\{t\}\times F(t))\cap \varphi(B) \neq \emptyset\} \\ &= p_T(\varphi(B)), \end{aligned}$$

where p_T is the projection of $T\times X$ onto T. Thus $G^{-1}(B)$ is a Souslin subset of T, and is therefore measurable with respect to $\mathcal{A}^*$. By Theorem 5.1 there is a selector g: $T\to P$ for G which is measurable with respect to $\mathcal{A}^*$. Then, if p_X denotes projection of $T\times X$ onto X, $f = p_X\circ\varphi\circ g$: $T\to X$ is a selector for F measurable with respect to $\mathcal{A}^*$. Finally, by a routine use of Lusin's Theorem, change the values of f on a set A of measure 0 to obtain a function f measurable with respect to $\mathcal{A}$ and satisfying $f(t)\in F(t)$ for all $t\in T-A$.

We will apply Theorem 5.1 to generalize Castaing's results on dense families of selectors [C, Theorem 5.4], and we will generalize Theorem 5.2 to hold when X is a Souslin space.

LEMMA 5.3. *Let X be separable metric, F: $T\to X$ a measurable multifunction with finite values, and f: $T\to X$ a measurable selector for F. Then $t\to F(t)-\{f(t)\}$ defines a measurable relation.*

Proof. Recall that $t\to \{f(t)\}\times F(t)$ is a weakly measurable multifunction into $X\times X$. Let U be an open subset of X, and let Δ be the diagonal of $X\times X$. Then

$$\begin{aligned}\big(F(t)-\{f(t)\}\big)\cap U\neq\emptyset &\Leftrightarrow x\neq f(t)\quad \text{for some } x\in F(t)\cap U\\ &\Leftrightarrow \big(f(t),x\big)\notin\Delta\quad \text{for some } x\in F(t)\cap U\\ &\Leftrightarrow \{f(t)\}\times\big(F(t)\cap U\big)-\Delta\neq\emptyset\\ &\Leftrightarrow \{f(t)\}\times F(t)\cap(X\times U-\Delta)\neq\emptyset .\end{aligned}$$

But $X\times U-\Delta$ is open in $X\times X$, so $t\to F(t)-\{f(t)\}$ is weakly measurable and hence measurable by the weak measurability of $t\to\{f(t)\}\times F(t)$.

THEOREM 5.4. *Let X be separable metric, and $F\colon T\to X$ a measurable multifunction with finite values. Then there exists an at most countable family $\{f_1,\ldots,f_i,\ldots\}$ of measurable selectors for F such that*

$$F(t)=\{f_1(t),\ldots,f_i(t),\ldots\}$$

for each $t\in T$. If each $F(t)$ has at most n elements, then at most n of the functions f_i are needed.

Proof. Let f_1 be a measurable selector for F. (One exists by Theorem 5.1.) Define $F_1\colon T\to X$ by

$$F_1(t)=\begin{cases}\{f_1(t)\}, & \text{if}\quad F(t)=\{f_1(t)\},\\ F(t)-\{f_1(t)\}, & \text{if}\quad F(t)-\{f_1(t)\}\neq\emptyset .\end{cases}$$

Then F_1 is measurable since $\{t|\ F(t)-\{f_1(t)\}\neq\emptyset\}=(F-\{f_1\})^{-1}(X)$ is measurable by Lemma 5.3.

Next, let f_2 be a measurable selector for F_1 and define $F_2\colon T\to X$ by

$$F_2(t)=\begin{cases}\{f_2(t)\}, & \text{if}\quad F_1(t)=\{f_2(t)\},\\ F_1(t)-\{f_2(t)\}, & \text{if}\quad F_1(t)-\{f_2(t)\}\neq\emptyset .\end{cases}$$

Continuing in this way the functions $f_1,\ldots,f_n,\ldots$ must eventually exhaust every value of F.

COROLLARY 5.5. *Let X be separable metric, and $F\colon T\to X$ a measurable multifunction such that each value $F(t)$ has at most n members. Then there exist a measurable function $f\colon T\to X^n$, and a continuous finite valued multifunction $G\colon X^n\to X$ such that $F=G\circ f$. G is given by $G(x_1,\ldots,x_n)=\{x_1,\ldots,x_n\}$, and continuity is with respect to the Hausdorff metric on the compact subsets of X.*

We can now prove the following generalization of a theorem of Castaing [C, Theorem 5.4]. If U is a family of functions from T to X, then $U(t)$ denotes the set $\{u(t)|\ u\in U\}$ for each $t\in T$.

THEOREM 5.6. *Let X be separable metric and let $F\colon T\to X$ be a multifunction with complete values. Then F is weakly measurable if and only if there exists a countable family U of measurable selectors for F such that $F(t)=\overline{U(t)}$ for all $t\in T$. If X is also σ-compact, then F need only have closed values.*

Proof. If U is a countable family of measurable selectors as in the theorem, then $t\to U(t)$ defines a multifunction which is measurable since it is the countable union of measurable multifunctions. It follows that $t\to F(t)=\overline{U(t)}$ is weakly measurable.

Now suppose F is weakly measurable, and, changing the metric on X if necessary, let Y be a compact metric space containing X as a dense subspace. (Of course F need not have complete values in the new metric. But it will be sufficient at the end of the proof for F to have complete values in some metric.) To find U it is sufficient for each $\varepsilon>0$ to find finitely many measurable selectors $u_1,\ldots,u_n\colon T\to X$ such that $\{u_1(t),\ldots,u_n(t)\}$ is an ε-net in $F(t)$ for each $t\in T$.

So let $\varepsilon>0$, let $Y_1,\ldots,Y_n$ be a cover of Y by open sets of diameter less than ε, and let T_i be the measurable set $F^{-1}(Y_i)$ for each i. Define $F_i\colon T_i\to Y$ by $F_i(t)=\overline{F(t)\cap Y_i}^{Y}$. ($\bar{A}^{Y}$ denotes the Y-closure of A.) Then F_i is a measurable multifunction, since it has compact non-empty values and $t\to F(t)\cap Y_i$ defines a weakly measurable multifunction on T_i. It follows that the corresponding function $\hat{F}_i\colon T_i\to 2^Y$ is measurable.

Define a multifunction $\psi\colon T\to 2^Y$ by

$$\psi(t)=\{\hat{F}_i(t)|\ t\in T_i\}=\{F_i(t)|\ F_i(t)\neq\emptyset\},\qquad \text{for each } t\in T .$$

Then ψ, being the union of measurable functions, each with measurable domain, is measurable. By Theorem 5.4, there exist n measurable selectors $\psi_1,\ldots,\psi_n\colon T\to 2^Y$ for ψ such that $\psi(t)=\{\psi_1(t),\ldots,\psi_n(t)\}$ for all $t\in T$. (Note then that each $\psi_i(t)$ is an $F_j(t)$ and that $\bigcup_i\psi_i(t)=\overline{F(t)}^{Y}$.) Now for each i, define a multifunction $\varphi_i\colon T\to X$ by $\varphi_i(t)=F(t)\cap Y_j$, where Y_j is any of the sets such that $\psi_i(t)=\overline{F(t)\cap Y_j}^{Y}$. (For each t, there may be several such Y_j. Choose any one of them.) Then $\varphi_i\colon T\to X$ is weakly measurable, since ψ_i is. Hence also the multifunction $t\to\overline{\varphi_i(t)}^{X}$ is weakly measurable. Moreover, $F(t)=X\cap\overline{F(t)}^{Y}=X\cap\bigcup_i\overline{\varphi_i(t)}^{Y}=\bigcup_i\overline{\varphi_i(t)}^{X}$, and each $\overline{\varphi_i(t)}^{X}$ has diameter less than ε.

By Theorem 5.1, the multifunction $t\to\overline{\varphi_i(t)}^{X}$ has a measurable selector u_i. Clearly $\{u_1(t),\ldots,u_n(t)\}$ is an ε-net in $F(t)$ for each $t\in T$.

To prove the last assertion in the theorem, let $X\bigcup_n X_n$, where each X_n is compact, and let $T_n=F^{-1}(X_n)$. Each multifunction $F_n=F\cap(T_n\times X_n)\colon T_n\to X_n$ has a measurable selector f_n. Also $f\colon T\to X$, defined by $f(t)=f_n(t)$ if $t\in T_n-(T_1\cup\ldots\cup T_{n-1})$, is a measurable selector for F. Next, let U_n be a countable family of measurable selectors for F_n such that $F_n(t) = \overline{U_n(t)}$ for all $t \in T_n$. Define $U_n^* = \{u^* | \ u \in U_n\}$ by $u^*(t) = u(t)$ if $t \in T_n$, $u^*(t) = f(t)$ if $t \in T - T_n$. Then $U = \bigcup_n U_n^*$ is the desired family of measurable selectors for F.

THEOREM 5.7. *If T is σ-finite, X is a Souslin space, and F: $T \to X$ is a multifunction with measurable graph, then there is a measurable function f: $T \to X$ such that $f(t) \in F(t)$ for all t except those in some set of measure* 0.

Proof. Let φ: $P \to X$ be a continuous function from a Polish space P onto X. Then, as in the proof of Theorem 3.4, (i, φ): $T \times P \to T \times X$ is measurable, and so $\mathrm{Gr}(\varphi^{-1} \circ F) = (i, \varphi)^{-1}(\mathrm{Gr}(F))$ is measurable. By Theorem 5.2 there is a measurable function g: $T \to P$ such that $g(t) \in \varphi^{-1}(F(t))$ for all t except those in a set T_0 of measure 0. Then $\varphi \circ g$: $T \to X$ is measurable and $\varphi(g(t)) \in F(t)$ except when $t \in T_0$.

In applications, say to control theory or game theory, one often has the situation of Theorem 6.5, where $Y =$ the real numbers and F has compact values. In this case a measurable selector γ for F is sought such that $f(t, \gamma(t)) = \max f(t \times F(t))$ for all $t \in T$ (or such that $f(t, \gamma(t)) = \min f(t \times F(t))$ for all $t \in T$). That such selectors exist follows from Theorem 6.5, Theorem 7.1 of the next section, and the following consequence of Theorem 5.6.

THEOREM 6.6. *Let X be separable metric, and F: $T \to R^1$ a weakly measurable multifunction (with not necessarily closed values). Then $t \to \sup F(t)$, $t \to \inf F(t)$ define measurable functions into the space $R^1 \cup \{-\infty, +\infty\}$ of extended real numbers.*

Proof. We consider only the function $t \to \sup F(t)$. Since $t \to \overline{F(t)}$ is also weakly measurable, and $\sup F(t) = \sup \overline{F(t)}$, we may assume, without loss of generality, that F has closed values.

Let U be a countable family of measurable selectors for F as given by Theorem 5.6, and let r be a real number. Then the measurability of $t \to \sup F(t)$ follows from

$$\{t| \ \sup F(t) > r\} = \bigcup_{u \in U} \{t| \ u(t) > r\}.$$

6. Functions of two variables and a superposition theorem. Let f: $T \times X \to Y$ be a function such that $f(t, x)$ is measurable in t for each x and continuous in x for each t. In this section we concern ourselves with the measurability of f and of various relations defined from f. The proof of the following theorem is essentially from Kuratowski [K-1, p. 378].

THEOREM 6.1. *Let X be separable metric and Y metric, and let f: $T \times X \to Y$ be measurable in t and continuous in x. Then f is measurable; in fact, for each closed subset B of Y, $f^{-1}(B)$ is the countable intersection of countable unions of basic rectangles in $\mathcal{A} \times \mathcal{B}$.*

Proof. Let B be a closed subset of Y, and let A be a countable dense subset of X. Let d denote the metric for both X and Y, and let $B_n = \{y| \ d(y, B) < 1/n\}$. Then $f(t, x) \in B$ iff for every n there exists $a \in A$ such that $d(x, a) < 1/n$ and $f(t, a) \in B_n$. Hence

$$f^{-1}(B) = \bigcap_n \bigcup_{a \in A} \{t| \ f(t, a) \in B_n\} \times \{x \in X| \ d(x, a) < 1/n\},$$

and so $f^{-1}(B)$ is the countable intersection of countable unions of basic rectangles in $\mathcal{A} \times \mathcal{B}$.

THEOREM 6.2. *Let X be separable metric, Y metric, f: $T \times X \to Y$ measurable in t and continuous in x, and U an open subset of Y. Then $F(t) = \{x \in X| \ f(t, x) \in U\}$ defines a measurable relation from T to X. In particular, if f is real-valued, then $t \to \{x| \ f(t, x) > \lambda\}$ and $t \to \{x| \ f(t, x) < \lambda\}$ are measurable.*

Proof. Let $B \subset X$ and let A be a countable dense subset of B. Then $F^{-1}(B)$ is measurable, since

$$\begin{aligned} F^{-1}(B) &= \{t| \ F(t) \cap B \neq \emptyset\} \\ &= \{t| \ f(t, x) \in U \text{ for some } x \in B\} \\ &= \{t| \ f(t, a) \in U \text{ for some } a \in A\} \\ &= \bigcup_{a \in A} \{t| \ f(t, a) \in U\}. \end{aligned}$$

It does not follow from the previous theorem that $t \to \{x \in X| \ f(t, x) \leqslant \lambda\}$ is measurable when f is real valued. However, it does follow that $t \to \mathrm{Cl}\{x \in X| \ f(t, x) < \lambda\}$ is weakly measurable, and sometimes this is all that is needed, as for example in Theorem 7.1 and in the following corollary.

COROLLARY 6.3. *Let X be separable metric, and let f: $T \times X \to R^1$ be measurable in t and continuous in x. Then the relation F: $T \to X$ defined by $F(t) = \{x| \ f(t, x) = 0\}$ has measurable graph.*

Proof. Define F_n: $T \to X$ be $F_n(t) = \{x| \ |f(t, x)| < 1/n\}$. By the preceding theorem each F_n is measurable. It follows then from Theorem 3.3 that $t \to \overline{F_n(t)}$ has measurable graph. But clearly $F(t) = \bigcap_n \overline{F_n(t)}$ (since $\overline{F_n(t)} \subset \{x| \ |f(t, x)| \leqslant 1/n\}$). Hence $\mathrm{Gr}(F) = \bigcap_n \mathrm{Gr}(\overline{F}_n)$ is measurable.

By assuming more about T and X we get the following

THEOREM 6.4. *Let T be complete, X Souslin, Y metric, f: $T \times X \to Y$ measurable in t and continuous in x, and B a closed subset of Y. Then $t \to F(t) = \{x \in X| \ f(t, x) \in B\}$ defines a measurable relation. In particular, if f is real valued, then*

$$t \to \{x| \ f(t, x) \geqslant \lambda\}, \quad t \to \{x| \ f(t, x) \leqslant \lambda\}, \quad t \to \{x| \ f(t, x) = \lambda\}$$

are all measurable.

Proof. Let $B_n = \{y \in Y|\ d(y, B) < 1/n\}$, and define

$$F_n(t) = \{x \in X|\ f(t, x) \in B_n\}.$$

Then $F(t) = \bigcap_n F_n(t)$, and each F_n: $T \to X$ is a weakly measurable relation by Theorem 6.2. Now, for each n, $\overline{F_n}$ has measurable graph by Theorem 3.3. Clearly $F = \bigcap_n \overline{F_n}$, since $x \in \overline{F_n(t)} \Rightarrow f(t, x) \in \overline{B_n} \Rightarrow d(f(t, x), B) \leqslant 1/n < 1/n-1$. Hence F has measurable graph and is thus measurable, by Theorem 3.5.

Finally, using Theorem 6.1 and a selection theorem from the previous section we are able to prove the following superposition theorem for multifunctions.

THEOREM 6.5. *Let X be a separable metric space, Y a metric space, f: $T \times X \to Y$ a function measurable in t for each x and continuous in x for each t, and F: $T \to X$ a measurable multifunction with complete values. Then the multifunction G: $T \to Y$ defined by $G(t) = f(t \times F(t))$ is weakly measurable.*

Proof. Applying Theorem 5.6, let U be a countable set of measurable selectors for F such that $\overline{U(t)} = F(t)$ for each $t \in T$. Let B be an open subset of Y. Then

$$\begin{aligned} G^{-1}(B) &= \{t|\ f(t, F(t)) \cap B \neq \emptyset\} \\ &= \{t|\ f(t, \overline{U(t)}) \cap B \neq \emptyset\} \\ &= \{t|\ f(t, x) \in B \text{ for some } x \in \overline{U(t)}\} \\ &= \{t|\ f(t, u(t)) \in B \text{ for some } u \in U\} \\ &= \bigcup_{u \in U} \{t|\ f(t, u(t)) \in B\}. \end{aligned}$$

So it remains to show that $\{t|\ f(t, u(t)) \in B\}$ is measurable. But f is measurable, by Theorem 6.1, and the map φ_u: $T \to T \times X$, defined by $\varphi_u(t) = (t, u(t))$, is clearly measurable, in the sense that $\varphi_u^{-1}(\mathcal{A} \times \mathcal{B}_X) \subset \mathcal{A}$. It follows that $\{t|\ f(t, u(t)) \in B\}$ is measurable, since it is equal to $\varphi_u^{-1}(f^{-1}(B))$.

7. Implicit function theorems. Using the results of the preceding sections it is now easy to prove very general implicit function theorems of the type first proved by Filippov [F]. We give several samples. It is easy to obtain others with slight changes in hypotheses.

THEOREM 7.1. *Let X be separable metric, Y metric, f: $T \times X \to Y$ a function measurable in t and continuous in x, Γ: $T \to X$ a measurable multifunction with compact values, and g: $T \to Y$ a measurable function such that $g(t) \in f(\{t\} \times \Gamma(t))$ for all $t \in T$. Then there exists a measurable selector γ: $T \to X$ for Γ such that $g(t) = f(t, \gamma(t))$ for all $t \in T$.*

If X is also σ-compact, then Γ need only have closed values.

Proof. Define H: $T \to X$ by $H(t) = \Gamma(t) \cap \{x|\ d(f(t, x), g(t)) = 0\}$. The desired function g is any measurable selector for H. Thus, since H has compact non-empty values, it is sufficient, by Theorem 5.1, to show that H is measurable. To this end define F_n: $T \to X$ by

$$F_n(t) = \{x|\ d(f(t, x), g(t)) < 1/n\}.$$

F_n is measurable for each n by Theorem 6.2, and so $t \to \overline{F_n(t)}$ defines a weakly measurable multifunction. Clearly $\{x|\ d(f(t, x), g(t)) = 0\} = \bigcap_n \overline{F_n(t)}$ (since $\overline{F_n(t)} \subset \{x|\ d(f(t, x), g(t)) \leqslant 1/n\}$ for each n). Thus $t \to H(t) = \Gamma(t) \cap \bigcap_n \overline{F_n(t)}$ is measurable by Theorem 4.1.

To prove the last assertion in the theorem, use the same proof as above, except now use Corollary 4.2 and the last assertion in Theorem 5.6.

If we strengthen the hypotheses on T and X, then Γ need not have closed values. In fact we have the following.

THEOREM 7.2. *Let T be σ-finite, X Souslin, Y metric, f: $T \times X \to Y$ a function measurable in t and continuous in x, Γ: $T \to X$ a multifunction with measurable graph, and g: $T \to Y$ a measurable function such that $g(t) \in f(t \times \Gamma(t))$ for all $t \in T$. Then there exists a measurable function γ: $T \to X$ such that $\gamma(t) \in \Gamma(t)$ and $g(t) = f(t, \gamma(t))$ for almost all $t \in T$.*

Proof. Define H: $T \to X$ as in the proof of Theorem 7.1. By Corollary 6.3, $t \to \{x|\ d(f(t, x), g(t)) = 0\}$ has measurable graph. Hence so does H. We then obtain the desired selector (almost everywhere) by Theorem 5.7.

If in the preceding two theorems the function f depends on x alone, we have "lifting" theorems of the sort proved in [MW] and [HV-2, 3]. Two other lifting theorems are the following.

THEOREM 7.3. (Assume the continuum hypothesis for this theorem.) *Let X be separable metric, Y a Hausdorff space, f: $X \to Y$ a continuous function, Γ: $T \to Y$ a $\mathcal{C}$-measurable function with closed values, and g: $T \to Y$ a $\mathcal{C}$-measurable function such that $g(t) \in f(\Gamma(t))$ for all $t \in T$. Then there is a $\mathcal{C}$-measurable selector γ: $T \to X$ for Γ such that $g(t) = f(\gamma(t))$ for all $t \in T$.*

Proof. Define a multifunction F: $T \to X$ by $F(t) = f^{-1}(g(t))$. Then F has closed values and is $\mathcal{C}$-measurable. Also $F(t) \cap \Gamma(t) \neq \emptyset$ for all $t \in T$. By Corollary 4.3, $F \cap G$ is a $\mathcal{C}$-measurable multifunction. Hence, by [HV-2, Theorem 3], $F \cap G$ has a $\mathcal{C}$-measurable selector. It is clearly the desired function.

THEOREM 7.4. *Let T be σ-finite, X Souslin, Y-metric, f: $X \to Y$ a measurable function, Γ: $T \to X$ a multifunction with measurable graph, and g: $T \to Y$ a measurable function such that $g(t) \in f(\Gamma(t))$ for all $t \in T$. Then there exists a measurable function γ: $T \to X$ such that $\gamma(t) \in \Gamma(t)$ and $g(t) = f(\gamma(t))$ for almost all $t \in T$.*

Proof. As for Theorem 7.3, define $F: T \to X$ by $F(t) = f^{-1}(g(t))$. Then F has measurable graph, since $\mathrm{Gr}(F) = \{(t, x) | f(x) = g(t)\} = \varphi^{-1}(\Delta)$, where Δ is the diagonal in $Y \times Y$, and $\varphi: T \times X \to Y \times Y$ is the measurable function defined by $\varphi(t, x) = (g(t), f(x))$. Hence the multifunction defined by $t \to \Gamma(t) \cap f^{-1}(g(t))$ has measurable graph. By Theorem 5.7, there is a measurable function $\gamma: T \to X$ such that $\gamma(t) \in \Gamma(t) \cap f^{-1}(g(t))$ for almost all t. It is clearly the desired function.

8. Extension of measurable selectors. Let S be a subset of T. A function $f: S \to X$ is defined to be measurable iff f is measurable when S is assigned the trace σ-algebra $\mathcal{A}_S = \{A \cap S | A \in \mathcal{A}\}$. In this section we prove that it is always possible to extend such a function to a measurable selector for a complete valued measurable multifunction $F: T \to X$, provided X is separable metric and $f(t) \in F(t)$ for all $t \in S$. Note that we do not require $S \in \mathcal{A}$. We will need the following theorem.

THEOREM 8.1. *Let X be a Lusin space and $f: S \to X$ be measurable. Then there exists a measurable extension $f^*: T \to X$ of f.*

Proof. First suppose X is a complete separable metric space. In this case the theorem must be generally known. (For example, in [K-1, pp. 434] there is a proof that every function of class α extends to one of class $\alpha+1$.) However, since we know no precise reference to it, we give a proof here. Note that it is sufficient to find an extension g of f over some measurable subset A of T containing S. For then we define f^* to be constant on $T-A$ and to agree with g on A.

Begin by partitioning X into the union of a countable family $\mathcal{B}_1$ of pairwise disjoint Borel sets, each of diameter less than or equal to 1. For each $B \in \mathcal{B}_1$ choose $A_B \in \mathcal{A}$ such that $f^{-1}(B) = A_B \cap S$. Moreover, since $\mathcal{B}_1$ is countable, we may assume that any two A_B's are disjoint. Then let $\mathcal{A}_1 = \{A_B | B \in \mathcal{B}_1\}$ and $A = \bigcup_{B \in \mathcal{B}_1} A_B$.

Continuing by induction, for each $n = 1, 2, \ldots$, partition X into the union of a countable family $\mathcal{B}_n$ of pairwise disjoint Borel sets each of diameter less than or equal to $1/n$, and find a subfamily $\mathcal{A}_n = \{A_B | B \in \mathcal{B}_n\}$ of $\mathcal{A}$ such that $\mathcal{B}_{n+1}$ refines $\mathcal{B}_n$ and $\mathcal{A}_{n+1}$ refines $\mathcal{A}_n$, for all n, $f^{-1}(B) = A_B \cap S$ for all $B \in \mathcal{B}_n$, and $\mathcal{A}_{n+1}$ partitions A for all n.

Define $g_n: A \to X$ by making g_n constant on A_B with value in B for each $B \in \mathcal{B}_n$. Then

(i) g_n is measurable on A,

(ii) $d(g_n(t), f(t)) \leqslant 1/n$, if $t \in S$, and

(iii) $d(g_n(t), g_m(t)) \leqslant 1/n$, if $t \in A$ and $m \geqslant n \geqslant 1$.

Since the sequence $(g_n(t))$ is Cauchy for each $t \in A$, apply the completeness of X to define $g: A \to X$ by $g(t) = \lim_{n\to\infty} g_n(t)$. By (ii), g extends f.

Since g_n converges uniformly to g on A, we apply [KR, Lemma] to deduce that g is measurable. This concludes the proof for X complete.

Now suppose X is a Lusin space, and let $\bar{X}$ be the completion of X. By the first part of the proof, f extends to a measurable function $\bar{f}: T \to \bar{X}$. But X is a Borel subset of X^*, so $A = \bar{f}^{-1}(X) \in \mathcal{A}$. Clearly the measurable set A contains S; so $\bar{f}|A$ is a measurable extension of f with values in X. Define $f^*: T \to X$ to be constant on $T-A$ and to agree with $\bar{f}$ on A.

THEOREM 8.2. *Let X be separable metric, $F: T \to X$ a measurable multifunction with complete values, and $f: S \to X$ a measurable function such that $f(t) \in F(t)$ for all $t \in S$. Then there exists a measurable selector $g: T \to X$ for such that g extends f.*

Proof. Let $\bar{X}$ be the completion of X, and let $\bar{f}: T \to \bar{X}$ be the extension of f given by Theorem 8.1. By Theorem 4.1, the relation $H: T \to X \subset \bar{X}$, defined by $H(t) = \{\bar{f}(t)\} \cap F(t)$, is measurable, and so the set $A = H^{-1}(\bar{X})$ is measurable. But, $H^{-1}(\bar{X}) = \{t \in T | \bar{f}(t) \in F(t)\} \supset S$. So $\bar{f}(A) \subset X$, and $\bar{f}|A: A \to X$ is an extension of f over A. Next use Theorem 5.1 to find a measurable selector $\varphi: T-A \to X$ for $F|(T-A)$. Then the function g which agrees with $\bar{f}|A$ on A and with φ on $T-A$ is the desired function.

Remark. The previous theorem is also true for closed valued F, if it is assumed that X is a Lusin space, and that F is $\mathcal{B}$-measurable rather than measurable. The proof is essentially unchanged, except that it is unnecessary to take the completion of X, and we must now use [HV-2, Theorem 4] to get a selector on $T-A$.

9. Relations with values in linear spaces. We conclude with some application of the general results of the preceding sections to the case of multifunctions with values in a locally convex linear space. The following two theorems are generalizations of theorems from [C, section 6]. The convex hull and closed convex hull of a set A are denoted by $\mathrm{co}A$ and $\overline{\mathrm{co}}A$, respectively.

THEOREM 9.1. *Let X be a separable Fréchet space and $F: T \to X$ a weakly measurable relation with closed values. Then the multifunctions $\mathrm{co}F$, $\overline{\mathrm{co}}F$ defined by $t \to \mathrm{co}F(t)$, $t \to \overline{\mathrm{co}}F(t)$ are also weakly measurable. (X need only be a separable metric locally convex space if each $F(t)$ is assumed to be complete.)*

Proof. By Theorem 5.6 there is a countable collection $U = \{u_1, \ldots, u_n, \ldots\}$ of measurable selectors for F such that $F(t) = \overline{U(t)}$ for all $t \in T$. Let Q be the set of all sequences $(q_1, \ldots, q_n, \ldots)$ of non-negative rational numbers such that all but finitely many q_n's are 0 and $\sum_n q_n = 1$. The set Q is countable and so is $V = \{\sum_n q_n u_n | (q_1, \ldots, q_n, \ldots) \in Q\}$. V is a countable collection of measurable functions such that $\overline{V(t)}$

$= \overline{\mathrm{co}}\, U(t) = \overline{\mathrm{co}}\, F(t)$ for all $t \in T$. Hence, again applying Theorem 5.6, $\mathrm{co}F$ and $\overline{\mathrm{co}}F$ are weakly measurable.

THEOREM 9.2. *Let T be complete, X a separable Fréchet space, and $F\colon T \to X$ a multifunction with compact convex values. Then the following two statements are equivalent.*

a) *F is measurable.*

b) *For every continuous linear functional z' on X, the function $M_{z'}\colon T \to R^1$ defined by $M_{z'}(t) = \max_{x \in F(t)} \langle z', x\rangle$ is measurable.*

Proof. a $\Rightarrow$ b. Let z' be a continuous linear functional. The multifunction $z' \circ F\colon T \to R^1$ is weakly measurable. So by Theorem 5.6, there is a countable collection U of measurable selectors such that $z'(F)(t) = \overline{U(t)}$ for all $t \in T$. Let λ be a real number. Then

$$\begin{aligned} M_{z'}^{-1}\big((\lambda, \infty)\big) &= \{t|\ \max z'\big(F(t)\big) > \lambda\} \\ &= \{t|\ u(t) > \lambda \text{ for some } u \in U\} \\ &= \bigcup_{u \in U} u^{-1}\big((\lambda, \infty)\big) \in \mathcal{A}\,. \end{aligned}$$

b $\Rightarrow$ a. By [C, Theorem 6.1] there exists a countable family $\mathcal{F}$ of continuous linear functionals on X such that

$$F(t) = \bigcap_{z' \in \mathcal{F}} H_{z'}(t)\,,$$

where

$$H_{z'}(t) = \{x \in X|\ \langle z', x\rangle \leqslant M_{z'}(t)\}\,.$$

$H_{z'}\colon T \to X$ is measurable for each $z' \in \mathcal{F}$ by Theorem 6.4. Thus each $H_{z'}$ has measurable graph by Theorem 3.3. It follows that F has measurable graph, and hence, by Theorem 3.4, that F is measurable.

If F is a relation with compact, convex values in a locally convex space, we denote by F° the relation whose value at t is the set of extreme points of $F(t)$. We conclude with two theorems concerning the measurability of F° and the existence of implicitly defined measurable selectors for F°. The proof of the first is the same as the proof of [HV-1, Theorem 4], and hence is omitted.

THEOREM 9.3. *Let E be a locally convex space with separable dual space, let $X \subset E$ be the union of an increasing sequence (X_n) of compact convex metrizable subsets of E, and let $F\colon T \to X$ be a multifunction with compact convex values.*

a) *If F is measurable, then F° has measurable graph. If in addition T is complete, then F° is measurable.*

b) *If F° is measurable, then F is measurable. If F° has measurable graph and T is complete, then F is measurable.*

THEOREM 9.4. *Let T be complete, X the metrizable union of an increasing sequence of compact convex subsets of a locally convex space with separable dual space, Y a metric space, $f\colon T \times X \to Y$ a function measurable in t and continuous in x, $\Gamma\colon T \to X$ a measurable multifunction with compact convex values, and $g\colon T \to Y$ a measurable function such that $g(t) \in f\big(t \times \Gamma^\circ(t)\big)$ for all $t \in T$. Then there exists a measurable function $\gamma\colon T \to X$ such that $\gamma(t) \in \Gamma^\circ(t)$ and $g(t) = f\big(t, \gamma(t)\big)$ for all $t \in T$.*

Proof. Define $H\colon T \to X$ by

$$H(t) = \Gamma^\circ(t) \cap \{x|\ d\big(f(t, x), g(t)\big) = 0\}\,.$$

Γ° has measurable graph by the previous theorem, and the relation $t \to \{x|\ d\big(f(t, x), g(t)\big) = 0\}$ has measurable graph by Corollary 6.3. Hence H has measurable graph, and has a measurable selector γ, by Theorem 5.7. This is clearly the desired function.

References

[A-1] R. J. Aumann, *Measurable utility and the measurable choice theorem*, Proc. Int. Colloq., La Decision, C.N.R.S., Aix-en-Provence (1967), pp. 15–26.

[A-2] — *Integrals of set-valued functions*, J. Math. Anal. Appl. 12 (1965), pp. 1–12.

[C] C. Castaing, *Sur les multi-applications mesurables*, Rev. Francaise d'Informatique et de Recherche Opérationnelle 1 (1967), pp. 91–126.

[D] G. Debreu, *Integration of correspondences*, Proc. Fifth Berkeley Symposium on Mathematical Statistics and Probability Vol. II, Part I, pp. 351–372, University of California Press, Berkeley, 1966.

[F] A. F. Filippov, *On certain questions in the theory of optimal control*, Vestnik Moskov. Univ. Ser. I Mat. Meh. 2 (1959), pp. 25–32. English translation in J. Soc. Indust. Appl. Math. Ser. A Control 1 (1962), pp. 76–84.

[HV-1] C. Himmelberg and F. S. Van Vleck, *Extreme points of multifunctions*, Indiana Univ. Math. J. 22 (1973), pp. 719–729.

[HV-2] — — *Some selection theorems for measurable functions*, Can. J. Math. 21 (1969), pp. 394–399.

[HV-3] — — *Selection and implicit function theorems for multifunctions with Souslin graph*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 19 (1971), pp. 911–916.

[J] M. Q. Jacobs, *Measurable multivalued mappings and Lusin's theorem*, Trans. Amer. Math. Soc. 134 (1968), pp. 471–481.

[K-1] K. Kuratowski, *Topology*, Vol. I, New York–London–Warszawa 1966.

[K-2] — *Topology*, vol. II, New York–London–Warszawa 1968.

[KR] — and C. Ryll-Nardzewski, *A general theorem on selectors*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 13 (1965), pp. 397–403.

[MW] E. J. McShane and R. B. Warfield, Jr., *On Filippov's implicit functions lemma*, Proc. Amer. Math. Soc. 18 (1967), pp. 41–47.

[R] R. T. Rockafellar, *Measurable dependence of convex sets and functions on parameters*, J. Math. Anal. Appl. 28 (1969), pp. 4–25.
[V] M. Valadier, *Multi-applications mesurables à valeurs convexes compactes*, J. Math. Pures et Appl. 50 (1971), pp. 265–297.

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λ-complete near-rings *

by

R. Gorton (Dayton, Ohio)

Abstract. Let N be a near-ring. For each cardinal λ, a radical like ideal $C_\lambda(N)$ is introduced and used to describe the structure of N in terms of λ-complete near-rings of transformations. The radical $J_2(N)$ of Betsch is extended to near-rings in which $0n = 0$ is not assumed and it is shown that $J_2(N) \subseteq C_1(N) \cap N_c$. Finally, the result of Berman and Silverman on simplicity of near-rings of transformations is extended for infinite groups. and several illustrative examples are given.

1. Introduction. The most natural example of a near-ring is given by the collection of all transformations of a group. Several authors (for example [3, 4, 7, 8]) have studied the structure of near-rings by extending well known radical concepts of rings to near-rings. In this paper a radical like ideal $C_\lambda(N)$ is introduced and used to describe the structure of near-rings in terms of λ-complete near-rings of transformations. The radical $J_2(N)$ of Betsch is extended to near-rings in which $0n = 0$ is not assumed and it is shown that $J_2(N) \subseteq C_1(N) \cap N_c$. Finally, the principal result of [2] is extended for infinite groups and several illustrative examples are given.

2. Definitions. A *near-ring* N is a system (containing at least two elements) with two binary operations $+$ and $\cdot$ satisfying

(i) $(N, +)$ is a group.
(ii) $(N, \cdot)$ is a semigroup.
(iii) $a(b+c) = ab+ac$ for all $a, b, c \in N$.

If N is a near-ring then an additive group Γ ($\neq \{0\}$) is an *N-group* if and only if for all $\gamma \in \Gamma$ and $n \in N$, γn belongs to Γ and

(i) $\gamma(m+n) = \gamma m+\gamma n$ for all $\gamma \in \Gamma$ and $m, n \in N$.
(ii) $\gamma(mn) = (\gamma m)n$ for all $\gamma \in \Gamma$ and $m, n \in N$.

A subgroup Δ of an N-group Γ is an *N-subgroup* if and only if $\Delta N \subseteq \Delta$. Observe that any N-subgroup of Γ must contain $\Gamma_0 = 0N$ (where 0 is the identity element of Γ). If Γ and Γ' are N-groups and

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