On extending congruences from partial algebras

by

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Abstract. A new proof, very much shorter and easier than those in the literature, that every congruence on a partial universal algebra is the restriction of some congruence on a containing total algebra, leads to a generalization in which the latter may be chosen in an equational class defined by identities of a certain type whenever these hold in the former. The analogue for strong congruences is pushed even further: viz, to any total algebra in which the partial one is embedded in a not too special way. A third yield is the extendability of every congruence on a partial distributive lattice to some containing total distributive lattice, which settles a problem posed by Grätzer.

What follows had its origin in the observation that Theorem 13.3 of Grätzer's "Universal Algebra", to the effect that every congruence on a partial algebra is the restriction of a congruence on a containing total algebra (given an independent proof in [GW]; see also [W] for the extension to infinitary algebras) or rather its more precise and stronger version, Theorem 15.1, that there exists a single containing algebra (namely the freely generated one) which will do the job for all congruences; and whose proof extends across five pages of that work, can be very easily proved as follows: Embed the quotient of the partial algebra P modulo its congruence θ in a total algebra F in any way (either by extending the operations for undefined arguments arbitrarily in P — this is what [W] comes to — or by adjoining new elements as their values — cf. 13.1 in [G]) and extend the quotient map to a homomorphism into F of the total algebra $F(P)$ freely generated by P: the kernel of this homomorphism is then a congruence on $F(P)$ which meets P in θ. In addition, the elaborate construction [1] of $F(P)$ occupying the seven pages of the book immediately preceding 15.1 is not needed for the purpose at hand, inasmuch as $F(P)$, qua total algebra generated by P to which all homomorphisms of P into total algebras extend, may be obtained as the subdirect product generated by the image of P in a product of sufficiently varied total algebras generated by images of P — the fact that P is embedded in some total algebra (as seen above) ensuring that it will also be in such a product.

This efficient argument begs to be generalized: Does a given P admit

[1] A more graceful one can be found in [W].

embeddings into total algebras F, other than the freely generated one, to which every congruence θ (possibly of a special type) extends? By the above this means being able to embed P/θ (for such congruences) in a total algebra so as to allow an extension of the quotient homomorphism to F. Three results in this line are presented below.

1. Every partial algebra can be completed to be total by adjoining one new element to serve as value for all not already defined operation values (including those for which it is an argument). If the quotient homomorphism on P were to extend to a homomorphism between these respective one-element completions by making the new elements correspond, P would become saturated for the enlarged kernel. Now any subset P of an algebra, saturated for one of its congruences θ, must contain, with any operation-value of a tuple of its elements, also all other values obtained by $\operatorname{mod}\theta$ changes of argument which remain in P — in the terminology of [G], θ induces a "strong" congruence on P. More generally, a subset of a partial algebra F is saturated only if it contains all such values for those $\operatorname{mod}\theta$ argument changes which remain in P as well as in the domain of the operation's definition in F — this might be termed *strong relative to* the assignment to each (n-argument) operation of those (n-tuple) changes in P not leading out of the operation's domain of definition in F.

Conversely let θ be strong in P: Since the domains of definition of operations in P are closed under $\operatorname{mod}\theta$ argument changes, so also are their complements, and it follows that the extension of the quotient homomorphism to the one-element completions obtained by making correspond the new elements is indeed a homomorphism of these total algebras. Thus every strong congruence extends to the one-element completion. But more is true: for if a congruence on P extends to a (possibly partial) containing algebra F, so does it to every other containing algebra which can be mapped homomorphically into F so as to induce the identity on P (as the inverse image of the extending congruence); moreover, P will retain the saturatedness it had in F if elements outside it are mapped on outside elements; and this must happen for an algebra generated by P mapped to the one-element completion over the identity on P. Now the partial algebra extensions of P which can be mapped in this way on the one-element completion are just those which share with this completion the property of having operations take values in P at most for arguments in P. This justifies

THEOREM S.C. *Let the partial algebra P be embedded in the (possibly partial) algebra F so that operations take values in P only for arguments in P. Then every strong (relative to argument changes preserving performability in F) congruence on P is the restriction of a congruence on F for which P is saturated.*

Inasmuch as P is so embedded in its absolutely free completion, this effects a broad generalization of Theorem 16.2 in [G].

2. When attempting to extend a congruence to containing total algebras in an equational class, it is natural to replace the absolutely freely generated algebra by the one freely generated in the class. The required homomorphism extending the quotient map would then be forthcoming by freeness; whence it would only be needed to be able to embed P/θ in a member of the class. Such embeddability is also necessary: the congruences θ which extend to congruences in some containing total algebra of an equational class are just those for which P/θ is (like P) embeddable in a total algebra of the class. (The same characterization holds for the θ modulo whose extended congruence the total algebra is required to belong to a subclass.)

That this condition is not automatically fulfilled may be seen in the partial group of positive integers: every semigroup congruence is a partial group congruence (even a strong one) but there are certainly quotients not embeddable in a group.

Success can clearly be expected where the conditions for embedding a partial algebra in a total one from the class are preserved under passage to quotient partial algebras: e.g. conditions which can be put in the form of an implication whose hypothesis can be carried back from the quotient (an inequality, or the performability of an operation) and whose conclusion carried forward to it (a form built up using arbitrary compositions of the operations, equality, conjunction, disjunction, and quantification — but not negation). Indeed, according to unpublished results of M. Makkai [M] (for information about which I am indebted to Gonzalo Reyes) this is effectively the only type of condition preserved under quotients.

An equational class may be singled out by stipulating that its members satisfy a certain set of identities, i.e. equations between polynomials. These may be construed here as "polynomial functions" which are of course not functions in the usual sense, but formal entities having a functional interpretation in every algebra, on a par with the operations. In fact, they are generated from the operations (including the unary identity) by composition and equating of arguments; put another way, they constitute the smallest class including these operations and closed under substitution for an argument of a member of the class or of another argument. (Alternatively, they may also be obtained from the word algebra built with the operations, as abstractions of the functions it induces in every total algebra equipped with these operations: thus formally modulo the equivalence of inducing in every such algebra the same function. The more elaborate definition [G] just comes to allowing extra (spurious) arguments. To get ordinary ring polynomials, one must have recourse to his more general "algebraic functions".) As the result of such a substitution, the polynomial no longer *depends on* the argument via that place, depending instead on the substituted argument or on all those arguments on which the substituted polynomial depends. The point of emphasizing this is that in the one-element completion of P an identity will hold if (and only if except for the trivial case that neither side ever takes a value in P) both sides depend on the same arguments and whenever one side can be evaluated in P, so can the other and the two are equal. (Indeed, a polynomial takes a value in P if and only if the arguments on which it depends do and it can be evaluated in P for them). Thus with identities of this form holding (in this sense) in it, P can be embedded in a total algebra in their equational class, hence in the algebra it freely generates in this class.

However, it is not to be expected that these identities will also hold in P/θ — unless the evaluability at a tuple of P/θ of a polynomial appearing on one side of an identity were to entail its evaluability at some tuple of P mapping on it. For pure operations (in which no substitutions have been made) this follows from their definition in P/θ; more generally it will follow if θ is strong relative to an assignment, to each operation which occurs in one of the identities with some substitutions made, of all those changes in P for its substituted arguments which would result in an instance of the occurring form.

THEOREM IS. *If each of a set of identities, both sides of which depend on the same arguments, holds (in the above sense) in the partial algebra P, then it is embedded in $F(P)$, the total algebra it freely generates in the equational class of the set. Moreover a congruence on P will extend to $F(P)$ if it is strong relative to the assignment, for those arguments substitution for which yields a polynomial form occurring in an identity, of their changes in P which would realize this form.*

Since the identity map on P extends to a homomorphism from $F(P)$ to the one-element completion, the case of (unqualified) strong congruences already falls under the previous Theorem SC. It may be worthwhile to state separately the case in which the identities equate operations in which no substitutions have been made:

COROLLARY. *If identities between (pure) operations depending on the same arguments hold in P, then each of its congruences extends to a congruence of the algebra freely generated by P in their equational class.*

The commutative law for a binary operation is an example of such

an identity. The corollary also contains (for the void set of identities) the original Theorem 15.1 of [G].

3. This by no means exhausts the applicability of the proposed proof strategy, since the embeddability criterion may well be preserved by passage to quotients in an equational class to which the one-element completion does not belong. Such is the case for the class of distributive lattices construed as algebras with two binary operations (even though one side of the absorption law depends on more variables then the other) provided that the partial algebras have one of them — say $\vee$ — everywhere defined. Specifically they are to be $\vee$-semilattices in which an additional binary operation $\wedge$ is partially defined by assigning certain pairs having a greatest lower bound in the $\vee$-derived order (not necessarily all such) this g.l.b. as value.

Necessary for embedding such a structure, with preservation of $\vee$ and all defined $\wedge$, in a distributive lattice is clearly: if $\alpha\wedge\beta$ is defined, then for every γ, $(\alpha\wedge\beta)\vee\gamma$ is the g.l.b. of $\alpha\vee\gamma$ and $\beta\vee\gamma$; and then one may as well extend $\wedge$ to such pairs since the embedding will certainly preserve this assignment.

Conversely, this condition is then sufficient (1). One has to show that for any $\alpha \nleq \beta$ there is a $\vee$ and $\wedge$ preserving map to the two-element chain sending α on 1 and β on 0; or, which comes to the same, that α and β can be separated by a decomposition into an ideal and a complementary filter (the latter must be understood here as a non-void subset closed under performable $\wedge$'s of its elements and passage to larger elements). Just as in the classical case, this results from an ideal maximal with respect to excluding a filter having a filter complement: if α, β are in the complement then their $\vee$ with some — hence all sufficiently large — γ of the ideal are in the filter; thus if $\alpha\wedge\beta$ is defined, the filter contains $(\alpha\vee\gamma)\wedge(\beta\vee\gamma)=(\alpha\wedge\beta)\vee\gamma$, whence $\alpha\wedge\beta$ cannot be in the ideal. (Note that the embedding preserves smallest and greatest elements — they belong to all ideals and filters respectively — hence also any existing complements).

Moreover, modulo a congruence for both operations, all of this subsists: the quotient is again a $\vee$-semilattice; one in which the quotient $\wedge$ functions (where defined) as g.l.b: for from $(\gamma\vee\alpha)\;\theta\;\alpha$, $(\gamma\vee\beta)\;\theta\;\beta$, and the existence of $\alpha\wedge\beta$ follows $(\gamma\vee\alpha)\wedge(\gamma\vee\beta)=\gamma\vee(\alpha\wedge\beta)\;\theta\;(\alpha\wedge\beta)$; and the condition is of the desired type: an implication of which the hypothesis is the performability of an operation and the conclusion an equality.

(1) This has also been observed by others: See B. M. Schein, *On the definition of distributive semilattices*, Alg. Univ. 2 (1972), pp. 1–2. An even more general result goes back to H. M. MacNeille, Trans. Am. Math. Soc. 42 (1937), p. 446 ff.

THEOREM DL. *Let P be a $\vee$-semilattice with a partial $\wedge$ taking g.l.b. as values such that the existence of $\alpha\wedge\beta$ implies $(\alpha\vee\gamma)\wedge(\beta\vee\gamma) = (\alpha\wedge\beta)\vee\gamma$ for every γ. Then P is embedded in the distributive lattice F it freely generates, and every congruence (for both the total and partial operation) of P is the restriction of a lattice congruence on F.*

This furnishes in particular a solution for the distributive case of Problem 20 in [G′].

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Topologically nondegenerate functions

by

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Abstract. Let M_n be a compact, connected topological manifold and F a continuous real valued function on M_n that is topologically nondegenerate in the sense of Morse [12]. Let c be an arbitrary value of F and set

$$F_c = \{p \in M_n | \; F(p) \leqslant c\}.$$

The "topological critical points" of F on F_c are finite in number and can be related to the invariants of the homology groups of F_c as in the differentiable case (Morse and Cairns [14]). F-deformations and F-tractions make this possible. F-tractions are our extensions of retracting deformations of Borsuk [1]. Kirby and Siebenmann in [7] have affirmed the existence of topologically nondegenerate functions on M_n when $n \neq 4$ or 5. For the differentiable case see [15], Milnor [9] and Cerf [3]. Paper [16] reorganizes the classical group structure of the singular homology theory of Eilenberg [5] for use in this paper.

Introduction. This paper is concerned with *continuous*, real-valued, *topologically* nondegenerate functions F, as distinguished from *differentiably* nondegenerate functions. (See § 1 for definitions.) The domain of F is taken as a compact topological manifold M_n. The paper [14] of Morse and Cairns is here extended from the differentiable case to the topological case. A brief abstract of this paper is found in [13].

Singular homology theory is used of the type first introduced by Eilenberg in 1944. See reference [6]. No "triangulations" are needed. Deformations termed "tractions", are fundamental; they relax the conditions on "retracting deformations" as commonly defined. For original concepts see Borsuk [10]. The theorem of Kirby and Siebenmann on the existence of topologically nondegenerate functions, when $n \neq 4$ or 5, is a starting point. This paper draws heavily on Morse [12] in which topologically nondegenerate functions were first defined. Paper [16] reorganizes the classical group structure for use in the necessary homology theory.

To avoid complexity in a first treatment this study has been subjected to many restrictions that can be readily removed. In particular, one could greatly lighten the condition that the manifold be compact. One could also remove the condition that ~~the~~ topological critical values be of *singleton* type in the sense of § 0.