# On Cartesian factors and the topological classification of linear metric spaces

by

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**Abstract.** In one part of the paper we improve the results of [26] to show that if $E$ is a locally convex metrizable TVS such that $E \cong \Sigma E$ (resp. $E \cong E^{\infty}$) then $X \times E \cong E$ for each retract (resp. complete retract) $X$ of $E$; see Theorem 3.1. The remaining part is concerned with the topological classification of metrizable TVS's. We show that if $\{X_i: i \in N\}$ is any family of such spaces and $E$ is a coordinate space homeomorphic to $l_2$, then $\Pi_E X_i$ is homeomorphic to the Cartesian product of the $X_i$'s. If, in addition, $X_1 \cong X_2 \cong \ldots \cong X$ and $X$ is a complete $\mathrm{AR}(\mathfrak{M})$ then $\Pi_E X_i$ is topologically a Hilbert space; see Theorems 1.1, 1.7, and 4.1.

We shall be concerned here with two topics, as indicated in the title. The first one is the characterization of Cartesian factors of a given infinite-dimensional linear metric space; in this respect the paper is a continuation of author's papers [25] and [26], cf. also J. E. West [29]. We prove a theorem slightly more general than the following:

(A) Let $E$ be a locally convex linear metric space and let $X$ be a retract of $E$. If either $E \cong \Sigma E = \{(x_i) \in E^{\infty}: x_i = 0$ for almost all $i\}$ or $X$ is complete-metrizable and $E \cong E^{\infty}$, then $X \times E \cong E$ ($\cong$ stands for "is homeomorphic to").

We use this theorem to extend the results of D. W. Henderson and J. E. West on representing a manifold as a product of its model with some locally finite-dimensional simplicial complex (metric topology) and also to obtain some other corollaries on infinite-dimensional manifolds; see § 3. It should be noted that (A), when combined with some embedding theorems (see 1.4 in [26]), contains as a special case the main result of [26].

The second topic is related to the topological classification of linear metric spaces. Continuing the investigations of C. Bessaga [1] and W. E. Terry [22], we show in § 1 that if a coordinate space $E$ is homeomorphic to $l_2$, then the strong $E$-product of any countable family of linear metric spaces is homeomorphic to the Cartesian product of that family. This fact is proved by using a factor theorem of [26] and, on the other hand, it is a tool in proving the general factor theorem (A). In turn, (A) allows us to establish in § 4 the following topological characterization of the Hilbert spaces: A complete linear metric space $E$ is homeomorphic to an infinite-dimensional Hilbert space iff $E \in \mathrm{AR}(\mathfrak{M})$ and $E \cong E^\infty$. A corollary of this fact is that for any Fréchet space $E$ the product $E^\infty$ is topologically a Hilbert space [1].

In the paper we prove also a theorem on embedding a retract of a linear metric space $E$ as a regular retract [2] of an $F$-normed space homeomorphic to $E^\infty$ (§ 2); this theorem enables us to apply the results of [25] in proving (A).

**§ 0. Preliminaries.** The notation and terminology not specified below is as in [26]. By $X^A$ we denote the set of all functions of $A$ into $X$. If $x \in X^A$ then we write also $x_a = x(a)$, $a \in A$, and $x = (x_a)$; similarly, if $x$ is a point of a product $\boldsymbol{P}X_a$ of a family of spaces, then $x_a$ is the $a$th coordinate of $x$ and we write $x = (x_a)$. If $f\colon X \to \boldsymbol{P}X_a$ is a function, then we write $f = (f_a)$ where $f_a(x) = \big(f(x)\big)_a$; "map" means "continuous function". $R$ denotes the set of reals.

Let $E$ be a linear space and let $A$ be a set. We consider $E^A$ as an $R^A$-module with natural operations, and for $\lambda, \mu \in R^A$ we let $\lambda \leqslant \mu$ if $\lambda(a) \leqslant \mu(a)$ for all $a \in A$. $\chi_B \in R^A$ is the characteristic function of a set $B \subset A$.

A linear metric space means here a metrizable topological vector space. After [14, p. 163] by an $F$-norm on a vector space $X$ we mean a function $x \mapsto \|x\|$ such that $\|x\| > 0$, $\lim_{\mu\to 0}\|\mu x\| = 0$ and $\|\lambda x + y\| \leqslant \|x\| + \|y\|$ for all $\lambda \in [-1, 1]$ and $x, y \in X$, $x \neq 0$. An "$F$-normed space" means a pair $(X, \|\ \|)$ where $\|\ \|$ is an $F$-norm on the vector space $X$ and $X$ is given the $\|\ \|$-topology. It is known that every linear metric space $X$ admits an $F$-norm which generates the topology of $X$; moreover, a theorem of V. L. Klee asserts that if $X$ is complete-metrizable (briefly: complete), then each admissible $F$-norm on $X$ is complete [14, p. 165].

The Tichonov product of a family of topological spaces is denoted by $\prod_{a\in A} X_a$ or $\Pi X_a$ in contrast to $\boldsymbol{P}_{a\in A} X_a$, which is treated as a set only. If $A = N$, the set of integers, and $(X_i, \|\ \|_i)$, $i \in N$, are $F$-normed spaces, then $\Pi X_i$ will always be considered in the $F$-norm

$$|||(x_i)||| = \sum_{i\in N} \min(\|x_i\|_i, 2^{-i+1}), \tag{0}$$

and $X_1 \times X_2$ in the $F$-norm $\|(x_1, x_2)\| = \|x_1\|_1 + \|x_2\|_2$.

The subspace $\{(x_a) \in \Pi X_a\colon x_a = 0 \text{ for almost all } a\}$ of $\Pi X_a$ is denoted by $\sum_{a\in A} X_a$ or $\Sigma X_a$. We abbreviate $\sum_{a\in A} X_a$ to $\Sigma X$ and $\prod_{a\in A} X_a$ to $X^\infty$ if $A = N$ and $X_a = X$ for all $a$. The following facts hold:

(1) This corollary was announced in [26]. Independently, it has been proved also by W. E. Terry [23].

(2) See [25] for the definition.



0.1. LEMMA. *If $X_i$, $i \in N$, are non-degenerate linear metric spaces, then $\Pi X_i \cong R^\infty \times \Pi X_i \cong l_2 \times \Pi X_i$ and $\Sigma X_i \cong \Sigma R \times \Sigma X_i$.*

Proof. By a result of E. Michael (see [16], p. 753) for each $i \in N$ there is a linear metric space $Y_i$ with $X_i \cong R \times Y_i$. Thus $\Pi X_i \cong R^\infty \times \Pi Y_i \cong R^\infty \times R^\infty \times \Pi Y_i \cong R^\infty \times \Pi X_i$ and similarly $\Sigma X_i \cong \Sigma R \times \Sigma X_i$. Now the assertion follows from Anderson's theorem [0], stating that $R^\infty \cong l_2$.

0.2. (Dugundji theorem [6, p. 77]). *If $X$ is a convex subset of a locally convex linear metric space then $X \in \mathrm{AR}(\mathfrak{M})$.*

By a Fréchet space we mean any locally convex linear metric space.

**§ 1. The topological type of coordinate products.** Following [1] and [5] we call an $F$-normed space $(E, \|\ \|)$ a *coordinate space over a set $A$* if $E$ is a linear subspace of $R^A$ and

(i) Given $f \in E$ and $\lambda \in R^A$ with $\sup\{|\lambda(a)|\colon a \in A\} \leqslant 1$ we have $\lambda \cdot f \in E$ and $\|\lambda \cdot f\| \leqslant \|f\|$;

(ii) For every $\varepsilon > 0$ and $f \in E$ there is a finite set $B \subset A$ with $\|\chi_{A\setminus B} f\| < \varepsilon$;

(iii) $E$ is contained in no hyperplane $H_a = \{\lambda \in R^A\colon \lambda(a) = 0\}$, $a \in A$.

It follows from (iii) and (i) that $E \supset \{\lambda \in R^A\colon \lambda(a) = 0 \text{ for almost all } a\}$. Thus, by (ii) and (i), the space $E$ is separable iff $A$ is a countable set.

Let $\{(X_a, \|\ \|_a)\colon a \in A\}$ be a family of $F$-normed spaces. Given $x = (x_a) \in \boldsymbol{P} X_a$, denote

$$|x| = (\|x_a\|_a) \in [0, \infty)^A.$$

If $(E, \|\ \|)$ is a coordinate space over $A$, we consider the following linear spaces [3]

$$\Pi_E(X_a, \|\ \|_a) = \{x \in \boldsymbol{P} X_a\colon |x| \in E\}$$

and

$$\Sigma_E(X_a, \|\ \|_a) = \{(x_a) \in \boldsymbol{P} X_a\colon x_a = 0 \text{ for almost all } a\},$$

both eqquipped with the $F$-norm

$$|||x||| = \|f\| \quad \text{where} \quad f = |x| \in E.$$

These spaces will be called, respectively, the *strong* and the *weak coordinate product of the $X_a$'s in the sense of $E$* or, briefly, the (*strong* and *weak*) *$E$-products* of the $X_a$'s. If the $F$-norms $\|\ \|_a$, $a \in A$, are fixed or are not important in the discussion, then we shall write $\Pi_E X_a$ instead of $\Pi_E(X_a, \|\ \|_a)$ and $\Sigma_E X_a$ instead of $\Sigma_E(X_a, \|\ \|_a)$, and we shall abbreviate $\Pi_E X_a$ to $\Pi_E X$ and $\Sigma_E X_a$ to $\Sigma_E X$ if $(X_a, \|\ \|_a) = (X, \|\ \|)$ for all $a \in A$.

(3) The notation is different from that of [1] and [5].

The aim of this section is to show that if $E$ is a complete separable coordinate space then, under rather weak additional assumptions, the topological types of $\Pi_E X_a$ and $\Sigma_E X_a$ depend only on the family $\{X_a\}$ and not on the linear-topological type of $E$. Earlier, results in this direction were obtained by C. Bessaga [1] who showed that if the $X_a$'s are all Banach spaces then $\Pi_E X_a \cong \Pi_F X_a$ for $E, F \in \{c_0, l_p, \text{Köthe spaces}\}$, and by W. E. Terry who recently showed in [22] that Bessaga's result remains true if the $X_a$'s are complete linear metric spaces with strictly increasing $F$-norms, and that under these assumptions $l_1 \times \Pi_{l_1} X_a \cong \Pi X_a$.

1.1. THEOREM. *Let $E$ be a complete coordinate space over a countable set $A$ and assume that the cone*

$$C_E = \{f \in E\colon f(a) > 0 \text{ for all } a \in A\}$$

*is an* $\mathrm{AR}(\mathfrak{M})$*-space. Then, given a family $\{(X_a, \|\ \|_a)\colon a \in A\}$ of $F$-normed spaces, the pairs $(l_2 \times \Pi_E X_a, l_2 \times \Sigma_E X_a)$ and $\big(l_2 \times \prod_{a\in A} X_a, l_2 \times \sum_{a\in A} X_a\big)$ are homeomorphic. In particular, $l_2 \times \Pi_E X_a \cong \prod_{a\in A} X_a$ if all the $X_a$'s are non-degenerate.*

1.2. Remark. In 1.1, the cone $C_E$ is an $\mathrm{AR}(\mathfrak{M})$ if and only if $E$ is.

In the proofs we shall assume without loss of generality $A = N$. First of all we observe

1.3. LEMMA. *If $E$ is a complete coordinate space over $N$ then the space $C_E$ is non-empty and complete.*

Proof. Denote

$$E^+ = \{f \in E\colon f(i) \geqslant 0 \text{ for all } i\} \quad \text{and} \quad H_i = \{f \in E\colon f(i) = 0\};$$

we have to show that $\{H_i\colon i \in N\}$ is not a cover of $E^+$. Assume the contrary; then for some $n \in N$ the set $H_n$ contains a relatively open subset of $E^+$. The map $f \overset{p}{\mapsto} |f|$ being a continuous retraction (see (i)) of $E$ onto $E^+$ with $p^{-1}(H_n) \subset H_n$, $H_n$ contains an open subset of $E$. Since $H_n$ is a closed linear subspace of $E$, we infer that $H_n \cap E = E$. This contradicts (iii). The completeness of $C_E$ follows from the fact that $C_E$ is a $G_\delta$-subspace of a complete space.

We need also

1.4. LEMMA. *Let $(F, \|\ \|)$ be a coordinate space over $A$, let $\{(H_a, |||\ |||_a)\colon a \in A\}$ be a family of $F$-normed spaces and let $Z$ be a topological space. If $v\colon Z \to F^+ = \{f \in F\colon f(a) \geqslant 0 \text{ for all } a\}$ and $u\colon Z \to \prod_{a\in A} H_a$ are maps such that*

$$|||u_a(z)|||_a \leqslant v_a(z) \quad \text{for all } (a, z) \in A \times Z,$$

*then* $\mathrm{image}(u) \subset \Pi_E H_a$ *and $u$ is continuous as a map into $\Pi_E H_a$.*



Proof. The first assertion is clear. To prove the second, fix $z_0 \in Z$ and $\varepsilon > 0$ and let $B \subset A$ be a finite set such that $C = A \setminus B$ satisfies

$$\|\chi_C |u(z_0)|\,\| < \tfrac{1}{4}\varepsilon \quad \text{and} \quad \|\chi_C v(z_0)\| < \tfrac{1}{4}\varepsilon.$$

Then for all $z \in Z$ we have

$$\begin{aligned} |u(z) - u(z_0)| - \chi_B |u(z) - u(z_0)| &\leqslant \chi_C |u(z_0)| + \chi_C |u(z)| \\ &\leqslant \chi_C |u(z_0)| + \chi_C v(z_0) + \chi_C\big(v(z) - v(z_0)\big), \end{aligned}$$

whence, by (i),

$$|||u(z) - u(z_0)||| \leqslant \tfrac{1}{2}\varepsilon + \|\chi_B |u(z) - u(z_0)|\,\| + \|v(z) - v(z_0)\|.$$

Thus, if $V$ is a neighbourhood of $z_0$ such that the (continuous) function $z \mapsto \|\chi_B |u(z) - u(z_0)|\,\| + \|v(z) - v(z_0)\|$ does not exceed $\frac{1}{2}\varepsilon$ on $V$, then $|||u(z) - u(z_0)||| < \varepsilon$ for all $z \in V$.

Finally, the third step in proving 1.1 is

1.5. PROPOSITION. *Let $(E, \|\ \|_E)$ and $(F, \|\ \|_F)$ be coordinate spaces over $N$, let $(X_i, \|\ \|_i)$, $i \in N$, be $F$-normed spaces and let $f = (f_1, f_2)\colon l_2 \times C_E \xrightarrow{\text{onto}} l_2 \times C_F$ be a homeomorphism. Then the formulas*

(1) $\lambda(t, s, x) \in [0, \infty)^N, \quad \lambda(t, s, x)\cdot s + |\lambda(t, s, x)\cdot x| = f_2(t, s + |x|),$

*and*

(2) $\hat{f}(t, s, x) = \big(f_1(t, s + |x|),\ \lambda(t, s, x)\cdot s,\ \lambda(t, s, x)\cdot x\big)$

*define a homeomorphism $\hat{f}\colon l_2 \times C_E \times \Pi_E X_i \xrightarrow{\text{onto}} l_2 \times C_F \times \Pi_F X_i$ such that $\hat{f}(l_2 \times C_E \times \Sigma_E X_i) = l_2 \times C_F \times \Sigma_F X_i$.*

Proof. Observe that for each $n \in N$ and $(s, x) \in C_E \times \Pi_E X_i$ the map $\gamma \mapsto \gamma s_n + \|\gamma x_n\|_n$ is strictly increasing and unbounded on $[0, \infty)$. Therefore formula (1) uniquely determines a function $\lambda\colon l_2 \times C_E \times \Pi_E X_i \to R^N$; thus $\hat{f}$ is well-defined.

Now set $Z = l_2 \times C_E \times \Pi_E X_i$, $(H_i, |\ |_i) = (R \times X_i$, the "sum" norm) (4), $v(t, s, x) = f_2(t, s + |x|)$ and $w(t, s, x) = \big((s_i, x_i)\big) \in \boldsymbol{P} H_i$. Given $i \in N$ the function $\lambda_i\colon Z \to R$ is defined by $|\lambda_i(z) w_i(z)|_i = v_i(z)$, $z \in Z$, and thus is continuous by Lemma 4 of Terry [22] (5). Thus, by 1.4, $z \mapsto \lambda(z)\cdot w(z)$ is a map of $Z$ into $\Pi_E H_i$. This is equivalent to saying that $\hat{f}$ is a map of $Z$ into $l_2 \times C_F \times \Pi_F X_i$.

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(4) I.e. $|(y, \lambda)|_i = \|y\|_i + |\lambda|$.

(5) For the sake of completeness we insert here a reformulation of this lemma: let $(H, |\ |)$ be an $F$-normed space, let $y_n \in H$ satisfy $\lim_{n\to\infty} y_n = y_1$ and suppose that $|\ |$ is strictly increasing on the ray $[0, \infty)\cdot y_1$. If $(\mu_n) \in [0, \infty)^N$ is a sequence with $\lim_{n\to\infty} |\mu_n y_n| = |\mu_1 y_1|$, then $\lim \mu_n = \mu_1$. A proof follows immediately from the fact that $(\mu_n)$ cannot have any cluster point in $[0, \infty] \setminus \{\mu_1\}$.

Denoting by $g\colon l_2\times C_F\to l_2\times C_E$ the inverse of $f$, we infer (by symmetry) that $\hat{g}$ is also well-defined and continuous. Moreover, given $(t,s,x)\in l_2\times C_E\times \Pi_E X_i$, we have $\hat{f}(t,s,x)=\big(f_1(t,s+|x|),\ \lambda s,\ \lambda x\big)$ and $\hat{g}\hat{f}(t,s,x)=\big(g_1(f_1(t,s+|x|),\ \lambda s+|\lambda x|),\ \mu\lambda s,\ \mu\lambda x\big)$, where $\lambda,\mu\in[0,\infty)^N$ satisfy $\lambda s+|\lambda x|=f_2(t,s+|x|)$ and $\mu\lambda s+|\mu\lambda x|=g_2\big(f_1(t,s+|x|),\ f_2(t,s+|x|)\big)=s+|x|$. The latter defines $\mu\lambda\in[0,\infty)^N$ uniquely (for the maps $\gamma\mapsto\gamma s_i+\|\gamma x_i\|_i$ are strictly increasing on $[0,\infty)$), whence $\mu\lambda=\mathbf{1}$, $\hat{g}\hat{f}=\mathrm{id}$ and, by symmetry, $\hat{f}\hat{g}=\mathrm{id}$. Since $\hat{f}(l_2\times C_E\times\Sigma_E X_i)\subset l_2\times C_F\times\Sigma_F X_i$ and $\hat{g}(l_2\times C_F\times\Sigma_F X_i)\subset l_2\times C_E\times\Sigma_E X_i$, the result follows.

Now we finish the proof of 1.1. Set $F=R^\infty$ and $\|(\lambda_i)\|_F=\sum\min(2^{-i+1},|\lambda_i|)$. Theorem 3.2 of [26] and 1.3 imply that $l_2\times C_E\cong l_2\times C_F\cong l_2$. Combining this with 1.5, we obtain the first assertion of 1.1, and applying a part of 0.1, we get the second one.

Before passing to the proof of 1.2, we note that neither 1.2 nor the results 1.7–1.8 below are used in the subsequent sections and the following observation is quite sufficient for our purposes:

1.6. Remark. If $E$ is a locally convex coordinate space then $C_E\in\mathrm{AR}(\mathfrak{M})$ by Dugundji theorem 0.2.

Proof of 1.2. We use the notation of 1.3. If $E$ is a complete $\mathrm{AR}(\mathfrak{M})$ then so is $E^+$ as a retract of $E$. Hence in order to prove that $C_E=E^+\setminus\bigcup_{n\in N}H_n\in\mathrm{AR}(\mathfrak{M})$ it sufficies to show that every map $f\colon[0,1]^\infty\to E^+$ is a uniform limit of $C_E$-valued maps (see [27]). To this end fix $f\colon[0,1]^\infty\to E^+$ and let $t$ be a point of $C_E$; it is clear that the maps

$$f_n(x)=f(x)+n^{-1}t,\qquad x\in[0,1]^\infty,$$

have the property that $\sup_x\|f_n(x)-f(x)\|\leqslant\|n^{-1}t\|$ and $\mathrm{image}(f_n)\subset C_E$ for all $n\in N$.

Conversely, if $C_E\in\mathrm{AR}(\mathfrak{M})$ then $l_2\times\Pi_E R\cong R^\infty$ by 1.1, whence $E$ is a retract of the space $R^\infty\in\mathrm{AR}(\mathfrak{M})$.

A partial refinement of 1.1 is the following:

1.7. Theorem. *Let $E$ be a coordinate space over a countable set $A$ and assume that $E$ is homeomorphic to a Fréchet space. Then, given $F$-normed spaces $(X_a,\|\ \|_a)$, $a\in A$, we have $\Pi_E X_a\cong\prod_{a\in A}X_a$.*

Proof. Without loss of generality assume that $A=N$ and all the $X_i$'s are non-degenerate. For each $i\in N$ choose an $a_i\in X_i\setminus\{0\}$, denote by $(Y_i,|||\ |||_i)$ the quotient space $X_i/Ra_i$ and by $p_i\colon X_i\to Y_i$ the projection. We need the following.

Lemma. *There are maps $q_i\colon Y_i\to X_i$ $i\in N$, such that $p_iq_i=\mathrm{id}$ and $\|q_i(y)\|_i\leqslant 4|||y|||_i$ for $y\in Y_i$.*



Proof. Fix $i\in N$. By Michael's theorem ([16], p. 753) there is a map $r\colon Y_i\to X_i$ with $p_ir=\mathrm{id}$. Put for $y\in Y_i\setminus\{0\}$

$$\psi(y)=\sup\{\lambda\colon \|r(y)+\lambda a_i\|_i<2|||y|||_i\},$$
$$\varphi(y)=\inf\{\lambda\colon \|r(y)+\lambda a_i\|_i<2|||y|||_i\}$$

where $\sup R=\infty$ and $\inf R=-\infty$. The set

$$\{(y,\lambda)\in(Y_i\setminus\{0\})\times R\colon \varphi(y)<\lambda<\psi(y)\}$$

is open in $Y_i\times R$. Therefore there exists a map $\gamma\colon Y_i\setminus\{0\}\to R$ such that $\varphi<\gamma<\psi$ (see [10], p. 170). We let $q_i(0)=0$ and $q_i(y)=r(y)+\gamma(y)a_i$ for $y\in Y_i\setminus\{0\}$. Given $y\in Y_i\setminus\{0\}$ there are $\lambda_1,\lambda_2\in R$ such that $\lambda_1\leqslant\gamma(y)\leqslant\lambda_2$ and $\|r(y)+\lambda_ja_i\|_i<2|||y|||_i$ for $j=1,2$. Writing $\gamma(y)=t\lambda_1+(1-t)\lambda_2$ with $t\in[0,1]$, we get

$$\|q_i(y)\|_i\leqslant|||t(r(y)+\lambda_1a_i)|||_i+|||(1-t)(r(y)+\lambda_2a_i)|||<4|||y|||_i,$$

as required.

We continue the proof of 1.7. Let $q_i$ be the maps of the Lemma. It follows from 1.4 that the map

$$(x_i)\mapsto\Big(\big(p_i(x_i)\big),\ \big(x_i-q_ip_i(x_i)\big)\Big)$$

is a homeomorphism of $\Pi_E X_i$ onto $G\times H$, where $G=\Pi_E Y_i$ and $H=\{(x_i)\in\Pi_E X_i\colon x_i\in Ra_i \text{ for all } i\}$. Thus it remains to show that $H$ has $R^\infty$ as a factor (for by Anderson's theorem we would then get $\Pi_E X_i\cong G\times H\cong G\times H_1\times R^\infty\cong G\times H_1\times R^\infty\times l_2\cong\Pi_E X_i\times l_2\cong\Pi X_i$, see 1.1, 1.2 and 0.2).

Clearly, $H\cong\Pi_E(R,v_i)$ where $v_i(\lambda)=\|\lambda a_i\|_i$ for $i\in N$, $\lambda\in R$. Writing $E_w=\Pi_E(R,w_i)$ for any system $w=(w_i)$ of $F$-norms on $R$ we infer from 1.4 that, given two such systems $w$ and $w'$, the following facts hold:

(1) If, for each $i\in N$, $w_i$ and $w_i'$ are strictly increasing on $[0,\infty)$ and satisfy

$$\sup\{w_i(\lambda)\colon\ \lambda\in R\}=\sup\{w_i'(\lambda)\colon\lambda\in R\},$$

then

$$(\lambda_i)\mapsto\big(\mathrm{sign}(\lambda_i)w_i^{-1}w_i'(|\lambda_i|)\big)$$

is a homeomorphism of $E_w$ onto $E_{w'}$.

(2) If, for each $i\in N$, $w_i$ and $w_i'$ satisfy $\frac{1}{2}w_i\leqslant w_i'\leqslant 2w_i$, then the identity map is a homeomorphism of $E_w$ onto $E_{w'}$.

Also note that if $|\ |$ is an $F$-norm on $R$ then $c=\inf\{|\lambda|/|\lambda|\colon\lambda\in[0,1]\}>0$ and the $F$-norm $|\lambda|'=|\lambda|+c|\lambda|(1+|\lambda|)^{-1}$ is strictly increasing on $[0,\infty)$ and satisfies $|\ |\leqslant|\ |'\leqslant 2|\ |$. Therefore, applying successively (2),

(1) and (2), we infer that for some system $(c_i) \in (0, \infty]^N$ we have $E_v \cong E_u$, where

$$u_i(\lambda) = \min(|\lambda|, c_i) \quad \text{for} \quad \lambda \in R, \quad i \in N.$$

Let $\bar{c}_i = \sup\{\|\lambda e_i\|: |\lambda| < c_i\}$, where $e_i \in E$ is the $i$th versor of $R^\infty$, and let $c = \inf\{\bar{c}_i: i \in N\}$. We consider two cases:

(a) $c = 0$. Let $A$ be an infinite set with $\sum_{i \in A} \bar{c}_i < \infty$, set $B = N \setminus A$ and let $F_A = \{\chi_A f: f \in E_u\}$ and $F_B = \{\chi_B f: f \in E_u\}$ be the subspaces of $E_u$. By (i) and (ii), $E_u \cong F_A \times F_B$ and $F_A$ coincides (as a topological space) with $R^\infty$. Hence $H \cong E_u \cong R^\infty \times F_B$ has $R^\infty$ as a factor.

(b) $c > 0$. Then, the identity map $i: E \to E_u$ is an isometry on every ball of radius $\frac{1}{2}c$ and hence it is a homeomorphism of $E$ *onto* $E_u$ (use the linearity of $i$). Thus $H \cong E_u \cong E \cong R^\infty$ by the Kadec–Anderson theorem. This completes the proof.

1.9. Corollary. *If in addition to the data of* 1.7, *all the $X_a$'s are homeomorphic to a space $X \in \mathrm{AR}(\mathfrak{M})$, then $\Sigma_E X_a \cong \Sigma X$.*

Proof. By 1.1, 1.2 and the Anderson theorem we have $R^\infty \times \Sigma_E X_a \cong R^\infty \times \Sigma X$, whence $\Sigma R^\infty \times \Sigma_E X_a \cong \Sigma R^\infty \times \Sigma X \overset{\mathrm{df}}{=} Y$. Now, by [24, § 7] we have $\Sigma R^\infty = R^\infty \times \Sigma R$ and therefore $Y_1 \cong Y_2 \cong Y$, where $Y_1 = R^\infty \times \Sigma R \times \Sigma_E X_a$ and $Y_2 = R^\infty \times \Sigma R \times \Sigma X$. Since $Y$ is a linear metric space with $\Sigma Y \cong Y \in \mathrm{AR}(\mathfrak{M})$, the sets $M_1 = \Sigma R \times \Sigma R \times \Sigma_E X_a$ and $M_2 = \Sigma R \times \Sigma R \times \Sigma X$ are $Z$-absorbing in $Y_1$ and $Y_2$, respectively (see [24], § 7), whence $M_1 \cong M_2$ by [24, § 3]. Thus $\Sigma R \times \Sigma_E X_a \cong \Sigma X$, and it remains to show that $\Sigma_E X_a$ has $\Sigma R$ as a factor. This is done in much the same way as in the proof of 1.7, by using in case (b) an argument involving absorbing sets (the details are left to the reader).

**§ 2. Some embedding theorems.** The following is related to a well-known lemma on existence of $\mathfrak{U}$-fine pseudometrics on paracompact spaces (see [15], p. 165 and [7], p. 527):

2.1. Lemma. *Let $X$ be a metric space, let $\mathfrak{U}$ be an open cover of $X$ and let $E$ be a linear metric space with* $\mathrm{dens}(E) \geqslant \mathrm{card}(\mathfrak{U})$. *Then, there are an $F$-normed space $(H, |\,|)$ and a map $h: X \to H$ such that $H \cong E^\infty$ and*

(*) *if $K \subset H$ satisfies* $\mathrm{diam}_{|\,|} K < 1$ *then $h^{-1}(K) \subset U$ for some $U \in \mathfrak{U}$.*

Proof. By the paracompactness of metric spaces we may assume that the cover $\mathfrak{U}$ is locally finite and $\sigma$-discrete (see [7], p. 529). Let $(\lambda_U)_{U \in \mathfrak{U}}$ be a system of $[0,1]$-valued maps such that $\bigcup_{U \in \mathfrak{U}} \lambda_U^{-1}(1) = X$ and $\lambda_U | X \setminus U = 0$ for $U \in \mathfrak{U}$. Express $\mathfrak{U} = \bigcup \mathfrak{U}_n$ where the $\mathfrak{U}_n$'s are discrete families; without loss of generality we may assume $\mathfrak{U}_n \cap \mathfrak{U}_m = \emptyset$ for $n \neq m$.



2.2. Sublemma. *There are an $F$-normed space $(H_1, \|\,\|_1)$ and a set $A \subset H_1$ such that $H_1 \cong E^\infty$,* $\mathrm{card}(A) = \mathrm{dens}(E)$ *and $\|a - \lambda b\|_1 \geqslant 1$ for all $\lambda \in [0,1]$ and $a, b \in A$, $a \neq b$.*

Suppose 2.2 has been shown. Set $H_2 = \Pi_{l_1} H_1$ and $A_n = \{(x_i) \in H_2: x_n \in A \text{ and } x_i = 0 \text{ for } i \neq n\}$, $n \in N$, and let $U \mapsto a_U$ be an injection of $\mathfrak{U}$ into $A_n$ such that $a_U \in A_n$ iff $U \in \mathfrak{U}_n$. Define

$$g(x) = \sum_{U \in \mathfrak{U}} \lambda_U(x) a_U \quad \text{for} \quad x \in X.$$

If $K \subset H_2$ satisfies $\mathrm{diam}_{|||\,|||} K < 1$, then choose $x \in g^{-1}(K)$ and $(n, U) \in N \times \mathfrak{U}$ with $x \in \lambda_U^{-1}(1)$ and $U \in \mathfrak{U}_n$. For all $y \in X \setminus U$ we have $|||g(y) - g(x)||| \geqslant \|a_U - z(y)\|_1 \geqslant 1$, where

$$z(y) = \begin{cases} 0 & \text{if} \quad y \notin \bigcup_{V \in \mathfrak{U}_n} V, \\ \lambda_V(y) a_V & \text{if} \quad y \in V \in \mathfrak{U}_n. \end{cases}$$

Thus $g^{-1}(K) \subset U$. If we now let $H = l_2 \times H_2$ and $h = 0 \times g$, then $(H, h)$ will satisfy all the required conditions (see 1.1 and 1.6).

Proof of 2.2. Fix an $F$-norm $\|\,\|$ on $E$ and, given $\alpha = (k, l) \in N \times N$, let $B_\alpha$ be a maximal subset of $E$ such that

$$\|a\| \geqslant k^{-1} \quad \text{and} \quad \inf\{\|a - \lambda b\|: \lambda \in [0,1]\} \geqslant l^{-1} \quad \text{for} \quad a, b \in B_\alpha, \ a \neq b.$$

The set $B = \bigcup_{\alpha \in N \times N} B_\alpha$ is linearly dense in $E$ and hence its cardinality is equal to $\mathrm{dens}(E)$ (Proof. Put $E_0 = \mathrm{cl\,span} B$ and suppose there is an $a \in E \setminus E_0$. Choose $k \in N$ with $k^{-1} < \inf\{\|a - x\|: x \in E_0\}$ and let $l > k$ satisfy $l^{-1} < \inf\{\|x - \lambda a\|: x \in E_0, \|x\| \geqslant k^{-1} \text{ and } \lambda \in [0,1]\}$. The set $B_{k,l} \cup \{a\}$ still lies outside the ball $\{x \in E: \|x\| \geqslant k^{-1}\}$ and we have $\inf\{\|b - \lambda a\|: \lambda \in [0,1]\} \geqslant l^{-1}$ and $\inf\{\|\lambda b - a\|: \lambda \in [0,1]\} \geqslant k^{-1} \geqslant l^{-1}$ for all $b \in B_{k,l}$. This contradicts the maximality of $B_{k,l}$).

Set $L = l_1(N \times N)$ and $H_0 = \Pi_L(E_\alpha, \|\,\|_\alpha)$, where for $\alpha = (k, l) \in N \times N$ we let $E_\alpha = E$ and $\|\,\|_\alpha = kl\|\,\|$. Finally, define $A_\alpha = \{x \in H_0: x_\alpha \in B_\alpha \text{ and } x_\beta = 0 \text{ for } \beta \neq \alpha\}$ and $A_0 = \bigcup A_\alpha$. It is clear that $A = \{0\} \times A_0$ and $H_1 = l_2 \times H_0$ satisfy the required conditions.

Let us recall that if $(F_i, \|\,\|_i)$, $i \in N$, are $F$-normed spaces and we refer to $\Pi F_i$ as a metric space, then we shall always consider it under the $F$-norm defined by (0). So, when speaking e.g. about $(\Pi F_i)^\infty$, we shall have a specific $F$-norm on this space in mind.

The proof of the next result is a "geometric interpretation" of the proof of Theorem 2.3 of [26].

2.3. Proposition. *Let $(F, \|\,\|)$ be an $F$-normed space and let $f: X \to F$ be a closed embedding into $F$ of an $\mathrm{AR}(\mathfrak{M})$-space $X$. Then there are an*

*$F$-normed space $(G, |||\;|||)$ with $G \cong F^\infty$ and a map $g: X \to G$ such that*

$$x \mapsto (f(x), g(x))$$

*is an embedding of $X$ onto a regular retract of $F \times G$.*

Proof. Without loss of generality we may assume (see [17]) that $X$ is a closed subset of a normed linear space $L$; let $r: L \xrightarrow{\text{onto}} X$ be a continuous retraction.

SUBLEMMA. *Let $\varrho_{i-1}$ be a continuous pseudometric on $X$. Then there are an $F$-normed space $(G_i, |||\;|||_i)$ and a map $g_i: X \to G_i$ such that $G_i \cong F^\infty$ and, denoting*

$$(**) \qquad \varrho_i(x_1, x_2) = |||g_i(x_1) - g_i(x_2)|||_i, \quad x_1, x_2 \in X,$$

*we have*

(**) *for any $n \in N$ and $A \subset X$, $\operatorname{diam}_{\varrho_i} A \leqslant 2^{-n}$ implies $\operatorname{diam}_{\varrho_{i-1}} r(\operatorname{conv} A) \leqslant n^{-1}2^{-n}$.*

Proof. Given $n \in N$, let $\mathcal{U}_n$ be a cover of $X$ consisting of (relatively) open sets which are so small that $\operatorname{diam}_{\varrho_{i-1}} r(\operatorname{conv} A) \leqslant n^{-1}2^{-n}$ for all $U \in \mathcal{U}_n$, and let $(H_n, |\;|_n)$ and $h_n: X \to H_n$ be as asserted in the statement of Lemma 2.1, taking $\mathcal{U} = \mathcal{U}_n$ and $E = F$. We set $G_i = \Pi H_n$ and $g_i = (h_n)$; it is clear that if $n \in N$ and $A \subset X$ satisfy $\operatorname{diam}_{\varrho_i} A \leqslant 2^{-n}$, then $A \subset U$ for some $U \in \mathcal{U}_n$, whence $\operatorname{diam}_{\varrho_{i-1}} r(\operatorname{conv} A) \leqslant n^{-1}2^{-n}$.

Proof of 2.3. Put $\varrho_0(x_1, x_2) = \|f(x_1) - f(x_2)\|$; $x_1, x_2 \in X$. Using the sublemma, construct inductively a sequence $((G_i, |||\;|||_i, g_i))_{i \in N}$ such that, for each $i \in N$, $(G_i, |||\;|||_i)$ is an $F$-normed space homeomorphic to $F^\infty$ and conditions (**) are satisfied. Let $G = \Pi G_i$, $g = (g_i)$ and

$$\varrho(x_1, x_2) = \varrho_0(x_1, x_2) + \sum_{i \in N} \min\big(\varrho_i(x_1, x_2), 2^{-i+1}\big).$$

The map $x \mapsto (f(x), g(x))$ is a closed isometric embedding of $(X, \varrho)$ into $F \times G$ and one easily verifies that, for every $A \subset X$, $\operatorname{diam}_\varrho A < \frac{1}{2}$ implies $\operatorname{diam}_\varrho r(\operatorname{conv} A) \leqslant 10 \operatorname{diam}_\varrho A$ (see the proof of 2.1 in [26]). Thus the image of $X$ is a regular retract of $F \times G$ by [26, Proposition 2.2].

Finally, we need

2.4. PROPOSITION. *Let $E$ be a linear metric space and let $X$ be a metric space with $\operatorname{dens}(X) \leqslant \operatorname{dens}(E)$. Then $X$ is homeomorphic to a subset of $E^\infty$, and if, moreover, $X$ is complete-metrizable, then there are an $F$-normed space $(F, \|\;\|)$ and an embedding $f: X \to F$ such that $F \cong E^\infty$ and the set $f(X)$ is complete in the $F$-norm $\|\;\|$.*



Proof. Fix a metric $\varrho$ for $X$ and let $\mathcal{U}_n$ denote the cover of $X$ by open $\varrho$-balls of radius $1/n$. For each $n \in N$ let an $F$-normed space $(H_n, |\;|_n)$, with $H_n \cong E^\infty$, and a map $h_n: X \to H_n$, satisfy condition $(*)$ with $\mathcal{U} = \mathcal{U}_n$; it is clear that $h = (h_n)$ is an embedding of $X$ into $H = \Pi H_n \cong E^\infty$. Let $(\hat{H}, \|\;\|)$ be the completion of $H$. If $X$ is complete-metrizable, then, by a theorem of W. Sierpiński [21], $h(X)$ is a $G_\delta$-set in $\hat{H}$ and therefore there is a map $g: X \to R^\infty$ such that $x \overset{f}{\mapsto} (g(x), h(x))$ is an embedding of $X$ onto a closed subset of $R^\infty \times \hat{H}$. Then the set $f(X)$ is contained in $F = R^\infty \times H$ and is complete in the $F$-norm of $F$. Since $F \cong R^\infty \times E^\infty \cong E^\infty$ (see 0.1), the result follows.

QUESTION. Let $E$ be a Banach space. Is $E^\infty$ homeomorphic to a subset of $E$?

## § 3. Factors of linear metric spaces.

The main result here is:

3.1. THEOREM. *Let $E$ be a linear metric space and let $X$ be an $\mathrm{AR}(\mathfrak{M})$-space with $\operatorname{dens}(X) \leqslant \operatorname{dens}(E)$. In either of the following cases we have $X \times F \cong F$:*

(a) *$X$ is complete-metrizable and $F = E^\infty$,*

(b) *$X$ admits a closed embedding into $E$, and either $F = \Sigma E^\infty$ or $E \in \mathrm{AR}(\mathfrak{M})$ and $F = \Sigma E$.*

We note that theorem (A) mentioned in the introduction follows from 3.1 and the Dugundji theorem 0.2.

Proof of 3.1. First consider case (a). By 2.4 and 2.3 there are an $F$-normed space $(H, \|\;\|)$ and an embedding $h: X \to H$ such that $H \cong E^\infty$ and the set $h(X)$ is complete in the $F$-norm $\|\;\|$ and is a regular retract of $(H, \|\;\|)$. By Proposition 1a and Remark 1 of [25] there is a concave homeomorphism $v: [0, \infty) \xrightarrow{\text{onto}} [0, \infty)$ such that, writing,

$$|||t||| = v(\|t\|), \qquad t \in H,$$
$$L = \{(\lambda_i) \in R^N: \|(\lambda_i)\|_L = \sum 2^i |\lambda_i| < \infty\},$$

we have $X \times \Pi_L(H, |||\;|||) \cong \Pi_L(H, |||\;|||)$. Since $L$ is a Banach space and $|||\;|||$ is an admissible $F$-norm for $H$, we infer from 1.1 and 1.6 that $l_2 \times \Pi_L(H, |||\;|||) \cong H^\infty \cong E^\infty$. Thus $X \times E^\infty \cong E^\infty$.

Now consider the other case. Arguing as above, we get $X \times l_2 \times \Sigma E^\infty \cong l_2 \times \Sigma E^\infty$, whence $X \times \Sigma E^\infty \cong \Sigma E^\infty$ by 1.6. If now $E \in \mathrm{AR}(\mathfrak{M})$, then by [24, § 7] we have $\Sigma E^\infty \cong E^\infty \times \Sigma R$ and $X \times E^\infty \times \Sigma R \cong E^\infty \times \Sigma R$. Using results of [24, § 7] again, we infer that both $X \times \Sigma E \times \Sigma R$ and $\Sigma E \times \Sigma R$ can be interpreted as $\mathcal{K}_E$-absorbing sets in $X \times E^\infty \times \Sigma R$ and $E^\infty \times \Sigma R$, respectively, where $\mathcal{K}_E$ is the family of all the $Z$-sets in the space which admit a closed embedding into a finite power of $E$. Thus $X \times E^\infty \times \Sigma R \cong E^\infty \times \Sigma R$ implies $X \times \Sigma E \times \Sigma R \cong \Sigma E \times \Sigma R$ by [21], § 3. Combining this with 0.1, we get $X \times \Sigma E \cong \Sigma E$.

In the sequel we denote by $CX$ the closed metric cone over $X$, as defined in [26]. The following is a consequence of 3.1.

3.2. Theorem. *Let $E$ be a linear metric space, let $X$ be an* $\mathrm{ANR}(\mathfrak{M})$ *with* $\operatorname{dens}(X) \leqslant \operatorname{dens}(E)$ *and suppose that either condition* (a) *or condition* (b) *of* 3.1 *is satisfied. Then $X \times F$ is homeomorphic to an open set $U \subset F$ such that* $(F, F\setminus U) \cong (CX \times F, \{0\} \times F)$.

The proof is the same as that of [26, Theorem 4.2], with the aid of the fact that, by 3.1, $F \times R \cong F \cong F \times (-\infty, 1]$.

Now we apply 3.1 and 3.2 to get some corollaries on infinite-dimensional manifolds. We first note

3.3. Proposition. *Let $E$ be a linear metric space such that $E \cong E^\infty$ or $E \cong \Sigma E \in \mathrm{AR}(\mathfrak{M})$. If $K$ is a locally finite-dimensional simplicial complex such that the star of each vertex of $K$ contains at most* $\operatorname{dens}(E)$ *simplexes, then $|K| \times E$ is an $E$-manifold ($|K|$ is taken in the metric topology).*

Proof. Without loss of generality assume that the complex $K$ is finite-dimensional and $K$ contains at most $\operatorname{dens}(E)$ simplexes. Then $|K|$ is a complete $\mathrm{ANR}(\mathfrak{M})$ with $\operatorname{dens}(|K|) \leqslant \operatorname{dens}(E)$ and therefore $|K| \times E^\infty$ is an $E^\infty$-manifold; thus we can restrict ourselves to the case $E \cong \Sigma E \in \mathrm{AR}(\mathfrak{M})$. The assertion will be proved by induction with respect to $n = \dim K$ (it clearly holds true for $n = 0$). Suppose we have shown that for every $(n-1)$-dimensional simplicial complex $L$ containing at most $\operatorname{dens}(E)$ simplexes, the space $|L| \times E$ is an $E$-manifold, and let $v$ be a vertex of $K$. Then $|\mathrm{st}(v)|$ is the closed cone over an $(n-1)$-dimensional simplicial complex $|L|$ and, by our inductive assumption, $|L|$ admits a closed embedding into an $E$-manifold. Using a theorem of Henderson ([11], Theorem 2), we infer that $|L|$ admits a closed embedding into $E$ and $|\mathrm{st}(v)|$ admits a closed embedding into $CE$. Since $CE$ is a retract of $E$ (see [26], Lemma 4.1), 3.1 implies that $|\mathrm{st}(v)| \times E \cong E$. Hence, by the arbitrariness of $v \in K^0$, $|K| \times E$ is an $E$-manifold.

3.4. Theorem (Representation of manifolds). *Let $M$ be a connected paracompact $E$-manifold where $E \in \mathrm{AR}(\mathfrak{M})$ is a linear metric space such that $E \cong E^\infty$ or $E \cong \Sigma E$. Then:*

(a) *$M$ is homeomorphic to a product $|K| \times E$, where $K$ is a locally finite-dimensional simplicial complex;*

(b) *$M$ is homeomorphic to an open set $U \subset E$ such that* $(E, E\setminus U) \cong (CY \times E, \{0\} \times E)$ *for some space $Y$.*

Proof. As shown in the proof of [29, Theorem 5], there is a locally finite-dimensional metric simplicial complex $K$ which contains at most $\operatorname{dens}(E)$ simplexes and has the homotopy type of $M$. Then, $|K| \times E$ and $M$ are homotopy-equivalent paracompact $E$-manifolds and the theorem of Henderson [11, Theorem 6] (7) shows them to be homeomorphic. Now, (b) follows from (a) and 3.2.



Let us make some comments on Theorem 3.4. Assertion (a) was proved by D. W. Henderson [10] for $E = l_2$ and by J. E. West [29] for $E =$ the non-separable Hilbert space (8); by using the absorbing sets it was next extended to certain pre-Hilbert spaces in [8], [12] and [24]. A part of (b), Henderson's Open Embedding Theorem, was proved in its final form [11] in a slightly more general setting; we include here another version of this theorem since it gives additional information on the embedding (for instance, the results of [10] can easily be derived from 3.4). Our proof of 3.4 depends on Henderson's theory; however, if $E$ is complete-metrizable or $E \cong \Sigma E$, then assertion (b) follows directly from 3.2 and the stability theorem of [20].

For the terminology used below see e.g. [26], § 6.

3.5. Theorem. *Let $E$ be a locally convex linear metric space such that $E \cong E^\infty$ or $E \cong \Sigma E \in \mathrm{AR}(\mathfrak{M})$, let $X$ be an $\mathrm{ANR}(\mathfrak{M})$-space and assume that $X$ is complete-metrizable if $E \cong E^\infty$. Then, given a $Z$-embedding $h: X \to E$, $h$ admits a trivial tubular neighbourhood $(\bar{h}, E, U)$ such that* $(E, E\setminus U, h(X)) \cong (CX \times E, \{0\} \times E, X \times \{1\} \times \{0\})$. *If, in addition, $X$ is a contractible space, then $E$ is itself a trivial tubular neighbourhood of $h$.*

3.6. Corollary. *Let $E$ and $X$ be as in* 3.5. *If $M$ is an $E$-manifold, then every $Z$-embedding $h: X \to M$ admits a trivial tubular neighbourhood.*

The proofs are the same as those of Theorem 6.3 and Corollary 6.4 of [26].

**§ 4. Linear metric spaces homeomorphic to a Hilbert space.** One of the problems in the infinite-dimensional topology is to identify the spaces which are homeomorphic to a Hilbert space. It is known that the class $\mathcal{H}$ of linear metric spaces homeomorphic to a Hilbert space includes all separable Fréchet spaces (Kadec–Anderson, see [0] and [5]), all reflexive Banach spaces (Bessaga [2]), the spaces of the form $c_0(A)$ (Troyanski [28]) and $l_\infty(A)$, certain spaces of continuous functions, and also the non-locally convex space $L_0(0,1)$ of measurable functions (Bessaga–Pełczyński [3]–[5]); a well known conjecture is that $\mathcal{H}$ includes all Fréchet spaces. Following a suggestion of C. Bessaga, we shall use Theorem 3.1 to get the following description of $\mathcal{H}$:

(7) Although stated in [11] for manifolds with a locally convex model only, that theorem remains true if $E \in \mathrm{AR}(\mathfrak{M})$.

(8) Compare West's seemingly more general setting with the result of § 4.

4.1. THEOREM. *An infinite-dimensional linear metric space $X$ is homeomorphic to a Hilbert space if and only if $X$ is a complete* $\mathrm{AR}(\mathfrak{M})$*-space with* $X \cong X^\infty$.

Proof. If $X$ is an infinite-dimensional Hilbert space, then (a) $X$ is a complete $\mathrm{AR}(\mathfrak{M})$ (Dugundji theorem 0.2), and (b) $X \cong X^\infty$ (Bessaga and Pełczyński [3], p. 266; use 1.1 with $E = l_2$ to get a quick proof). Conversely, let $X$ satisfy conditions (a) and (b) and let $H$ be an infinite-dimensional Hilbert space with $\mathrm{dens}(H) = \mathrm{dens}(X)$. Then, both $X^\infty$ and $H^\infty$ are complete-metrizable $\mathrm{AR}(\mathfrak{M})$'s, whence, by 3.1, $X^\infty \times H^\infty = H^\infty$ and $X^\infty \times H^\infty \cong X^\infty$. Thus $H^\infty \cong X^\infty$; since $H^\infty \cong H$ and (by assumption) $X^\infty \cong X$, the proof is complete.

4.2. COROLLARY. *If $F$ is a non-degenerate Fréchet space with $F \cong F^\infty$, then $F \cong l_2(A)$, where $A$ is a set of cardinality* $\mathrm{dens}(F)$.

4.3. COROLLARY. *If $E$ is any complete locally bounded linear metric space, then $E^\infty$ is homeomorphic to a Hilbert space.*

Proof. (We recall that $E$ is said to be *locally bounded* if 0 has any bounded neighbourhood). By a theorem of Aoki and Rolewicz (see [19], p. 61), there is an admissible $F$-norm $\|\ \|$ on $E$ which is $p$-homogeneous for a certain $p > 0$ (i.e. $\|\lambda x\| = |\lambda|^p \|x\|$ for $\lambda \in R$ and $x \in E$). Then the maps

$$\big((\lambda_1, \ldots, \lambda_n), (x_1, \ldots, x_n)\big) \mapsto \sum_{i=1}^{n} |\lambda_i|^{1/p} x_i$$

satisfy the conditions required in Theorem 3 of [13] and therefore, by that theorem and the contractibility of $E$, $E \in \mathrm{AR}(\mathfrak{M})$. Now the assertion follows from 4.1.

The condition $X \cong X^\infty$ used in 4.1–4.3 is in general fairly difficult to verify (even if $X$ is a Hilbert space). With the aid of 1.7, however, 4.1 can provide a tool in proving that certain spaces are homeomorphic to a Hilbert space.

4.4. EXAMPLE. Let $(X, \|\ \|)$ be an $F$-normed Fréchet space, let $A$ be an infinite set and consider the space

$$Y = c_0(A; X) = \{(x_a) \in X^A \colon \{a \in A \colon \|x_a\| > \varepsilon\} \text{ is finite for each } \varepsilon > 0\},$$

equipped with the $F$-norm $|||(x_a)||| = \sup\{\|x_a\| \colon a \in A\}$. Then, $\Pi_{c_0} Y$ is isomorphic to $Y$, whence by means of 1.7 (or 1.1 and the Bartle-Graves theorem) we have $Y \cong Y^\infty$. Moreover, $Y$ is easily seen to be a Fréchet space and therefore we infer from 4.2 that $c_0(A; X)$ is topologically a Hilbert space. This includes the result of [28] as a special case.

For another application of the factor theorems to the topological classification of linear metric spaces see [18].



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