On movability and other similar shape properties

by

Juliusz Olędzki (Warszawa)

Abstract. The hereditary shape property called the $\mathfrak{R}$-*movability* has been defined. Some relations between the $\mathfrak{R}$-movability and other shape properties: the movability, the n-movability and the A-movability have been established. There are answers the following questions:

1° Is it true that if compacta X and Y are $\mathfrak{R}$-movable, then $X\circ Y$ is $\mathfrak{R}$-movable?

2° Is it true that if a compactum X is A-movable and B-movable then X is $A\circ B$-movable?

where the binary operation $\circ$ is the Cartesian product or the join or the one-point union or the topological sum.

1. Introduction. K. Borsuk introduced hereditary shape properties: n-movability and A-movability ([3] and [4]). Let $\mathfrak{R}$ be a family of compacta. In this paper we define $\mathfrak{R}$-movability, which is a generalization of those shape properties. The aim of this paper is to study the properties of $\mathfrak{R}$-movability and to determine the relations between n-movability, A-movability, $\mathfrak{R}$-movability and movability.

2. $\mathfrak{R}$-movability. Let A and X be compacta and let $X\subset M\in \mathrm{AR}(\mathfrak{M})$. X is said to be A-*movable* if for every neighborhood U of X in M there exists a neighborhood U_0 of X in M such that for every neighborhood $\hat{U}$ of X in M every map $\alpha\colon A\to U_0$ is homotopic in U to a map with values in $\hat{U}$. K. Borsuk showed ([4]) that A-movability does not depend upon the choice of a space M and that A-movability is a hereditary shape property, i.e., if $\mathrm{Sh}(X)\leqslant \mathrm{Sh}(Y)$ and Y is A-movable, then X is A-movable. To generalize this property, consider a family $\mathfrak{R}$ of compacta. A compactum $X\subset M\in \mathrm{AR}(\mathfrak{M})$ is said to be $\mathfrak{R}$-*movable* if for every neighborhood U of X in M there exists a neighborhood U_0 of X in M such that for every neighborhood $\hat{U}$ of X in M and for every $A\in\mathfrak{R}$ every map α: $A\to U_0$ is homotopic in U to a map with values in $\hat{U}$. By a slight change in the proofs of an analogous theorem for A-movability ([4]) one proves that

(2.1) *The choice of the space M and the embedding of X into M are not important for the definition of $\mathfrak{K}$-movability*

and that

(2.2) *If X is $\mathfrak{K}$-movable and* $\mathrm{Sh}(X) \geqslant \mathrm{Sh}(Y)$, *then Y is $\mathfrak{K}$-movable.*

It is easy to see that

(2.3) *If a compactum X is movable, then X is $\mathfrak{K}$-movable for every family $\mathfrak{K}$.*

A compactum X lying in the Hilbert cube Q is said to be *n-movable* ([3], p. 859) if for every neighborhood U of X in Q there exists a neighborhood U_0 of X in Q such that for every compactum $A \subset U_0$ with $\dim A \leqslant n$ and for every neighborhood $\hat{U}$ of X in Q there exists a homotopy $\varphi\colon A \times [0,1] \to U$ satisfying conditions: $\varphi(x,0) = x$ and $\varphi(x,1) \in \hat{U}$ for every point $x \in A$. Put $M = Q$ in the definition of $\mathfrak{K}$-movability. Since for every compactum $A \subset Q$ and its neighborhood U_0 in Q every map $\alpha\colon A \to U_0$ is homotopic in U_0 to an embedding of A into U_0, we get

(2.4) *If $\mathfrak{K}$ is a family of all compacta of dimension $\leqslant n$, then $\mathfrak{K}$-movability is equivalent to n-movability.*

Let $\mathfrak{K}$ and $\mathfrak{K}'$ be families of compacta. $\mathfrak{K}$ is said to be *M-dominated* by $\mathfrak{K}'$ ($\mathfrak{K} \underset{M}{\leqslant} \mathfrak{K}'$) if every $\mathfrak{K}'$-movable compactum X is also $\mathfrak{K}$-movable. If $\mathfrak{K} \underset{M}{\leqslant} \mathfrak{K}'$ and $\mathfrak{K}' \underset{M}{\leqslant} \mathfrak{K}$, then $\mathfrak{K}$ and $\mathfrak{K}'$ are said to be *M-equivalent* ($\mathfrak{K} \underset{M}{=} \mathfrak{K}'$). If families $\mathfrak{K}$ and $\mathfrak{K}'$ consist of single elements A and A' respectively, then we write $A \underset{M}{\leqslant} A'$ ($A \underset{M}{=} A'$) instead of $\mathfrak{K} \underset{M}{\leqslant} \mathfrak{K}'$ ($\mathfrak{K} \underset{M}{=} R'$).

(2.5) THEOREM. *Let $\mathfrak{K}$ and $\mathfrak{K}'$ be families of compacta. If for every $A \in \mathfrak{K}$ there exists $B \in \mathfrak{K}'$ such that* $\mathrm{Sh}(A) \leqslant \mathrm{Sh}(B)$, *then* $\mathfrak{K} \underset{M}{\leqslant} \mathfrak{K}'$.

Proof. Suppose that a compactum X is $\mathfrak{K}'$-movable and that X is lying in the Hilbert cube Q. It is sufficient to prove that X is $\mathfrak{K}$-movable. Let U be a neighborhood of X in Q. Since X is $\mathfrak{K}'$-movable, there exists a neighborhood U_0 of X in Q such that for every $B \in R'$ and for every map $\beta\colon B \to U_0$ and for every neighborhood $\hat{U}$ of X in Q there exists a homotopy $\varphi\colon B \times [0,1] \to U$ satisfying the conditions: $\varphi(p,0) = \beta(p)$ and $\varphi(p,1) \in \hat{U}$ for every point $p \in B$. We shall show that U_0 satisfies the required condition in the definition of the $\mathfrak{K}$-movability of X. Take $A \in \mathfrak{K}$ and a map $\alpha\colon A \to U_0$. We can find an embedding $\alpha'\colon A \to U_0$ such that

$$\alpha \simeq \alpha' \quad \text{in} \quad U. \tag{2.6}$$

Let $A' = \alpha'(A)$. There exists a $B \in \mathfrak{K}'$ such that $\mathrm{Sh}(B) \geqslant \mathrm{Sh}(A) = \mathrm{Sh}(A')$. Assume that $B \subset N \in \mathrm{AR}(\mathfrak{M})$. Then there exist fundamental sequences $\underline{f} = \{f_n, A', B\}_{Q,N}$ and $\underline{g} = \{g_n, B, A'\}_{N,Q}$ such that $\underline{g} \circ \underline{f} \simeq \underline{i}_{A'}$.

Hence for the neighborhood U_0 of A' in Q there exists a neighborhood V_0 of B in N and an integer n_0 such that

$$g_n|_{V_0} \simeq g_{n+1}|_{V_0} \quad \text{in} \quad U_0, \quad \text{for} \quad n \geqslant n_0. \tag{2.7}$$

Let $\hat{U}$ be a neighborhood of X in Q. Since $g_{n_0}|_B$ carries B into U_0 and X is R'-movable, there exists a homotopy $\varphi\colon B \times [0,1] \to U$ such that $\varphi(p,0) = g_{n_0}(p)$ and $\varphi(p,1) \in \hat{U}$ for every point $p \in B$. Define a map $F\colon (B \times [0,1]) \cup (N \times \{0\}) \to U$ by

$$F(p,t) = \begin{cases} \varphi(p,t) & \text{for} \quad (p,t) \in B \times [0,1], \\ g_{n_0}(p) & \text{for} \quad (p,t) \in N \times \{0\}. \end{cases}$$

Since $B \times [0,1] \cup N \times \{0\}$ is a compactum and U is open in Q, there exists a neighborhood W of $B \times [0,1] \cup N \times \{0\}$ in $N \times [0,1]$ and a map $\bar{F}\colon W \to U$ extending F. There exists a neighborhood V of B in N satisfying three conditions: $V \subset V_0$, $V \times [0,1] \subset W$ and $\bar{F}(V \times \{1\}) \subset \hat{U}$. Since $\underline{f}$ is a fundamental sequence and $\underline{g} \circ \underline{f} \simeq \underline{i}_{A'}$, there exists an $n_1 \geqslant n_0$ such that

$$f_{n_1}(A') \subset V \tag{2.8}$$

and

$$g_{n_1} \circ f_{n_1}|_{A'} \simeq \mathrm{id}_{A'} \quad \text{in} \quad U. \tag{2.9}$$

By (2.7) and (2.8) it follows that

$$g_{n_0} \circ f_{n_1}|_{A'} \simeq g_{n_1} \circ f_{n_1}|_{A'} \quad \text{in} \quad U. \tag{2.10}$$

$\bar{F}|_{f_{n_1}(A') \times [0,1]}\colon f_{n_1}(A') \times [0,1] \to U$ is a homotopy satisfying conditions $\bar{F}(p,0) = g_{n_0}(p)$ and $\bar{F}(p,1) \in \hat{U}$ for every $p \in f_{n_1}(A')$. Let $\bar{F}_1\colon f_{n_1}(A') \to U$ be defined by the formula $\bar{F}_1(p) = \bar{F}(p,1)$. Therefore

$$g_{n_0} \circ f_{n_1}|_{A'} \simeq \bar{F}_1 \circ f_{n_1}|_{A'} \quad \text{in} \quad U. \tag{2.11}$$

By (2.6), (2.9), (2.10) and (2.11), α is homotopic in U to a map with values in $\hat{U}$, and thus X is $\mathfrak{K}$-movable.

(2.12) COROLLARY. *If* $\mathrm{Sh}(A) \leqslant \mathrm{Sh}(B)$, *then* $A \underset{M}{\leqslant} B$.

(2.13) EXAMPLE. Let S_j^i be i-dimensional spheres for $i = 1, 2$, $j = 1, 2$ and let $S_j^i \cap S_{j'}^{i'} = \emptyset$ for $(i,j) \neq (i',j')$. Let $A = S_1^1 \cup S_2^1 \cup S_1^2$ and $B = S_1^1 \cup S_1^2 \cup S_2^2$.

Shapes of A and B are not comparable but one can easily see that $A \underset{M}{=} B$.

If follows, by Corollary (2.12) and Example (2.13), that the classes of all M-equivalent compacta are larger than the classes of compacta of the same shape and that the relation of the fundamental domination is a proper subset of the relation of M-domination, where these relations are considered as subsets of the family of all pairs of compacta.

(2.14) EXAMPLE. Let $\mathfrak{R}$ be a family of solenoids and let a family $\{S^1\}$ consist of a single circle S^1. Then $\mathfrak{R} \underset{M}{\leqslant} \{S^1\}$.

Assume that a compactum $X \subset N \in \mathrm{AR}(\mathfrak{M})$ is S^1-movable. By Corollary (2.12), X is T-movable, where T is a solid torus. Let U be a neighborhood of X in N. There exists a neighborhood U_0 of X in N such that for every neighborhood $\hat{U}$ of X in N every map $\alpha\colon T \to U_0$ is homotopic in U to a map with values in $\hat{U}$. Let $\hat{U}$ be a neighborhood of X in N and take $S \in \mathfrak{R}$ and $\beta\colon S \to U_0$. The solenoid S can be described as an intersection of a decreasing sequence of solid tori T_i, $i = 1, 2, \ldots$ There exist an integer n_0 and a map $\beta'\colon T_{n_0} \to U_0$ extending β. Since β' is homotopic in U to a map with values in $\hat{U}$, β is homotopic in U to such a map. Thus X is $\mathfrak{R}$-movable.

(2.15) PROBLEM. *Is the family of all solenoids M-equivalent to a circle?*

3. Relations between movability, n-movability, A-movability and $\mathfrak{R}$-movability. S. Mardešić and J. Segal ([8], p. 651) proved the following

(3.1) LEMMA. *If $X = \varprojlim\{X_n, p_{nn'}\}$, where $X_n \in \mathrm{ANR}$ for $n = 1, 2, \ldots$, then X is movable if and only if for every integer n there exists an $n_0 \geqslant n$ such that for every $\hat{n} \geqslant n$ there exists a map $r\colon X_{n_0} \to X_{\hat{n}}$ satisfying the condition $p_{n\hat{n}} \circ r \simeq p_{nn_0}$.*

By a slight modification of the proof of this lemma one can easily show the following:

(3.2) LEMMA. *If $X = \varprojlim\{X_n, p_{nn'}\}$, where $X_n \in \mathrm{ANR}$ for $n = 1, 2, \ldots$ and $\mathfrak{R}$ is a family of compacta, then X is $\mathfrak{R}$-movable if and only if for every integer n there exists an $n_0 \geqslant n$ such that for every $\hat{n} \geqslant n$ and for every $A \in \mathfrak{R}$ and for every map $\alpha\colon A \to X_{n_0}$ there exists a map $\alpha'\colon A \to X_{\hat{n}}$ satisfying the condition:*

$$p_{n\hat{n}} \circ \alpha' \simeq p_{nn_0} \circ \alpha. \tag{3.3}$$

(3.4) COROLLARY. *If $X = \varprojlim\{X_n, p_{nn'}\}$, where $X_n \in \mathrm{ANR}$ for $n = 1, 2, \ldots$ and $\mathfrak{R}$ is a family of compacta and for almost all n and for every $A \in \mathfrak{R}$ every map $f\colon A \to X_n$ is homotopic to a constant map, then X is $\mathfrak{R}$-movable.*

(3.5) LEMMA. *Let $X = \varprojlim\{X_n, p_{nn'}\}$, where $X_n \in \mathrm{ANR}$ for $n = 1, 2, \ldots$ and let $\mathfrak{R} = \{X_n;\ n = 1, 2, \ldots\}$. If X is $\mathfrak{R}$-movable, then X is movable.*

Proof. Let n be an integer. It follows by Lemma (3.2) that there exists an $n_0 \geqslant n$ such that for every $\hat{n} \geqslant n$ and for every $A \in \mathfrak{R}$ and for every map $\alpha\colon A \to X_{n_0}$ there exists a map $\alpha'\colon A \to X_{\hat{n}}$ satisfying condition (3.3). Let $\hat{n} \geqslant n$. Take $A = X_{n_0} \in \mathfrak{R}$ and $\alpha = \mathrm{id}_{X_{n_0}}$. Hence there exists a map $\alpha'\colon X_{n_0} \to X_{\hat{n}}$ satisfying condition (3.3). Thus $p_{nn_0} \simeq p_{n\hat{n}} \circ \alpha'$. Put $r = \alpha'$. By Lemma (3.1), X is movable.

(3.6) THEOREM. *There exists a countable family $\mathfrak{W}$ of polyhedra such that $\mathfrak{W}$-movability is equivalent to movability.*

Proof. One knows that there are only countably many homotopy types of polyhedra. Let $\mathfrak{W}$ consist of elements taken singly from all homotopy types of polyhedra. Let a compactum X be $\mathfrak{W}$-movable. X can be described as an inverse limit of a sequence of polyhedra: let $X = \varprojlim\{W_n, p_{nn'}\}$. Therefore, by Theorem (2.5), $\mathfrak{R} = \{W_n;\ n = 1, 2, \ldots\}$ is M-dominated by $\mathfrak{W}$. Thus X is $\mathfrak{R}$-movable. By Lemma (3.5), X is movable.

Conversely, if X is movable, then by (2.3) X is $\mathfrak{R}$-movable for every family $\mathfrak{R}$ of compacta, in particular for $\mathfrak{W}$.

(3.7) COROLLARY. *There exists a compactum W such that W-movability is equivalent to movability.*

Proof. Let W be a one-point compactification of a disjoint union of elements of $\mathfrak{W}$. It is clear that the family consisting of the single element W is M-equivalent to $\mathfrak{W}$.

(3.8) COROLLARY. *There exists a maximal element (the family $\mathfrak{W}$ of compacta or the compactum W) in the partial ordering "$\underset{M}{\leqslant}$".*

(3.9) PROBLEM. *Does there exist, for every family $\mathfrak{R}$ of compacta, a compactum A such that $\{A\} \underset{M}{=} \mathfrak{R}$?*

(3.10) THEOREM. *For $n = 1, 2, \ldots$ there exists a countable family $\mathfrak{W}^n$ of polyhedra of dimension $\leqslant n$ such that $\mathfrak{W}^n$-movability is equivalent to n-movability.*

Proof. There is only a countable number of homotopy types of polyhedra of dimension $\leqslant n$. Let $\mathfrak{W}^n$ consist of polyhedra taken singly from all these types. Let $\mathfrak{R}$ be a family of all compacta of dimension $\leqslant n$. By (2.4), n-movability is equivalent to $\mathfrak{R}$-movability. Since $\mathfrak{W}^n \subset \mathfrak{R}$, n-movability implies $\mathfrak{W}^n$-movability. Assume now that X is $\mathfrak{W}^n$-movable. Let $X \subset N \in \mathrm{AR}(\mathfrak{M})$ and let U be a neighborhood of X in N. Since X is $\mathfrak{W}^n$-movable, there exists a neighborhood U_0 of X in N such that for every neighborhood $\hat{U}$ of X in N and for every $K \in \mathfrak{W}^n$ and for every map $\varphi\colon K \to U_0$ there exists a map $\varphi'\colon K \to U_0$ satisfying the conditions:

$$\varphi \simeq \varphi' \quad \text{in } U \quad \text{and} \quad \varphi'(K) \subset \hat{U}. \tag{3.11}$$

Take $A \in \mathfrak{R}$ and a map $\alpha\colon A \to U_0$ and let $\hat{U}$ be a neighborhood of X in N. Since $\dim A \leqslant n$, there exist polyhedra K_i for $i = 1, 2, \ldots$, and maps $p_{ii'}\colon K_{i'} \to K_i$ for $i' > i$ such that $\dim K_i \leqslant n$ and $A = \varprojlim\{K_i, p_{ii'}\}$. Let the maps $p_i\colon A \to K_i$ for $i = 1, 2, \ldots$ be projections such that $p_i = p_{ii'} \circ p_{i'}$ for $i < i'$. Since U_0 is open in $N \in \mathrm{AR}(\mathfrak{M})$, there exist an integer i_0 and a map $\bar{\alpha}\colon K_{i_0} \to U_0$ such that $\bar{\alpha} \circ p_{i_0} \simeq \alpha$.

Let K_{i_0} be homotopically equivalent to $K \in \mathfrak{W}^n$. Hence there exist maps $f\colon K_{i_0} \to K$ and $g\colon K \to K_{i_0}$ such that $g \circ f \simeq \mathrm{id}_{K_{i_0}}$. Take $\varphi = \bar{a} \circ g$. Since X is $\mathfrak{W}^n$-movable, there exists a $\varphi'\colon K \to U_0$ satisfying (3.11). Thus $\varphi' \circ f \circ p_{i_0} \simeq \bar{a} \circ g \circ f \circ p_{i_0} \simeq a$ in U and $\varphi' \circ f \circ p_{i_0}(A) \subset \hat{U}$. Then X is $\mathfrak{K}$-movable, and thus by (2.4) X is n-movable.

Let W^n be a one-point compactification of a disjoint union of polyhedra belonging to $\mathfrak{W}^n$. The family $\mathfrak{W}^n$ is M-dominated by the family consisting of the single element W^n. Combining this with Theorem 17 in [4] and with Theorem (3.10) we get the following

(3.12) COROLLARY. *The following conditions are equivalent*:

(a) *X is n-movable,*

(b) *X is A-movable for every compactum A of dimension $\leqslant n$,*

(c) *X is W^n-movable.*

By Corollary (2.12) we can replace "$\dim A$" by "$\mathrm{Fd}(A)$" in condition (b).

(3.13) THEOREM. *If a compactum X is n-movable and $\mathrm{Fd}(X) \leqslant n$, then X is movable.*

Proof. Since $\mathrm{Fd}(X) \leqslant n$, there exists a compactum Y such that $\mathrm{Sh}(Y) = \mathrm{Sh}(X)$ and $\dim Y \leqslant n$ ([9]). Hence there exist polyhedra Y_i for $i = 1, 2, \ldots$ and maps $p_{ii'}\colon Y_{i'} \to Y_i$ for $i' > i$ such that $\dim Y_i \leqslant n$ for $i = 1, 2, \ldots$ and $Y = \underleftarrow{\lim}\{Y_i, p_{ii'}\}$. Y is n-movable ([3], p. 860). Then it follows by Theorem (3.10) that Y is $\mathfrak{W}^n$-movable. Let $\mathfrak{K} = \{Y_i\colon i = 1, 2, \ldots\}$. By Theorem (2.5), Y is $\mathfrak{K}$-movable. Finally, by Lemma (3.5), Y is movable. Movability is a shape property ([1], p. 142), and thus X is movable.

It is easy to see that if a compactum X is $\mathfrak{K}$-movable, then X is A-movable for every $A \in \mathfrak{K}$. But the converse implication fails:

(3.14) EXAMPLE. There exist a family $\mathfrak{K}$ of compacta and a compactum X which is A-movable for every $A \in \mathfrak{K}$, but is not $\mathfrak{K}$-movable.

For every natural n let T_n be the orientable surface with n handles. Put $\mathfrak{K} = \{T_n\colon n = 1, 2, \ldots\}$ and let X be a non-movable continuum described by K. Borsuk in [2]. The compactum X can be obtained as an inverse limit of a sequence $\{T_n, p_{nn'}\}$ satisfying the condition: for $n \leqslant n'$ there exists a point $a_n \in T_n$ for which $p_{nn'}|_{p_{nn'}^{-1}(a_n)}$ is an embedding. By Lemma (3.5), X is not $\mathfrak{K}$-movable. It remains to prove that X is T_k-movable for every $T_k \in \mathfrak{K}$. Let n be an integer and let n_0 be greater than n and k. Take $\hat{n} \geqslant n_0$ and let a carry T_k into T_{n_0}. Since the number of handless of T_k is greater than the number of handles of T_{n_0}, a is homotopic to a map β with values in $T_{n_0} - \{a_{n_0}\}$. Define $\alpha'\colon T_k \to T_{\hat{n}}$ by $\alpha'(x) = p_{n_0\hat{n}}^{-1}(\beta(x))$; thus (3.3) is satisfied. By Lemma (3.2) X is T_k-movable.

4. Some properties of $\mathfrak{K}$-movability.

(4.1) EXAMPLE. For $n = 2, 3, \ldots$, there exists a continuum X_n which is $(n-1)$-movable but is not n-movable.

Let X_n be an inverse limit of a sequence $\{S_k^n, p_{kk'}\}$, where S_k^n is a n-dimensional sphere for $k = 1, 2, \ldots$ and the maps $p_{kk'}\colon S_{k'}^n \to S_k^n$, for $k' > k$, are such that $|\deg p_{kk'}| > 1$. X_1 is a solenoid and X_2 is the suspension of a solenoid. Since the homotopy classes of the maps $p_{kk'}$ are given, the shape of X_n is completely determined. X_n is non-movable ([8], p. 652); therefore by Theorem (3.13), X_n is not n-movable. By Corollary (3.4) and Theorem (3.10), X_n is $(n-1)$-movable. This example is an answer to the Problem (4.6) from [3], p. 864.

For a family $\mathfrak{K}$ of compacta and for an arbitrary binary operation $\circ$ in the family of all compacta, the following two problems arise:

1° Is it true that if X and Y are $\mathfrak{K}$-movable, then $X \circ Y$ is $\mathfrak{K}$-movable?

2° Is it true that if X is A-movable and B-movable, then X is $A \circ B$-movable?

First, we are going to answer these two questions for $\circ$ being the Cartesian product. By a slight change of the proof that if X and Y are movable, then $X \times Y$ is movable ([1], p. 142) one proves the following

(4.2) THEOREM. *X and Y are $\mathfrak{K}$-movable if and only if $X \times Y$ is $\mathfrak{K}$-movable.*

(4.3) COROLLARY. *X and Y are n-movable if and only if $X \times Y$ is n-movable.*

By Example (4.1), for $n = 1, 2, \ldots$ there exists a n-movable compactum which is not $(n+1)$-movable. Therefore

(4.4) *If X is n-movable and Y is m-movable, then $X \times Y$ is $\min(n, m)$-movable and the last number cannot be increased in general.*

The statement (4.4) is an answer to Problem (1.6) from [3], p. 860.

It is not true that if X is A-movable and B-movable, then X is $A \times B$-movable.

(4.5) EXAMPLE. There exists a compactum X which is S^1-movable but is not $S^1 \times S^1$-movable. Furthermore the non-movable compactum X is $\mathfrak{K}$-movable, where $\mathfrak{K}$ is a family of spheres of all dimensions.

Let $\{\lambda_k\}$ and $\{\lambda_k'\}$ be sequences of prime numbers greater than 1. Let S_k and S_k' be circles for $k = 1, 2, \ldots$ and let $a_k \in S_k$ and $a_k' \in S_k'$. Denote $(S_k \times \{a_k'\}) \cup (\{a_k\} \times S_k')$ by $S_k \dot{\cup} S_k'$. Let $S_k \dot{\cup} S_k'$, for $k = 2, 3, \ldots$

and $S_1\times S_1'$ be pairwise disjoint sets. Put $X_1 = S_1\times S_1'$ and $X_n = S_1\times S_1' \cup \bigcup_{k=2}^{n} S_k \dot\cup S_k'$ for $n\geqslant 2$. Define maps $p_{n,n+1}\colon X_{n+1}\to X_n$ by

$$p_{n,n+1}(x) = \begin{cases} h_{n+1}(x) & \text{for } x\in S_{n+1}\dot\cup S_{n+1}',\\ (p_n, p_n')(x) & \text{for } x\in S_1\times S_1',\\ x & \text{for } x\in \bigcup_{k=2}^{n} S_k\dot\cup S_k', \end{cases}$$

where the map $h_n\colon S_n\dot\cup S_n'\to S_1\dot\cup S_1$ is a homeomorphism and maps $p_n\colon S_1\to S_1$ and $p_n'\colon S_1'\to S_1'$ are such that $\deg p_n = \lambda_n$ and $\deg p_n' = \lambda_n'$ for $n = 1, 2, \ldots$ Put $p_{nn'} = p_{n,n+1}\circ\ldots\circ p_{n'-1,n'}\colon X_{n'}\to X_n$ for $n' > n$. Let $X = \varprojlim\{X_n, p_{nn'}\}$. Let $\mathfrak{R}$ be a family of spheres of all dimensions. We will prove that X is $\mathfrak{R}$-movable. Let n be an integer and put $n_0 = n$. Take $\hat n\geqslant n$ and a map $\alpha\colon S^m\to X_{n_0}$. If $m > 1$, then α is homotopic to a constant map. Let $\alpha'\colon S^m\to X_{\hat n}$ be a constant map such that the sets $p_{n\hat n}\circ\alpha'(S^m)$ and $\alpha(S^m)$ are both included in the same component of X_n. Then condition (3.3) is satisfied. In the case of $\hat n = n$, we put $\alpha' = \alpha$; then condition (3.3) is also satisfied. Consider $m = 1$ and $\hat n > n$. If $\alpha(S^1)\subset\bigcup_{k=2}^{n} S_k\dot\cup S_k'$, then define $\alpha'\colon S^1\to X_{\hat n}$ by $\alpha'(x) = \alpha(x)$ for $x\in S^1$. Thus $p_{n\hat n}\circ\alpha' = \alpha$. If $\alpha(S^1)\subset S_1\times S_1'$, then α is homotopic to some map $\bar\alpha\colon S^1\to X_n$ with values in $S_1\dot\cup S_1'$. Then define $\alpha'\colon S^1\to X_{\hat n}$ by $\alpha'(x) = h_n^{-1}(\bar\alpha(x))$. Thus $p_{n\hat n}\circ\alpha' = \bar\alpha\simeq\alpha$ and by Lemma (3.2), X is $\mathfrak{R}$-movable. We will now prove that X is not $S^1\times S^1$-movable. Put $n = 1$ and for $n_0\geqslant 1$ put $\hat n = n_0+1$. Let a map $\alpha\colon S^1\times S^1\to X_{n_0}$ be an embedding; then $\alpha(S^1\times S^1) = S_1\times S_1'$. Assume that a map $\alpha'\colon S^1\times S^1\to X_{n_0+1}$. Since $S^1\times S^1$ is connected, $\alpha'(S^1\times S^1)\subset S_1\times S_1'$ or $\alpha'(S^1\times S^1)\subset\bigcup_{k=2}^{n_0+1} S_k\dot\cup S_k'$. In the first case, since λ_{n_0} and λ_{n_0}' are greater than 1, $p_{1n_0}\circ\alpha$ and $p_{1,n_0+1}\circ\alpha'$ are not homotopic. In the second case, $p_{1\hat n}\circ\alpha'(S^1\times S^1)\subset S_1\dot\cup S_1'\not\subset S_1\times S_1'$. But $p_{1n_0}\circ\alpha$ is homotopic to no map with values in a proper subset of $S_1\times S_1'$. By Lemma (3.2) we infer that X is not $S^1\times S^1$-movable. Thus X is non-movable. Since X is $\mathfrak{R}$-movable, X is S^n-movable for $n = 1, 2, \ldots$ Example (4.5) is an answer to Problem 19 from [4].

Now consider the join of two spaces as the operation $\circ$. The *join* $X * Y$ of two compacta X, Y is the quotient space $(X\times Y\times[0,1])_{/G}$, where G is the decomposition of $X\times Y\times[0,1]$ into sets of the form $\{a\}\times Y\times\{1\}$ or $X\times\{b\}\times\{0\}$ (where $a\in X$ and $b\in Y$) and into single points. The shape of $X * Y$ depends only upon $\mathrm{Sh}(X)$ and $\mathrm{Sh}(Y)$ ([10], p. 854).

In general for the operation of the join the answers to questions 1° and 2° are negative. Indeed, the join $S * S^n$ of a solenoid S and a n-dimensional sphere S^n (i.e., the space X_n in Example (4.1)) is not movable. $S * S^n$ is a inverse limit of a sequence of $n+2$-dimensional spheres. By Lemma (3.5) $S * S^n$ is not S^{n+2}-movable, while by Corollary (3.4) S and S^n are S^{n+2}-movable. Also it is easy to see that the join $A * B$ of two two-point spaces A and B is a circle S^1. By Corollary (3.4) a solenoid is A-movable and B-movable, but is not S^1-movable.

(4.6) THEOREM. *Let $\mathfrak{R}$ be a family of compacta such that if $A\in\mathfrak{R}$ and a compactum B is the closure of an open subset of A, then $B\in\mathfrak{R}$. If compacta X and Y are $\mathfrak{R}$-movable, then the join $X * Y$ is $\mathfrak{R}$-movable.*

Proof. Let Q and Q' be the Hilbert cubes. Assume that $X\subset Q$ and $Y\subset Q'$. $M = Q * Q'\in AR$ ([10], p. 854). Let U be a neighborhood of $X * Y$ in M. There exists a neighborhood U^1 of X in Q and a neighborhood U^2 of Y in Q' and a number $\varepsilon\in(0,\frac{1}{6}]$ such that the sets $U^1 * U^2 = \{[(x,y,t)]\in M;\ x\in U^1, y\in U^2\}$, $K(U^1,\varepsilon) = \{[(x,y,t)]\in M;\ x\in U^1, y\in Q', 1-\varepsilon < t\leqslant 1\}$ and $K'(U^2,\varepsilon) = \{[(x,y,t)]\in M;\ x\in Q, y\in U^2, 0\leqslant t<\varepsilon\}$ are subset of U.

Since X and Y are $\mathfrak{R}$-movable, for U^1 and U^2 there exist neighborhoods: U_0^1 of X in Q and U_0^2 of Y in Q' satisfying required conditions of the definition of the $\mathfrak{R}$-movability. The set $U_0 = K(U_0^1,\varepsilon)\cup K'(U_0^2,\varepsilon)\cup U_0^1\times U_0^2\times[0,1]_{/G}$ is a neighborhood of $X * Y$ in M ([10], p. 854). Let $\hat U$ be a neighborhood of $X * Y$ in M. There exists a neighborhood $\hat U^1$ of X in Q and a neighborhood $\hat U^2$ of Y in Q' such that $\hat U^1 * \hat U^2\subset\hat U$. Take $A\in\mathfrak{R}$ and a map $\alpha\colon A\to U_0$. The sets $B = \overline{\alpha^{-1}(U_0^1\times U_0^2\times(\varepsilon,1]_{/G})}$ and $B' = \overline{\alpha^{-1}(U_0^1\times U_0^2\times[0,1-\varepsilon)_{/G})}$ belong to $\mathfrak{R}$. Let a map $p_1\colon Q\times Q'\times(0,1]_{/G}$ be defined by $p_1([(x,y,t)]) = x$ for $[(x,y,t)]\in Q\times Q'\times(0,1]_{/G}$ and let a map $p_2\colon Q\times Q'\times[0,1)_{/G}$ be defined by $p_2([(x,y,t)]) = y$ for $[(x,y,t)]\in Q\times Q'\times[0,1)_{/G}$.

Define maps: $\beta\colon B\to U_0$ by $\beta(b) = p_1(\alpha(b))$ for $b\in B$ and $\beta'\colon B'\to U_0$ by $\beta'(b) = p_2(\alpha(b))$ for $x\in B'$. Since X and Y are $\mathfrak{R}$-movable, there exists a homotopy $F\colon B\times[0,1]\to U^1$ satisfying conditions: $F(b,0) = \beta(b)$ and $F(b,1)\in\hat U^1$ for $b\in B$ and there exists a homotopy $F'\colon B'\times[0,1]\to U^2$ satisfying conditions: $F'(b,0) = \beta'(b)$ and $F'(b,1)\in\hat U^2$ for $b\in B'$. Define a map $q\colon M\to[0,1]$ by $q([(x,y,t)]) = t$ for $[(x,y,t)]\in M = Q\times Q'\times[0,1]_{/G}$. For $s\in[0,1]$ let a map $\varphi_s\colon [0,1]\to[0,1]$ be defined by:

$$\varphi_s(t) = \begin{cases} 0 & \text{for } 0\leqslant t\leqslant\frac{1}{3}s,\\ \dfrac{3t-s}{3-2s} & \text{for } \frac{1}{3}s<t<1-\frac{1}{3}s,\\ 1 & \text{for } 1-\frac{1}{3}s\leqslant t\leqslant 1. \end{cases}$$

Define a homotopy $H\colon A\times[0,1]\to U$ by the formula:

$$H(a,s)=\begin{cases}[p_1(\alpha(a)),\ p_2(\alpha(a)),\ \varphi_{2s}\circ q(\alpha(a))] & \text{if } 0\leqslant s\leqslant\frac12 \text{ and } 0<q(\alpha(a))<1,\\ \alpha(a) & \text{if } 0\leqslant s\leqslant\frac12 \text{ and } q(\alpha(a))=0 \text{ or } 1,\\ [F(a,2s-1),\ F'(a,2s-1),\ \varphi_1\circ q(\alpha(a))] & \text{if } \frac12<s\leqslant 1 \text{ and } \frac13<q(\alpha(a))<\frac23,\\ \{F(a,2s-1)\}\times Q'\times\{1\} & \text{if } \frac12<s\leqslant 1 \text{ and } q((\alpha(a))\geqslant\frac23,\\ Q\times\{F'(a,2s-1)\}\times\{0\} & \text{if } \frac12<s\leqslant 1 \text{ and } q(\alpha(a))\leqslant\frac13.\end{cases}$$

This homotopy satisfies conditions: $H(a,0)=\alpha(a)$ and $H(a,1)\in\hat U^1*\hat U^2\subset\hat U$ for every $a\in A$. Thus $X*Y$ is $\mathfrak{R}$-movable.

(4.7) Corollary. *If the compacta X and Y are n-movable, then the join $X*Y$ is n-movable.*

Proof. n-movability is equivalent to $\mathfrak{R}$-movability, where $\mathfrak{R}$ is a family of all compacta of the dimension $\leqslant n$ (cf. (2.4)). If $A\in\mathfrak{R}$ and a compactum $B\subset A$, then $\dim B\leqslant n$, then $B\in\mathfrak{R}$. Hence $\mathfrak{R}$ satisfies the assumption of Theorem (4.6).

(4.8) Example. There exist compacta X^1, X^2, A^1 and A^2 such that $X^1\cap X^2=\{x_0\}$, $A^1\cap A^2=\{a_0\}$, X^i is $A^1\cup A^2$-movable and $X^1\cup X^2$ is A^i-movable for $i=1,2$ but $X^1\cup X^2$ is not $A^1\cup A^2$-movable.

The main idea of this example is due C. Cox [5]. Let $i=1,2$. Let $\{\lambda_j^i\}$ be sequences of prime number different from 1. For $k=1,2,\dots$, let S_k^i be pairwise disjoint k-dimensional spheres, except the pair S_1^1, S_1^2 with the point a_0 in common. Let $f_j^i\colon S_1^i\to S_1^i$ be a map such that $\deg f_j^i=\lambda_j^i$ and $f_j^i(a_0)=a_0$ for $j=1,2,\dots$ and let $h_k^i\colon S_{k+1}^i\to S_1^i$ be a homeomorphism for $k=1,2,\dots$ Put $X_n^i=\bigcup_{k=1}^n S_k^i$. Define $p_{nn+1}^i\colon X_{n+1}^i\to X_n^i$ by

$$p_{nn+1}^i(x)=\begin{cases}f_n^i(x) & \text{for } x\in S_1^i,\\ x & \text{for } x\in\bigcup_{k=2}^n S_k^i,\\ h_n^i(x) & \text{for } x\in S_{n+1}^i.\end{cases}$$

Let $p_{nn'}^i=p_{nn+1}^i\circ\dots\circ p_{n'-1n'}^i$ for $n<n'$ and $p_{nn}^i=\mathrm{id}_{X_n^i}$. Put $X^i=\varprojlim\{X_n^i,p_{nn'}^i\}$. Let n be an integer and put $n_0=n$. Let $\hat n>n$. Define a map $r^i\colon X_{n_0}^i\to X_{\hat n}^i$ by

$$r^i(x)=\begin{cases}x & \text{for } x\in\bigcup_{k=2}^{n_0} S_k^i,\\ (h_n^i)^{-1}(x) & \text{for } x\in S_1^i.\end{cases}$$

Then $p_{n\hat n}^i\circ r^i=\mathrm{id}_{X_n^i}=p_{nn_0}^i$. By Lemma (3.1) X^i is movable; then X^i is B-movable for every compactum B. Let $A^1=S_1^1$ and $A^2=S_1^2$. It is easy to see that a compactum $X=X^1\cup X^2$ is A^1-movable and A^2-movable. It remains to prove that X is not $S_1^1\cup S_1^2$-movable. Let $S_1^1\cup S_1^2=A$. We have $X=\varprojlim\{X_n,q_{nn'}\}$, where $X_n=X_n^1\cup X_n^2$ and $q_{nn'}(x)=p_{nn'}^i(x)$ for $x\in X_{n'}^i$. Take $n=1$ and let $n_0\geqslant 1$. Put $\hat n=n_0+1$ and let $\alpha\colon A\to X_{n_0}$ be an inclusion map. Take a map $\alpha'\colon A\to X_{n_0+1}$. For $i=1,2$, S_1^i is reeled $(\lambda_1^i\cdot\dots\cdot\lambda_{n_0-1}^i)$ times in S_1^i by $p_{1n_0}^i\circ\alpha$. Since A is connected, $\alpha'(A)$ is contained in some component of X_{n_0+1}. If $\alpha'(A)\subset S_k^i$ for $k>1$, then $\alpha'|_{S_1^j}$ for $j\neq i$ is homotopic to a constant map. If $\alpha'(A)\subset S_1^1\cup S_1^2$, then for $i=1$, S_1^i is reeled λ^i-times in S_1^i by α' for some integer λ^i. Since $\lambda_{n_0}^i>1$, $\lambda_1^1\cdot\dots\cdot\lambda_{n_0-1}^i$ and $\lambda_1^1\cdot\dots\cdot\lambda_{n_0}^1\cdot\lambda^1$ are different. Thus $p_{1n_0}\circ\alpha$ and $p_{1\hat n}\circ\alpha'$ are not homotopic. By Lemma (3.2), X is not $A^1\cup A^2$-movable.

(4.9) Theorem. *If every component of a compactum X is $\mathfrak{R}$-movable, then X is $\mathfrak{R}$-movable.*

Proof. Assume that $X\subset N\in\mathrm{AR}(\mathfrak{M})$. Let U be a neighborhood of X in N. As in the proof of a similar theorem for movability ([1], p. 140) we can choose a finite system of components $X_1,\dots,X_n$ of X and pairwise disjoint open sets $U_1,\dots,U_n$ satisfying three conditions:

U_i is a neighborhood of X_i in N for $i=1,2,\dots,n$,

$U_0=\bigcup_{i=1}^n U_i$ is a neighborhood of X in N,

for $i=1,2,\dots,n$, for every neighborhood $\hat U_i$ of X_i in N and for every $A\in\mathfrak{R}$ every map $\alpha_i\colon A\to U_i$ is homotopic in U to a map with values in $\hat U_i$.

Let $\hat U$ be a neighborhood of X in N and take $A\in\mathfrak{R}$ and a map $\alpha\colon A\to U_0$. Put $A_i=\alpha^{-1}(U_i)$. Define $\alpha_i\colon A\to U_i$ for $i=1,2,\dots,n$ by

$$\alpha_i(x)=\begin{cases}\alpha(x) & \text{for } x\in A_i,\\ x_i & \text{for } x\notin A_i,\ \text{where } x_i \text{ is a fixed point of } X_i.\end{cases}$$

For $i=1,2,\dots,n$, let $H_i\colon A\times[0,1]\to U$ be a homotopy such that $H_i(a,0)=\alpha_i(a)$ and $H_i(a,1)\in\hat U$ for every $a\in A$. Define $H\colon A\times[0,1]\to U$ by $H(a,t)=H_i(a,t)$ for $a\in A_i$. $H(a,0)=\alpha(a)$ and $H(a,1)\in\hat U$ for every $a\in A$, then X is $\mathfrak{R}$-movable.

On the other hand, it is not true that if X is $\mathfrak{R}$-movable, then every component of X is $\mathfrak{R}$-movable. There exists a movable compactum with a solenoid as a component (K. Borsuk's Example [1], p. 140, also the compactum X^1 in Example (4.8)). As in Example (4.8), for every compactum X which is not $\mathfrak{R}$-movable one can construct an $\mathfrak{R}$-movable compactum Y with X as a component.

(4.10) THEOREM. *If $\mathfrak{R}$ is a family of all components of a compactum A, then $\{A\}$ and $\mathfrak{R}$ are M-equivalent.*

Proof. Assume that $X \subset N \in \mathrm{AR}(\mathfrak{M})$. Let U, U_0 and $\hat{U}$ be neighborhoods of X in N. Assume that a map $\alpha\colon A \to U_0$ and that for every $B \in \mathfrak{R}$ there exists a homotopy $\varphi_B\colon B \times [0,1] \to U$ such that $\varphi_B(a, 0) = \alpha(a)$ and $\varphi_B(a, 1) \in \hat{U}$ for every $a \in B$. Since a component B is closed in A and U is open in N, a homotopy φ_B can be extend over a set $B' \times [0,1]$ such that B' is a closed-open neighborhood of B. As in the proof of Theorem (4.9), one can choose a finite system of components $B_1, \ldots, B_n$ such that the sets $B_1', \ldots, B_n'$ constructed for them are pairwise disjoint and $\bigcup_{i=1}^{n} B_i' = A$. Define a homotopy $H\colon A \times [0,1] \to U$ by $H(a, t) = \varphi_{B_i'}(a, t)$ for $a \in B_i'$; then $H(a, 0) = \alpha(a)$ and $H(a, 1) \in \hat{U}$ for every $a \in A$. Conversely, assume that $B \in \mathfrak{R}$, a map $\beta\colon B \to U_0$ and that every map $\alpha\colon A \to U_0$ is homotopic in U to a map with values in $\hat{U}$. There exists a closed-open neighborhood B' of B in A and a map $\beta'\colon B' \to U_0$ extending β. Let $u_0 \in U_0$, and define $\alpha\colon A \to U_0$ by

$$\alpha(a) = \begin{cases} \beta'(a) & \text{for} \quad a \in B', \\ U_0 & \text{for} \quad a \in A - B'. \end{cases}$$

Terefore, $\beta = \alpha|_B$ is homotopic in U to a map with values in $\hat{U}$. Thus A-movability and $\mathfrak{R}$-movability are equivalent.

The notion of n-movability has recently been studied by Kodama and Watanabe and by Kozlowski and Segal (see [6] and [7]). They obtained independently the following results contained in the present paper: Theorems (3.6), (3.10) and (3.13), Example (4.1) and Corollary (4.3).

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