# A generalization of cohomotopy groups

by

**Jerzy Dydak** (Warszawa)

**Abstract.** The notion of the generalized cohomotopy group has been introduced by K. Borsuk in [4]. This paper gives another generalization of cohomotopy groups such that to every topological space one assigns a generalized cohomotopy group. It is shown that this generalization coincides in some special cases with Borsuk's concepts.

**Introduction.** In [4] K. Borsuk introduced the notion of the generalized cohomotopy group $\pi_k^n(X)$ for some class of spaces and $k < 2n-1$. We show in § 7 that there exist two ANR-sets $X$ and $S$ having the same homotopy type and such that the groups $\pi_2^2(X)$ and $\pi_2^2(S)$ are not isomorphic. Thus from the point of view of the homotopy theory the groups $\pi_k^n(X)$ are not satisfactory. The main purpose of this paper is to give a new generalization of cohomotopy groups. First we show that for $k < 2n-1$ there exists a contravariant functor $\pi_k^n$: $\underline{CW} \to \underline{G}$, where $\underline{WC}$ is a full subcategory of the homotopy category $\underline{H}$ and the objects of $\underline{CW}$ are all $CW$ complexes, and $\underline{G}$ is a category of Abelian groups, such that $\pi_k^n(W)$ is the Borsuk generalized cohomotopy group of each $CW$ complex $W$. If one considers $\underline{CW}$ as a full subcategory of the category of inverse systems in $\underline{CW}$, which we denote by $\operatorname{Inv}\underline{CW}$, then $\pi_k^n$: $\underline{CW} \to \underline{G}$ is extendable in a natural way over $\operatorname{Inv}\underline{CW}$. Next we consider the functors $F$: $\underline{S} \to \operatorname{Inv}\underline{CW}$, where $\underline{S}$ is the shape category (see [11]) such that the inverse system $F(X)$ is associated with a topological space $X$ (see [12]). Taking the composition $\pi_k^n F$, we obtain the contravariant functor from $\underline{S}$ to $\underline{G}$. The group $\pi_k^n F(X)$ will be refered to as a generalized cohomotopy group. The main properties of groups defined in this way are the following: if $\operatorname{Sh} X = \operatorname{Sh} Y$, then $\pi_k^n F(X)$ and $\pi_k^n F(Y)$ are isomorphic, and if $\operatorname{Sd} X \leqslant k < 2n-1$, then $\pi_k^n F(X)$ and $\pi^n(X)$ are isomorphic.

**1. Shape category and the category of inverse systems.** For any category $C$, let us denote by $\operatorname{Ob}\underline{C}$ the class of all objects of $\underline{C}$; by $f \in \underline{C}(X, Y)$ we mean that $f$ is a morphism from $X$ to $Y$ in $\underline{C}$.

Let $\underline{W}$ be the full subcategory of $\underline{H}$ whose objects are all topological spaces having the homotopy type of a $CW$ complex.

S. Mardešić introduced in [11] the *shape category* $\underline{S}$ as follows. The objects of $S$ are topological spaces. The morphisms of $\underline{S}$ are called *shapings*. Let $X$ and $Y$ be topological spaces. A shaping $\underline{f}\colon X\to Y$ is a class of functions $\underline{f}^Q\colon \underline{H}(Y,Q)\to\underline{H}(X,Q)$, $Q\in\operatorname{Ob}\underline{W}$, such that for $Q'\in\operatorname{Ob}\underline{W}$, $\eta\in\underline{H}(Y,Q)$, $\eta'\in\underline{H}(Y,Q')$ and $\mu\in\underline{W}(Q,Q')$ the equality $\mu\eta=\eta'$ implies $\mu\underline{f}^Q(\eta)=\underline{f}^{Q'}(\eta')$. If $X$, $Y$, $Z$ are topological spaces and $\underline{f}\colon X\to Y$, $\underline{g}\colon Y\to Z$ are shapings, then the composite $\underline{g}\cdot\underline{f}\colon X\to Z$ is defined by $(\underline{gf})^Q=\underline{f}^Q\underline{g}^Q$ for $Q\in\operatorname{Ob}\underline{W}$. The identical shaping $\underline{1}_X\colon X\to X$ is defined by $\underline{1}_X^Q(\eta)=\eta$ for $\eta\in\underline{H}(X,Q)$ with $Q\in\operatorname{Ob}\underline{W}$.

The *shape functor* $S\colon\underline{H}\to\underline{S}$ is defined as follows: $S(X)=X$ for every space $X$, and if $\Phi\in\underline{H}(X,Y)$, the shaping $S(\Phi)\colon X\to Y$ is defined by $S(\Phi)^Q(\eta)=\eta\Phi$ for $\eta\in\underline{H}(Y,Q)$ with $Q\in\operatorname{Ob}\underline{W}$. $S$ is a covariant functor.

Let $\underline{C}$ be an arbitrary category. The inverse system in $\underline{C}$ is a system $\{X_\alpha, p_\alpha^{\alpha'}, A\}$, where $(A,\leqslant)$ is a directed set, $X_\alpha\in\operatorname{Ob}\underline{C}$ for each $\alpha\in A$, $p_\alpha^{\alpha'}\in\underline{C}(X_{\alpha'},X_\alpha)$ for every $\alpha,\alpha'\in A$ with $\alpha\leqslant\alpha'$, $p_\alpha^\alpha$ is the identity of $X_\alpha$ in $\underline{C}$, $p_\alpha^{\alpha''}=p_\alpha^{\alpha'}p_{\alpha'}^{\alpha''}$ for every $\alpha,\alpha',\alpha''\in A$ with $\alpha\leqslant\alpha'\leqslant\alpha''$. Let $\{X_\alpha, p_\alpha^{\alpha'}, A\}$ and $\{Y_\beta, q_\beta^{\beta'}, B\}$ be two inverse systems in $\underline{C}$. A map $\underline{f}=(f,f_\beta)$ from $\{X_\alpha, p_\alpha^{\alpha'}, A\}$ to $\{Y_\beta, q_\beta^{\beta'}, B\}$ consists of a function $f\colon B\to A$ and of morphisms $f_\beta\in\underline{C}(X_{f(\beta)},Y_\beta)$ such that if $\beta\leqslant\beta'$ in $B$, then $q_\beta^{\beta'}f_{\beta'}p_{f(\beta')}^\alpha=f_\beta p_{f(\beta)}^\alpha$ for some $\alpha\in A$ with $f(\beta), f(\beta')\leqslant\alpha$. Two maps $(f,f_\beta)$ and $(g,g_\beta)$ are said to be *homotopic* if for each $\beta\in B$ there exists an $\alpha\in A$ such that $f(\beta), g(\beta)\leqslant\alpha$ and $f_\beta p_{f(\beta)}^\alpha=g_\beta p_{g(\beta)}^\alpha$. Let $\underline{X}=\{X_\alpha, p_\alpha^{\alpha'}, A\}$, $\underline{Y}=\{Y_\beta, q_\beta^{\beta'}, B\}$ and $\underline{Z}=\{Z_\gamma, r_\gamma^{\gamma'}, C\}$ be inverse systems in $\underline{C}$. If $\underline{f}=(f,f_\beta)$ is a map from $\underline{X}$ to $\underline{Y}$ and $\underline{g}=(g,g_\gamma)$ is a map from $\underline{Y}$ to $\underline{Z}$, then the composition $\underline{h}=\underline{gf}=(h,h_\gamma)$ of maps $\underline{f}$ and $\underline{g}$ is defined as follows: $h=fg\colon C\to A$ and for $h_\gamma\in\underline{C}(X_{h(\gamma)},Z_\gamma)$ we take the composition $g_\gamma f_{g(\gamma)}$. The identity map $\underline{1}_{\underline{X}}\colon\underline{X}\to\underline{X}$ is given by $1(\alpha)=\alpha$ and $1_\alpha=1_{X_\alpha}$ is the identity of $X_\alpha$ in $\underline{C}$. Evidently the inverse systems in $\underline{C}$ as objects and homotopy classes of maps of systems as morphisms form a category, which we shall denote by $\operatorname{Inv}\underline{C}$.

Let $\underline{X}=\{X_\alpha, p_\alpha^{\alpha'}, A\}$ be an inverse system in the category $\underline{H}$ or $\underline{W}$. We shall say that $\underline{X}$ is *associated* with a topological space $X$ if there are continuous maps $p_\alpha\colon X\to X_\alpha$ for $\alpha\in A$ such that the following conditions are satisfied:

(1.1) $[p_\alpha^{\alpha'}][p_{\alpha'}]=[p_\alpha]$ for $\alpha\leqslant\alpha'$.

(1.2) For any continuous map $f\colon X\to Q$ with $Q\in\operatorname{Ob}\underline{W}$ there exist an $\alpha\in A$ and a continuous map $f_\alpha\colon X_\alpha\to Q$ such that $[f]=[f_\alpha][p_\alpha]$.

(1.3) For $\alpha\in A$ and for two continuous maps $f_\alpha, g_\alpha\colon X_\alpha\to Q$ with $Q\in\operatorname{Ob}\underline{W}$ such that $[f_\alpha][p_\alpha]=[g_\alpha][p_\alpha]$ there exists an $\alpha'\in A$ with $\alpha\leqslant\alpha'$ such that $[f_\alpha][p_\alpha^{\alpha'}]=[g_\alpha][p_\alpha^{\alpha'}]$.

This notion is due to Morita [12].

An open covering $U$ of a topological space $X$ is said to be a *normal open covering* provided there exists a map $f\colon X\to M$ from $X$ to the metrizable space $M$ such that $f^{-1}(V)$ is a refinement of $U$ for some open covering $V$ of $M$. We say that $\dim X\leqslant n$ if every finite open covering $U$ of $X$ admits a finite normal open covering of order $\leqslant n+1$ as its refinement (see [14]). If $X$ is a normal space, $\dim X$ defined in this way coincides with the covering dimension of $X$ in the usual sense. Morita has proved in [12] that for any topological space $X$ there exists an inverse system $\{K_\alpha, [p_\alpha^{\alpha'}], A\}$ in $\underline{W}$ associated with $X$ such that $\{U_\alpha\}_{\alpha\in A}$ is the set of all locally finite normal open coverings of $X$ such that $\alpha\leqslant\alpha'$ means that $U_{\alpha'}$ is a refinement of $U_\alpha$ and $K_\alpha$ is a nerve of $U_\alpha$ ($K_\alpha$ is considered as a simplicial complex with the weak topology). Since the condition $\dim X\leqslant n$ implies that every locally finite normal open covering $U_\alpha$ of $X$ is refined by a locally finite normal open covering $U_{\alpha'}$ of order $\leqslant n+1$ (see [14] and [13], Theorem 1.2), we infer that the set $B=\{\alpha\in A\colon \dim K_\alpha\leqslant n\}$ is cofinal with $A$. Then it is easy to see that the inverse system $\{K_\alpha, [p_\alpha^{\alpha'}], B\}$ is associated with $X$. Thus we obtain the following



THEOREM 1.4. *For any topological space $X$ there exists an inverse system $\{X_\alpha, [p_\alpha^{\alpha'}], A\}$ in $\underline{W}$ associated with $X$ such that all $X_\alpha$ are CW complexes and if $\dim X\leqslant n$, then $\dim X_\alpha\leqslant n$ for each $\alpha\in A$.*

For compact Hausdorff spaces we have the following theorem due to Mardešić [11]:

THEOREM 1.5. *Let $\{X_\alpha, p_\alpha^{\alpha'}, A\}$ be an inverse system of compact Hausdorff spaces with $X$ as its limit. Then the inverse system $\{X_\alpha, [p_\alpha^{\alpha'}], A\}$ in $\underline{H}$ is associated with $X$.*

Let $\underline{X}=\{X_\alpha, [p_\alpha^{\alpha'}], A\}$ and $\underline{Y}=\{Y_\beta, [q_\beta^{\beta'}], B\}$ be two inverse systems in $\underline{W}$ associated with the spaces $X$ and $Y$ respectively and let $\underline{f}=(f,f_\beta)$ be a map from $\underline{X}$ to $\underline{Y}$. Then $\underline{f}$ induces the shaping $F(\underline{f})\colon X\to Y$ in the following way: if $\eta\colon Y\to Q$, $Q\in\operatorname{Ob}\underline{W}$ is a continuous map, then $F(\underline{f})^Q([\eta])=[\eta_\beta]f_\beta[p_{f(\beta)}]$, where $\eta_\beta\colon Y_\beta\to Q$ is a continuous map such that $[\eta]=[\eta_\beta][q_\beta]$. Conditions (1.2) and (1.3) imply that $F(\underline{f})^Q([\eta])$ does not depend on the choice of $\eta_\beta$ and that $F(\underline{f})$ is actually a shaping. It is easy to see that for two homotopic maps $\underline{f}, \underline{g}\colon\underline{X}\to\underline{Y}$ we have $F(\underline{f})=F(\underline{g})$ and it follows that we may consider $F$ as the function from $\operatorname{Inv}\underline{W}(\underline{X},\underline{Y})$ to $\underline{S}(X,Y)$. Moreover, the following statement is true (see [12]):

THEOREM 1.6. *The function $F\colon\operatorname{Inv}\underline{W}(\underline{X},\underline{Y})\to\underline{S}(X,Y)$ gives a one-to-one correspondence between shapings $\underline{f}\colon X\to Y$ and homotopy classes of maps from inverse system $\underline{X}$ to $\underline{Y}$ such that the identity shaping $\underline{1}_X\colon X\to X$ corresponds to the homotopy class of the identity map $(1_A, 1_{X_\alpha})$ and if $\underline{Z}=\{Z_\gamma, [r_\gamma^{\gamma'}], C\}$ is associated with $Z$, $\underline{f}\in\operatorname{Inv}\underline{W}(\underline{X},\underline{Y})$, $\underline{g}\in\operatorname{Inv}\underline{W}(\underline{Y},\underline{Z})$, then $F(\underline{gf})=F(\underline{g})F(\underline{f})$.*

Let $\{X_\alpha, [p_\alpha^{\alpha'}], A\}$ be an inverse system in $\underline{H}$ such that for a topological space $X$ there exist continuous maps $p_\alpha\colon X\to X_\alpha$ satisfying Condition (1.1). Then for any topological space $Z$ the functions $p_\alpha$ induce the functions $p_\alpha^{\#}\colon \underline{H}(X_\alpha, Z)\to\underline{H}(X, Z)$ given by the formulae $p_\alpha^{\#}([f]) = [fp_\alpha]$ for each $\alpha \in A$, where $[f] \in \underline{H}(X_\alpha, Z)$. Since for $\alpha \leqslant \beta$ we have $p_\alpha^{\#} = p_\beta^{\#}p_\alpha^{\beta\#}$, it follows that the functions $p_\alpha^{\#}$ induce a function

$$\psi\colon \varinjlim\{\underline{H}(X_\alpha, Z), p_\alpha^{\beta\#}, A\}\to\underline{H}(X, Z),$$

which will be called the *natural transformation*.

Let us observe that in the case where $\{X_\alpha, [p_\alpha^{\alpha'}], A\}$ is associated with $X$ and $Z \in \mathrm{Ob}\, \underline{W}$, we have $\underline{H}(X, Z) = \underline{S}(X, Z)$ (see Theorem 2.2 of [11]), $\mathrm{Inv}\, \underline{W}(\underline{X}, \underline{Z}) = \varinjlim\{\underline{H}(X_\alpha, Z), p_\alpha^{\beta\#}, A\}$ where $\underline{Z} = (Z, 1_Z, \{1\})$ is an inverse system whose directed set consists of only one point, and $\psi = F\colon \mathrm{Inv}\, \underline{W}(\underline{X}, \underline{Z})\to\underline{S}(X, Z)$. Thus, the next statement is an immediate consequence of Theorem 1.6.

COROLLARY 1.7. *If $\{X_\alpha, [p_\alpha^{\alpha'}], A\}$ is associated with $X$, then $\psi$ is a one-to-one function for each $Z \in \mathrm{Ob}\, \underline{W}$.*

The function $\Phi\colon \underline{H}(X, Z)\to\varinjlim\{\underline{H}(X_\alpha, Z), p_\alpha^{\beta\#}, A\}$ inverse to $\psi$ will also be called the *natural transformation*.

**2. The shape dimension.** In [12] Morita introduced the shape dimension $\mathrm{Sd}\, X$ of a topological space $X$ as number $\mathrm{Min}\{\dim Y\colon \mathrm{Sh}\, X \leqslant \mathrm{Sh}\, Y\}$.

If $W$ is a $CW$ complex, then by $W^n$ we denote the $n$-skeleton of $W$.

LEMMA 2.1. *If $\eta\colon X\to W$ is a continuous map such that $\mathrm{Sd}\, X \leqslant n$ and $W$ is a $CW$ complex, then there is a map $\mu\colon X\to W$ homotopic to $\eta$ with $\mu(X) \subset W^n$.*

Proof. Let $Y$ be a topological space such that $\dim Y \leqslant n$ and $\underline{h}\underline{g} = \underline{1}_X$ for some shapings $\underline{g}\colon X\to Y$ and $\underline{h}\colon Y\to X$. By Theorem 1.4 there exists an inverse system $\{Y_\alpha, [p_\alpha^{\alpha'}], A\}$ in $\underline{W}$ associated with $Y$ and $Y_\alpha^n = Y_\alpha$ for each $\alpha \in A$. Let $[\tilde{\eta}] = \underline{h}^W([\eta])$. It follows by Condition (1.2) that $[\eta_\alpha p_\alpha] = [\eta]$ for some map $\eta_\alpha\colon Y_\alpha\to W$. By the cellular approximation theorem (see [10], p. 72) we can suppose that $\eta_\alpha(Y_\alpha) \subset W^n$, i.e., that there is a map $\mu_\alpha\colon Y_\alpha\to W^n$ such that $i_{W^n\subset W}\mu_\alpha = \eta_\alpha$. Let $[\tilde{\mu}] = \underline{g}^{W^n}([\mu_\alpha p_\alpha])$. Then we get

$$\begin{aligned}[\eta] = (\underline{h}\underline{g})^W([\eta]) &= \underline{g}^W([\eta_\alpha][p_\alpha]) = \underline{g}^W([i_{W^n\subset W}\mu_\alpha][p_\alpha]) \\ &= \underline{g}^W([i_{W^n\subset W}][\mu_\alpha p_\alpha]) = [i_{W^n\subset W}]\underline{g}^{W^n}([\mu_\alpha p_\alpha]) = [i_{W^n\subset W}][\tilde{\mu}],\end{aligned}$$

i.e., $\mu = i_{W^n\subset W}\tilde{\mu}$ satisfies required conditions.

THEOREM 2.2. *Let $\{X_\alpha, [p_\alpha^{\alpha'}], A\}$ be an infinite inverse system in $\underline{CW}$ associated with a topological space $X$. If $\mathrm{Sd}\, X \leqslant n$, then there is an inverse system $\{Y_\beta, [q_\beta^{\beta'}], B\}$ in $\underline{CW}$ associated with $X$ such that the cardinal $kB$ is equal to $kA$ and each $Y_\beta$ is an $n$-skeleton of some $X_\alpha$.*

Proof. By Lemma 2.1 there are maps $q_\alpha\colon X\to X_\alpha^n$ such that $[i_{X_\alpha^n\subset X_\alpha}q_\alpha] = [p_\alpha]$ for each $\alpha \in A$. Let $\alpha$ be an arbitrary element of $A$. By (1.2) there exist an $\alpha_0 \in A$ and a map $\eta\colon X_{\alpha_0}\to X_\alpha^n$ such that $[\eta][p_{\alpha_0}] = [q_\alpha]$. Let $\alpha_1 \in A$, $\alpha_0, \alpha \leqslant \alpha_1$. Now,

$$[i_{X_\alpha^n\subset X_\alpha}\eta p_{\alpha_0}^{\alpha_1}][p_{\alpha_1}] = [i_{X_\alpha^n\subset X_\alpha}\eta][p_{\alpha_0}] = [i_{X_\alpha^n\subset X_\alpha}][q_\alpha] = [p_\alpha] = [p_\alpha^{\alpha_1}][p_{\alpha_1}]$$

and by (1.3) we have $\alpha' \in A$, $\alpha_1 \leqslant \alpha'$ with

$$[i_{X_\alpha^n\subset X_\alpha}\eta p_{\alpha_0}^{\alpha_1}][p_{\alpha_1}^{\alpha'}] = [p_\alpha^{\alpha_1}][p_{\alpha_1}^{\alpha'}] = [p_\alpha^{\alpha'}],$$

i.e., $[i_{X_\alpha^n\subset X_\alpha}\eta p_{\alpha_0}^{\alpha'}] = [p_\alpha^{\alpha'}]$. If we set $r_\alpha^{\alpha'} = \eta p_{\alpha_0}^{\alpha'}\colon X_{\alpha'}\to X_\alpha^n$, then we get a map such that $[i_{X_\alpha^n\subset X_\alpha}r_\alpha^{\alpha'}] = [p_\alpha^{\alpha'}]$ and

$$[r_\alpha^{\alpha'}p_{\alpha'}] = [\eta p_{\alpha_0}^{\alpha}p_{\alpha'}] = [\eta p_{\alpha_0}] = [q_\alpha].$$

Thus the following statement is true:

(1) For each $\alpha \in A$ there exist an $\alpha' \in A$ and a map $r_\alpha^{\alpha'}\colon X_{\alpha'}\to X_\alpha^n$ such that $\alpha \leqslant \alpha'$, $[i_{X_\alpha^n\subset X_\alpha}r_\alpha^{\alpha'}] = [p_\alpha^{\alpha'}]$ and $[r_\alpha^{\alpha'}p_{\alpha'}] = [q_\alpha]$.

For $(B, \leqslant)$ we take the set of all finite subsets $\beta = \{\alpha_1, \ldots, \alpha_k\}$ of $A$ ordered by inclusion, so that $\beta' \leqslant \beta$ means $\beta' \subset \beta$. We shall define by induction on the cardinal of $\beta$ an increasing function $a\colon B\to A$ such that

(2) $$a(\{\alpha\}) = \alpha \quad \text{for each } \alpha \in A,$$

(3) for every $\beta, \beta' \in B$ with $\beta < \beta'$ ($\beta < \beta'$ means $\beta \leqslant \beta'$ and $\beta \neq \beta'$) we have a map $s_\beta^{\beta'}\colon X_{a(\beta')}\to X_{a(\beta)}^n$ such that

$$[i_{X_{a(\beta)}^n\subset X_{a(\beta)}}s_\beta^{\beta'}] = [p_{a(\beta)}^{a(\beta')}],$$

(4) $$[s_\beta^{\beta'}][p_{a(\beta')}] = [q_{a(\beta)}],$$

(5) $$\beta < \beta' < \beta'' \quad \text{implies} \quad [s_\beta^{\beta''}] = [s_\beta^{\beta'}i_{X_{a(\beta')}^n\subset X_{a(\beta')}}s_\beta^{\beta''}].$$

Let us suppose that $a(\beta)$ and $s_\beta^{\beta'}$ are defined for each $\beta, \beta' \in B$ with $\beta < \beta'$ such that $\mathrm{card}\,\beta$, $\mathrm{card}\,\beta' \leqslant k$ $(1 \leqslant k)$ and let $\beta'' \in B$ be an element with $\mathrm{card}\,\beta'' = k+1$. Since the set of all predecessors of $\beta''$ is finite, by (1) there exist an $\alpha \in A$ and maps $r_{a(\beta')}^\alpha\colon X_\alpha\to X_{a(\beta')}^n$ for each $\beta' < \beta''$ such that $\alpha \geqslant a(\beta')$ for each $\beta' < \beta''$ and all maps $r_{a(\beta')}^\alpha$ satisfy Condition (1). Now, if $\beta, \beta' \in B$ and $\beta < \beta' < \beta''$, then we have $[r_{a(\beta)}^\alpha][p_\alpha] = [q_{a(\beta)}]$ and $[s_\beta^{\beta'}i_{X_{a(\beta')}^n\subset X_{a(\beta')}}r_{a(\beta')}^\alpha][p_\alpha] = [s_\beta^{\beta'}i_{X_{a(\beta')}^n\subset X_{a(\beta')}}q_{a(\beta')}] = [s_\beta^{\beta'}][p_{a(\beta')}] = [q_{a(\beta)}]$. Since the set of all pairs $(\beta, \beta')$ with $\beta < \beta' < \beta''$ is finite, by (1.3) there exists an $\alpha' \geqslant \alpha$ such that

(6) $\beta < \beta' < \beta''$ implies $[r_{a(\beta)}^\alpha p_\alpha^{\alpha'}] = [s_\beta^{\beta'}i_{X_{a(\beta')}^n\subset X_{a(\beta')}}r_{a(\beta')}^\alpha p_\alpha^{\alpha'}]$ for every $\beta, \beta' \in B$.

Let us put $a(\beta'') = \alpha'$ and $s_{\beta'}^{\beta''} = r_{a(\beta')}^\alpha p_\alpha^{\alpha'}\colon X_{\alpha'}\to X_{a(\beta')}^n$. Then

$$[i_{X_{a(\beta')}^n\subset X_{a(\beta')}}s_{\beta'}^{\beta''}] = [i_{X_{a(\beta')}^n\subset X_{a(\beta')}}r_{a(\beta')}^\alpha p_\alpha^{\alpha'}] = [p_{a(\beta')}^\alpha p_\alpha^{\alpha'}] = [p_{a(\beta')}^{\alpha'}] = [p_{a(\beta')}^{a(\beta'')}]$$

and $s_{\beta'}^{\beta''}$ satisfies (3),

$$[s_{\beta'}^{\beta''}][p_{a(\beta'')}] = [r_{a(\beta')}^{a} p_a^{a'} p_{a'}] = [r_{a(\beta')}^{a} p_a] = [q_{a(\beta')}]$$

and $s_\beta^{\beta''}$ satisfies (4); if $\beta < \beta' < \beta''$, then by (6)

$$[s_\beta^{\beta''}] = [r_{a(\beta)}^{a} p_a^{a'}] = [s_\beta^{\beta'} i_{X_{a(\beta')}^n \subset X_{a(\beta')}}][r_{a(\beta')}^{a} p_a^{a'}] = [s_\beta^{\beta'} i_{X_{a(\beta')}^n \subset X_{a(\beta')}^n} s_{\beta'}^{\beta''}]$$

and the maps $s_\beta^{\beta''}$ satisfy Conditions (5). Thus, the construction of $a\colon B \to A$ and of maps $s_\beta^{\beta'}$ is finished.

For every $\beta \in B$ we now put $Y_\beta = X_{a(\beta)}^n$ and for $\beta \leqslant \beta'$ we put $q_\beta^{\beta'} = s_\beta^{\beta'} i_{X_{a(\beta')}^n \subset X_{a(\beta')}}\colon Y_{\beta'} \to Y_\beta$ in the case where $q' \neq q$ and $q_\beta^{\beta'} = \mathrm{id}_{Y_\beta}$, if $\beta' = \beta$. If $\beta < \beta' < \beta''$, then

$$[q_\beta^{\beta'}][q_{\beta'}^{\beta''}] = [s_\beta^{\beta'} i_{X_{a(\beta')}^n \subset X_{a(\beta')}} s_{\beta'}^{\beta''}][i_{X_{a(\beta'')}^n \subset X_{a(\beta'')}}] = [s_\beta^{\beta''} i_{X_{a(\beta'')}^n \subset X_{a(\beta'')}}] = [q_\beta^{\beta''}]$$

and

$$[q_\beta^{\beta'}][q_{a(\beta')}] = [s_\beta^{\beta'} i_{X_{a(\beta')}^n \subset X_{a(\beta')}}][q_{a(\beta')}] = [s_\beta^{\beta'} p_{a(\beta')}] = [q_{a(\beta)}] .$$

Thus $\underline{Y} = \{Y_\beta, [q_\beta^{\beta'}], B\}$ is an inverse system in $\underline{CW}$ and we shall show that $\underline{Y}$ is associated with $X$.

Let $\eta\colon X \to Q$ be a continuous map with $Q \in \mathrm{Ob}\,\underline{W}$. By (1.2) there exist on $a \in A$ and a map $\eta_a\colon X_a \to Q$ such that $[\eta_a][p_a] = [\eta]$. Let $\beta = \{a\} \in B$. Then by (2) $Y_\beta = X_a^n$ and if we put $\eta_\beta = \eta_a i_{X_a^n \subset X_a}\colon Y_\beta \to Q$, then we get

$$[\eta_\beta][q_{a(\beta)}] = [\eta_a i_{X_a^n \subset X_a} q_a] = [\eta_a p_a] = [\eta] .$$

It follows that Condition (1.2) is satisfied.

Now, let $f_\beta, g_\beta\colon Y_\beta \to Q$, $Q \in \mathrm{Ob}\,\underline{W}$ be two maps for which $[f_\beta q_{a(\beta)}] = [g_\beta q_{a(\beta)}]$. Since $A$ is an infinite set, there is an $\beta' > \beta$ and we have the map $s_\beta^{\beta'}\colon X_{a(\beta')} \to X_{a(\beta)}^n = Y_\beta$ which satisfies Conditions (3), (4) and (5). Hence

$$[f_\beta s_\beta^{\beta'}][p_{a(\beta')}] = [f_\beta q_{a(\beta)}] = [g_\beta q_{a(\beta)}] = [g_\beta s_\beta^{\beta'}][p_{a(\beta')}]$$

and by (1.3) there exists an $a \geqslant a(\beta')$ such that $[f_\beta s_\beta^{\beta'} p_{a(\beta')}^{a}] = [g_\beta s_\beta^{\beta'} p_{a(\beta')}^{a}]$. Let $\beta'' = \beta' \cup \{a\}$. Then $a(\beta'') \geqslant a(\{a\}) = a$ and $a(\beta'') \geqslant a(\beta')$. It follows by (3) that

$$[f_\beta s_\beta^{\beta} p_{a(\beta')}^{a}][p_a^{a(\beta'')} i_{X_{a(\beta'')}^n \subset X_{a(\beta'')}}] = [f_\beta s_\beta^{\beta'} p_{a(\beta')}^{a(\beta'')} i_{X_{a(\beta'')}^n \subset X_{a(\beta'')}}]$$

$$= [f_\beta s_\beta^{\beta'} i_{X_{a(\beta')}^n \subset X_{a(\beta')}} s_{\beta'}^{\beta''} i_{X_{a(\beta'')}^n \subset X_{a(\beta'')}}] = [f_\beta s_\beta^{\beta''} i_{X_{a(\beta'')}^n \subset X_{a(\beta'')}}] = [f_\beta][q_\beta^{\beta''}]$$

and similarly

$$[g_\beta s_\beta^{\beta'} p_{a(\beta')}^{a}][p_a^{a(\beta'')} i_{X_{a(\beta'')}^n \subset X_{a(\beta'')}}] = [g_\beta][q_\beta^{\beta''}] .$$

Since $[f_\beta s_\beta^{\beta'} p_{a(\beta')}^{a}] = [g_\beta s_\beta^{\beta'} p_{a(\beta')}^{a}]$, we have $[f_\beta][q_\beta^{\beta''}] = [g_\beta][q_\beta^{\beta''}]$, i.e., Condition (1.3) is satisfied. We have $kB = kA$ ($A$ is an infinite set) and this completes the proof.



COROLLARY 2.3. *If $X$ is a compact metrizable space and* $\mathrm{Sd}\,X = k$, *then there exists a compact metrizable space $Y$ such that* $\dim Y = k$ *and* $\mathrm{Sh}\,X = \mathrm{Sh}\,Y$.

Proof. Let $\{X_n, p_n^m, N\}$ be an inverse sequence of polyhedra with $X$ as its inverse limit. By Theorem 1.5 $\{X_n, [p_n^m], N\}$ is associated with $X$ and by Theorem 2.2 there exists an inverse system $\{Y_\beta, [q_\beta^{\beta'}], B\}$ in $\underline{CW}$ associated with $X$ such that $B$ is an infinite countable set and every $Y_\beta$ is a polyhedron of dimension $\leqslant k$. Hence there exists a sequence of elements $\beta_n \in B$ such that $m < n$ implies $\beta_m < \beta_n$ and the set

$$A = \{\beta \in B\colon \beta = \beta_m \text{ for some } m\}$$

is cofinal with $B$. It is easy to see that there is an inverse sequence $\{Z_n, r_n^m, N\}$ in the category of topological spaces such that $Z_n = Y_{\beta_n}$ and $r_n^{n+1} = q_{\beta_n}^{\beta_{n+1}}$ for each $n \in N$. It follows that $[r_n^m] = [q_{\beta_n}^{\beta_m}]$ for each $m, n \in N$, $m > n$ and since $A$ is cofinal with $B$, it is clear that $\{Z_n, [r_n^m], N\}$ is associated with $X$. On the other hand, $\{Z_n, [r_n^m], N\}$ is associated with $Y = \varprojlim\{Z_n, r_n^m, N\}$ and by Theorem 1.6 we infer $\mathrm{Sh}\,X = \mathrm{Sh}\,Y$. Since $k \geqslant \dim Y \geqslant \mathrm{Sd}\,Y = \mathrm{Sd}\,X = k$, we have $\dim Y = k$ and this completes the proof.

Let us recall that K. Borsuk introduced the *fundamental dimension* $\mathrm{Fd}(X)$ of a compact metric space $X$ as the number

$$\mathrm{Min}\{\dim Y\colon \mathrm{Sh}(X) \leqslant \mathrm{Sh}(Y)\}$$

(by $\mathrm{Sh}(X)$ we mean the shape of $X$ in the sense of Borsuk).

COROLLARY 2.4. *If $X$ is a compact metrizable space, then* $\mathrm{Sd}\,X = \mathrm{Fd}(X)$.

Proof. Since for every two compact metrizable spaces $X$ and $Y$ the relation $\mathrm{Sh}(X) \leqslant \mathrm{Sh}(Y)$ is equivalent to $\mathrm{Sh}\,X \leqslant \mathrm{Sh}\,Y$ (see [11]), we have $\mathrm{Sd}\,X \leqslant \mathrm{Fd}(X)$. Let $\mathrm{Sd}\,X = k$. Then by Corollary 2.3 there is a compact metrizable space $Y$ such that $\mathrm{Sh}\,X = \mathrm{Sh}\,Y$ and $\dim Y = k$. But then $\mathrm{Sh}(X) \leqslant \mathrm{Sh}(Y)$ and $\mathrm{Fd}(X) \leqslant \dim Y = k$. Thus $\mathrm{Fd}(X) \leqslant \mathrm{Sd}\,X$ and the proof is finished.

Remark. Corollary 2.3 is a generalization of some result (unpublished) due to W. Holsztyński (see [15]).

**3. Cohomotopy groups.** Cohomotopy groups have been introduced by K. Borsuk in [2] and studied by E. Spanier in [16]. We recall the definition of the $n$th cohomotopy group of a space $X$ formulated by S. Godlewski in [7]. Let $S = S^n$ be an $n$-dimensional sphere. Let us choose a point $s_0 \in S$ and consider the subset $S \vee S = (S \times \{s_0\}) \cup (\{s_0\} \times S)$ of the Cartesian product $S \times S$. Let us define the map $\Omega\colon S \vee S \to S$ by the formula $\Omega(s, s_0) = \Omega(s_0, s) = s$ for $s \in s$. Take two arbitrary maps $\varphi_1, \varphi_2\colon X \to S$. A map $\Phi\colon X \times [0, 1] \to S \times S$ such that $\Phi(x, 0) = (\varphi_1(x), \varphi_2(x))$ and

$\Phi(x,1)\in S\vee S$ for $x\in X$ is called a *normalizing homotopy* for the maps $\varphi_1$ and $\varphi_2$. Then the map $\hat{\varphi}\colon X\to S\vee S$ defined by the formula $\hat{\varphi}(x)=\Phi(x,1)$ is said to be a *normalization* of the maps $\varphi_1$ and $\varphi_2$ (see [16], p. 210).

Let us suppose that a space $X$ satisfies the following conditions:

(3.1) For every two maps $\varphi_1,\varphi_2\colon X\to S$ there exists a normalizing homotopy.

(3.2) If $\hat{\varphi}$ is a normalization of maps $\varphi_1$ and $\varphi_2$, then the homotopy class $[\Omega\hat{\varphi}]$ of the map $\Omega\hat{\varphi}\colon X\to S$ depends only on the homotopy classes $[\varphi_1]$ and $[\varphi_2]$.

(3.3) The addition in the set $\underline{H}(X,S)$ defined by the formula $[\varphi_1]+[\varphi_2]=[\Omega\hat{\varphi}]$, where $\hat{\varphi}$ is a normalization of the maps $\varphi_1$ and $\varphi_2$, makes the set $\underline{H}(X,S)$ an Abelian group.

This group is called the *n-th cohomotopy group* of $X$ and is denoted by $\pi^n(X)$. The addition defined in (3.3) is called the *n-th cohomotopy addition*. It may be defined if Conditions (3.1) and (3.2) are satisfied. Then we say that the space $X$ admits the $n$th cohomotopy addition. Moreover, if Condition (3.3) is also satisfied we say that the space $X$ admits the existence of the $n$th cohomotopy group.

Let us suppose that $X$ and $Y$ admit the existence of the $n$th cohomotopy groups $\pi^n(X)$ and $\pi^n(Y)$ and let $\underline{f}\colon X\to Y$ be a shaping. Then we have the function $\underline{f}^{\#}\colon [S^Y]\to[S^X]$ which is equal to $\underline{f}^S$. The function $\underline{f}^{\#}$ is said to be induced by $\underline{f}$.

In [7] S. Godlewski has proved the following

THEOREM 3.4. *If topological spaces $X$ and $Y$ admit the existence of the n-th cohomotopy groups $\pi^n(X)$ and $\pi^n(Y)$ and $\underline{f}\colon X\to Y$ is a shaping, then the induced function $\underline{f}^{\#}\colon \pi^n(Y)\to\pi^n(X)$ is a homomorphism.*

The main result of this section is the following

THEOREM 3.5. *Let $\{X_a,[p_a^{a'}],A\}$ be an inverse system in $\underline{H}$ associated with $X$. If all $X_a$ admit the existence of the n-th cohomotopy group, then $X$ admits the existence of the n-th cohomotopy group and the natural transformation $\Phi\colon \pi^n(X)\to\varinjlim\{\pi^n(X_a),p_a^{a'\#},A\}$ is an isomorphism.*

Proof. Let $f,g\colon X\to S$ be continuous maps. By Condition (1.2) there exist $\gamma,\omega\in A$ and maps $f_\gamma\colon X_\gamma\to S$, $g_\omega\colon X_\omega\to S$ such that $[f_\gamma p_\gamma]=[f]$ and $[g_\omega p_\omega]=[g]$. Since $A$ is a directed set, there is an $a\in A$ with $\gamma,\omega\leqslant a$. Let $F\colon X_a\times[0,1]\to S\times S$ be a normalizing homotopy for the maps $f_\gamma p_\gamma^a$ and $g_\omega p_\omega^a$. Setting $G=F(p_a\times\mathrm{id}_{[0,1]})\colon X\times[0,1]\to S\times S$, we obtain a map for which

$$G(x,0)=F\big(p_a(x),0\big)=\big(f_\gamma p_\gamma^a p_a(x),\,g_\omega p_\omega^a p_a(x)\big)=\big(f_\gamma p_\gamma(x),\,g_\omega p_\omega(x)\big)$$

and

$$h(x)=G(x,1)=F\big(p_a(x),1\big)\in S\vee S$$



for each $x\in X$. Since $[f_\gamma p_\gamma]=[f]$ and $[g_\omega p_\omega]=[g]$, it is clear that there exists a map $H\colon X\times[0,1]\to S\times S$ such that $H(x,0)=\big(f(x),g(x)\big)$ and $H(x,1)=h(x)\in S\vee S$ for every $x\in X$, i.e., there is a normalizing homotopy for maps $f$ and $g$.

Let $\chi\colon X\to S\vee S$ be an arbitrary normalization of maps $f$ and $g$. By Condition (1.2) there exist a $\beta\geqslant a$ and a map $\chi_\beta\colon X_\beta\to S\vee S$ such that $[\chi_\beta p_\beta]=[\chi]$. Let $s_\beta\colon X_\beta\to S\times S$ be a map defined by the formula $s_\beta(x)=\big(f_\gamma p_\gamma^\beta(x),\,g_\omega p_\omega^\beta(x)\big)$ for $x\in X_\beta$. Now,

$$s_\beta p_\beta(x)=\big(f_\gamma p_\gamma^\beta p_\beta(x),\,g_\omega p_\omega^\beta p_\beta(x)\big)=\big(f_\gamma p_\gamma(x),\,g_\omega p_\omega(x)\big)\quad\text{for}\quad x\in X$$

and $s_\beta p_\beta$ is homotopic to the map $s\colon X\to S\times S$ given by the formula $s(x)=\big(f(x),g(x)\big)$ for $x\in X$. Since $\chi$ is a normalization of the maps $f$ and $g$, we have $[i_{S\vee S\subset S\times S}\chi]=[s]$ and this implies $[i_{S\vee S\subset S\times S}\chi_\beta][p_\beta]=[i_{S\vee S\subset S\times S}\chi]=[s]=[s_\beta][p_\beta]$. Then by Condition (1.3) we have $\sigma\in A$, $\beta\leqslant\sigma$ such that $[i_{S\vee S\subset S\times S}\chi_\beta p_\beta^\sigma]=[s_\beta p_\beta^\sigma]$. Since

$$s_\beta p_\beta^\sigma(x)=\big(f_\gamma p_\gamma^\beta p_\beta^\sigma(x),\,g_\omega p_\omega^\beta p_\beta^\sigma(x)\big)=\big(f_\gamma p_\gamma^\sigma(x),\,g_\omega p_\omega^\sigma(x)\big)\quad\text{for}\quad x\in X_\sigma,$$

then we infer that $\chi_\beta p_\beta^\sigma$ is a normalization of the maps $f_\gamma p_\gamma^\sigma$ and $g_\omega p_\omega^\sigma$. Hence

$$\begin{aligned}[\Omega\chi]=[\Omega\chi_\beta p_\beta]=[\Omega\chi_\beta p_\beta^\sigma p_\sigma]&=\psi(\{[\Omega\chi_\beta p_\beta^\sigma]\})=\psi(\{[f_\gamma p_\gamma^\sigma]+[g_\omega p_\omega^\sigma]\})\\&=\psi(\{[f_\gamma p_\gamma^\sigma]\}+\{[g_\omega p_\omega^\sigma]\})=\psi\big(\Phi([f])+\Phi([g])\big).\end{aligned}$$

Therefore the homotopy class $[\Omega\chi]$, where $\chi$ is a normalization of maps $f$ and $g$, depends only on the homotopy classes $[f]$ and $[g]$, and moreover $\Phi([\Omega\chi])=\Phi([f])+\Phi([g])$. Thus $X$ admits the $n$th cohomotopy addition. Since $\Phi$ is a one-to-one function (see Corollary 1.7), $\Phi([f]+[g])=\Phi([f])+\Phi([g])$ and $\varinjlim\{\pi^n(X_a),p_a^{a'\#},A\}$ is an Abelian group, it is easy to see that the $n$th cohomotopy addition makes the set $[S^X]$ an Abelian group and the natural transformation $\Phi$ is an isomorphism. Then $X$ admits the existence of $n$th cohomotopy group and the proof is finished.

Remark. Theorem 3.5 is a generalization of Theorem 3 in [5].

K. Borsuk proved in [1] (Theorem (11.10), p. 61) that every metric space $X$ with $\dim X<2n-1$ admits the existence of the $n$th cohomotopy group $\pi^n(X)$ and it is known that each compact Hausdorff space $X$ with $\dim X<2n-1$ admits the existence of the $n$th cohomotopy group (see [16]). In the next theorem we give a generalization of the above results.

THEOREM 3.6. *Every topological space $X$ with $\mathrm{Sd}X<2n-1$ admits the existence of the n-th cohomotopy group $\pi^n(X)$.*

Proof. Let $\varphi_1,\varphi_2\colon X\to S$. By $s\colon X\to S\times S$ we denote a map such that $s(x)=\big(\varphi_1(x),\varphi_2(x)\big)$ for each $x\in X$. Since $S\times S$ has a structure of a CW complex such that $S\vee S$ is a $2n-1$ skeleton of $S\times S$, by Lemma 2.1 the map $s\colon X\to S\times S$ is homotopic to the map $s_1\colon X\to S\times S$ with $s_1(x)\subset S\vee S$. Thus $\varphi_1$ and $\varphi_2$ have a normalizing homotopy and Condition (3.1) is satisfied.

Let us suppose that $\hat{\varphi}\colon X\to S\vee S$ and $\hat{\psi}\colon X\to S\vee S$ are two normalizations of maps $\varphi_1$ and $\varphi_2$. Evidently $i_{S\vee S\subset S\times S}\hat{\varphi}$ is homotopic to $i_{S\vee S\subset S\times S}\hat{\psi}$. Let $\{K_\alpha, [p_\alpha^{\alpha'}], A\}$ be an inverse system in $\underline{CW}$ associated with $X$ and such that $\dim K_\alpha \leqslant \operatorname{Sd} X < 2n-1$ for each $\alpha\in A$ (see Theorem 2.2). By Conditions (1.2) and (1.3) there exist an $\alpha\in A$ and maps $\hat{\varphi}_\alpha\colon K_\alpha\to S\vee S$, $\hat{\psi}_\alpha\colon K_\alpha\to S\vee S$ such that $[\hat{\varphi}_\alpha p_\alpha]=[\hat{\varphi}]$, $[\hat{\psi}_\alpha p_\alpha]=[\hat{\psi}]$ and $i_{S\vee S\subset S\times S}\hat{\varphi}_\alpha \simeq i_{S\vee S\subset S\times S}\hat{\psi}_\alpha$. Moreover, we may suppose that both $\hat{\varphi}_\alpha$ and $\hat{\psi}_\alpha$ are cellular. Let $H\colon K_\alpha\times I\to S\times S$ be a map such that $H(x,0)=\hat{\varphi}_\alpha(x)$ and $H(x,1)=\hat{\psi}_\alpha(x)$ for every $x\in K_\alpha$. By the cellular approximation theorem (see [10], p. 72) the map $H$ is homotopic to the cellular map $F\colon K_\alpha\times I\to S\times S$, and since $H|_{K_\alpha\times\{0\}\cup K_\alpha\times\{1\}}$ is cellular, we may suppose that $F|_{K_\alpha\times\{0\}\cup K_\alpha\times\{1\}} = H|_{K_\alpha\times\{0\}\cup K_\alpha\times\{1\}}$. Now, $\dim(K_\alpha\times I)\leqslant 2n-1$ and therefore $F(K_\alpha\times I)\subset (S\times S)^{2n-1}=S\vee S$. This means that $\hat{\varphi}_\alpha\simeq\hat{\psi}_\alpha$ and consequently $\hat{\varphi}\simeq\hat{\varphi}_\alpha p_\alpha\simeq\hat{\varphi}p_\alpha\simeq\hat{\psi}$. Thus Condition (3.2) is satisfied and $X$ admits the $n$th cohomotopy addition. We shall prove that this addition is associative. So let $f, g, h\colon X\to S$. By $s\colon X\to S\times S\times S$ we denote a map such that $s(x)=\big(f(x), g(x), h(x)\big)$ for each $x\in X$. Since

$$\begin{aligned}(S\times S\times S)^{2n-1} &= S\times\{s_0\}\times\{s_0\}\cup\{s_0\}\times S\times\{s_0\}\cup\{s_0\}\times\{s_0\}\times S\\ &= S\vee S\vee S,\end{aligned}$$

by Lemma 2.1 there is a map $F\colon X\times I\to S\times S\times S$ such that $F(x,0)=s(x)$ and $F(x,1)\in S\vee S\vee S$ for each $x\in X$. Let $F(x,1)=\big(f_1(x), g_1(x), h_1(x)\big)$ for every $x\in X$. We define the maps $\chi,\varphi,\psi,\omega\colon X\to S\vee S$ by the formulae

$$\begin{aligned}\chi(x) &= \big(f_1(x), g_1(x)\big),\\ \varphi(x) &= \big(\Omega\chi(x), g_1(x)\big),\\ \psi(x) &= \big(g_1(x), h_1(x)\big),\\ \omega(x) &= \big(f_1(x), \Omega\psi(x)\big)\end{aligned}$$

for each $x\in X$. It is easy to see that $\chi$ and $\psi$ are normalizations for maps $f, g$ and $g, h$ respectively. Similarly $\varphi$ and $\omega$ are normalizations of maps $\Omega\chi, g$ and $f, \Omega\psi$ respectively. Since $\Omega\varphi=\Omega\omega$, we have $([f]+[g])+[h]=[f]+([g]+[h])$, i.e., the addition is associative.

Let $e\colon X\to S$ be a map such that $e(x)=s_0$ for each $x\in X$. Evidently $[f]+[e]=[e]+[f]=[f]$ for every map $f\colon X\to S$. Since the $n$th cohomotopy addition is commutative, it remains only to show that for every map $f\colon X\to S$ there is a map $g\colon X\to S$ such that $[f]+[g]=[e]$. By Theorem (10.1) of [1] there are two maps $j, r\colon S\to S$ such that $r\simeq \mathrm{id}_S$, $r(x)=s_0$ or $j(x)=s_0$ for each $x\in S$ and the map $h\colon S\to S$ defined by the formula

$$h(x)=\begin{cases} r(x) & \text{if } j(x)=s_0,\\ j(x) & \text{if } r(x)=s_0\end{cases}$$



is null-homotopic. Then it is easy to see that setting $\chi(x)=\big(rf(x), jf(x)\big)$ we obtain the map $\chi\colon X\to S\vee S$, which is a normalization of maps $f$ and $jf$. We have $\Omega\chi=hf$ and consequently $[f]+[jf]=[e]$. This completes the proof.

**4. Homotopic $k$-skeletons and Borsuk's generalized cohomotopy groups.** Let us recall some notions introduced by K. Borsuk in [3] and [4].

Let $X_1$ and $X_2$ be two closed subsets of a space $X$ and let $i_{X_1\subset X}$ and $i_{X_2\subset X}$ denote the inclusion maps $i_{X_1\subset X}\colon X_1\to X$, $i_{X_2\subset X}\colon X_2\to X$. We say that the set $X_2$ *homotopically dominates* the set $X_1$ in the space $X$, written $X_1\underset{h}{\leqslant}X_2$ in $X$, provided there exists a continuous map $\alpha\colon X_1\to X_2$ such that $i_{X_1\subset X}\simeq i_{X_2\subset X}\alpha$. A closed subset $X_1$ of $X$ is said to be a *homotopic $k$-skeleton* of $X$ provided $\dim X_1\leqslant k$ and $X_2\underset{h}{\leqslant}X_1$ in $X$ for each closed subset $X_2$ of $X$ with $\dim X_2\leqslant k$.

Let $A$ be a closed subset of a binormal space $X$ (i.e., $X\times[0,1]$ is a normal space), and let $S=S^n$ be $n$-dimensional sphere. Let us denote by $S^{A\subset X}$ the subset of $S^A$ consisting of all maps $f\in S^A$ extendable over $X$. Since $S$ is an ANR-set, it is clear that all maps $g\in S^A$ homotopic to a map $f\in S^{A\subset X}$ belong to $S^{A\subset X}$. It follows that $S^{A\subset X}$ is the union of some homotopy classes belonging to $[S^A]$. Hence $[S^{A\subset X}]\subset[S^A]$. If $A$ admits the existence of the $n$th cohomotopy group, then the set $[S^{A\subset X}]$ generates a subgroup of $[S^A]$ denoted by $\pi^n(A\subset X)$. In [4] K. Borsuk has proved that if $X_1$ and $X_2$ are homotopic $k$-skeletons of a compact metric space $X$ and $k<2n-1$, then the groups $\pi^n(X_1\subset X)$ and $\pi^n(X_2\subset X)$ are isomorphic. We show that this statement is true in a more general case, namely when $X$ is a binormal space. The abstract group isomorphic to all groups $\pi^n(X_1\subset X)$, where $X_1$ is a homotopic $k$-skeleton of a binormal space $X$ ($k<2n-1$), is denoted by $\pi_k^n(X)$.

LEMMA 4.1. *Let $A$ and $B$ be such closed subsets of binormal spaces $X$ and $Y$, respectively, that both $A$ and $B$ admit the existence of the $n$-th cohomotopy group. If $f\colon X\to Y$ is a map, then for each map $\varphi\colon A\to B$ such that*

$$i_{B\subset Y}\varphi\simeq f i_{A\subset X} \tag{1}$$

*we have $\varphi^{\#}\big(\pi^n(B\subset Y)\big)\subset\pi^n(A\subset X)$. If $\eta\colon A\to B$ is another map satisfying (1), then $\eta^{\#}(b)=\varphi^{\#}(b)$ for each $b\in\pi^n(B\subset Y)$.*

Proof. Let $s\in S^{B\subset Y}$ and let $\tilde{s}\in S^Y$ be an extension of $s$. Then for every map $\psi\colon A\to B$ satisfying Condition (1) we have $s\psi=\tilde{s}i_{B\subset Y}\psi\simeq\tilde{s}fi_{A\subset X}$. Therefore for each $b\in[S^{B\subset Y}]$ we obtain $\varphi^{\#}(b)=\eta^{\#}(b)$ and consequently $\varphi^{\#}(b)=\eta^{\#}(b)$ for every $b\in\pi^n(B\subset Y)$. Now since $\tilde{s}fi_{A\subset X}$ has as extension a map $\tilde{s}f$ and $X$ is a binormal space, there exists an extension of the map $s\psi$, i.e., $\psi^{\#}([s])=[s\psi]\in[S^{A\subset X}]$. Hence $\psi^{\#}([s^{B\subset Y}])\subset[S^{A\subset X}]$ and this implies $\psi^{\#}\big(\pi^n(B\subset Y)\big)\subset\pi^n(A\subset X)$. Thus the proof is concluded.

By $f^*$: $\pi^n(B\subset Y)\to\pi^n(A\subset X)$ we denote a homomorphism given by the formula $f^*(b)=\varphi^{\#}(b)$ for $b\in\pi^n(B\subset Y)$. Evidently $f^*$ is defined if there exists a map $\varphi$: $A\to B$ satisfying Condition (1). Let us observe that if $g^*$: $\pi^n(C\subset Z)\to\pi^n(B\subset Y)$ is defined for some map $g$: $Y\to Z$, then $(gf)^*$ is also defined and $(gf)^*=f^*g^*$. Indeed, if $\mu$: $B\to C$ is a map such that $i_{C\subset Z}\mu\simeq gi_{B\subset Y}$, then $i_{C\subset Z}(\mu\varphi)\simeq gi_{B\subset Y}\varphi\simeq(gf)i_{A\subset X}$, i.e., $\mu\varphi$ satisfies (1) with respect to the map $gf$. Therefore $(gf)^*(c)=(\mu\varphi)^{\#}(c)=\varphi^{\#}(\mu^{\#}(c))=\varphi^{\#}(g^*(c))=f^*(g^*(c))$ for each $c\in\pi^n(C\subset Z)$. After these considerations it is easy to see that groups $\pi^n(X_1\subset X)$ and $\pi^n(X_2\subset X)$ are isomorphic for every two homotopic $k$-skeletons $X_1$ and $X_2$ of the binormal space $X$ and $k<2n-1$.

If $f$: $A\to B$ is a function and $A_0$, $B_0$ are such subsets of $A$ and $B$, respectively, that $f(A_0)\subset B_0$, then we have the function $f'$: $A_0\to B_0$ defined by $f$. If not stated otherwise, we shall denote $f'$ by $f$.

Let us prove the following

THEOREM 4.2. *Let $\{(X_a, X_{0a}), p_a^{a'}, A\}$ be an inverse system of compact Hausdorff pairs with $(X, X_0)$ as its inverse limit. If all $X_{0a}$ admit the existence of the $n$-th cohomotopy group, then*

$$\Phi\big(\pi^n(X_0\subset X)\big)=\varinjlim\{\pi^n(X_{0a}\subset X_a), p_a^{a'*}, A\},$$

*where $\Phi$: $\pi^n(X_0)\to\varinjlim\{\pi^n(X_{0a}), p_a^{a'\#}, A\}$ is a natural transformation.*

Proof. Let us observe that $\{X_{0a}, [p_a^{a'}], A\}$ is an inverse system associated with $X_0$ (see Theorem 1.5) and by Theorem 3.5 $X_0$ admits the existence if the $n$th cohomotopy group; so we may write $\pi^n(X_0\subset X)$. First let us show

$$\Phi([S^{X_0\subset X}])=\varinjlim\{[S^{X_{0a}\subset X_a}], p_a^{a'\#}, A\}. \tag{1}$$

In fact, if $f\in S^{X_0\subset X}$ and $\tilde f\in S^X$ is an extension of $f$, then by Condition (1.2) there is a map $\tilde f_a$: $X_a\to S$ for some $a\in A$ with $[\tilde f]=[\tilde f_a][p_a]$. We have $(\tilde f_a|_{X_{0a}})p_{0a}=\tilde f_ap_ai_{X_0\subset X}\simeq\tilde fi_{X_0\subset X}=f$, where $p_{0a}$: $X_0\to X_{0a}$ is a natural projection. Hence

$$\Phi([f])=\{[\tilde f_a|_{X_{0a}}]\}\in\varinjlim\{[S^{X_{0a}\subset X_a}], p_a^{a'\#}, A\}$$

(because $\tilde f_a|_{X_{0a}}\in S^{X_{0a}\subset X_a}$). Hence

$$\Phi([S^{X_0\subset X}])\subset\varinjlim\{[S^{X_{0a}\subset X_a}], p_a^{a'\#}, A\}.$$

On the other hand, if $f_a\in S^{X_{0a}\subset X_a}$ and $\tilde f_a\in S^{X_a}$ is an extension of $f_a$, then $\tilde f_ap_a\in S^X$ is an extension of $f_ap_{0a}=\tilde f_ap_ai_{X_0\subset X}$. Therefore $f_ap_{0a}\in S^{X_0\subset X}$ and $\Phi([f_ap_{0a}])=\{[f_a]\}$. Thus $\Phi([S^{X_0\subset X}])\supset\varinjlim\{[S^{X_{0a}\subset X_a}], p_a^{a'\#}, A\}$ and consequently

$$\Phi([S^{X_0\subset X}])=\varinjlim\{[S^{X_{0a}\subset X_a}], p_a^{a'\#}, A\}.$$



Let $r_\omega$: $\pi^n(X_{0\omega}\subset X_\omega)\to\varinjlim\{\pi^n(X_{0a}\subset X_a), p_a^{a'*}, A\}$ be a natural projection. Since $\Phi$ and $r_\omega$ are homomorphisms, $r_\omega^{-1}\big(\Phi(\pi^n(X_0\subset X))\big)$ is a subgroup which contains by (1) the set $[S^{X_{0\omega}\subset X_\omega}]$. Since $[S^{X_{0\omega}\subset X_\omega}]$ generates the group $\pi^n(X_{0\omega}\subset X_\omega)$, we infer $\pi^n(X_{0\omega}\subset X_\omega)=r_\omega^{-1}\big(\Phi(\pi^n(X_0\subset X))\big)$ for each $\omega\in A$. Consequently

$$\Phi\big(\pi^n(X_0\subset X)\big)\supset\varinjlim\{\pi^n(X_{0a}\subset X_a), p_a^{a'*}, A\}. \tag{2}$$

On the other hand, by (1) we infer that $\Phi^{-1}\big(\varinjlim\{\pi^n(X_{0a}\subset X_a), p_a^{a'*}, A\}\big)$ is a subgroup of $\pi^n(X_0)$ which contains the set $[S^{X_0\subset X}]$. Therefore

$$\Phi^{-1}\big(\varinjlim\{\pi^n(X_{0a}\subset X_a), p_a^{a'*}, A\}\big)\supset\pi^n(X_0\subset X)$$

and by (2)

$$\Phi\big(\pi^n(X_0\subset X)\big)=\varinjlim\{\pi^n(X_{0a}\subset X_a), p_a^{a'*}, A\}.$$

Thus the proof is finished.

Let $A$ be a closed subset of a topological space $X$ with $\dim A\leqslant k$. We say that $A$ is the *outer homotopic $k$-skeleton* provided for every map $f$: $Z\to X$ with $\dim\omega\leqslant k$ there exists a $g$: $Z\to A$ such that $[i_{A\subset X}g]=[f]$.

LEMMA 4.3. *Every outer homotopic $k$-skeleton of $X$ is a homotopic $k$-skeleton of $X$. If $X$ has an outer homotopic $k$-skeleton, then every homotopic $k$-skeleton of $X$ is an outer homotopic $k$-skeleton of $X$.*

Proof. Let $X_1\subset X$ be a closed subset of $X$ with $\dim X_1\leqslant k$. Then there exists a map $g$: $X_1\to A$ such that $i_{A\subset X}g\simeq i_{X_1\subset X}$, i.e., $A\underset{h}{\geqslant}X_1$, in $X$. Thus $A$ is a homotopic $k$-skeleton of $X$.

Let $B$ be an arbitrary homotopic $k$-skeleton of $X$. Since $B\underset{h}{\geqslant}A$ in $X$, there is a map $a$: $A\to B$ with $i_{B\subset X}a\simeq i_{A\subset X}$. Let $f$: $Z\to X$ be a map where $\dim Z\leqslant k$. Since $A$ is an outer homotopic $k$-skeleton of $X$, then there is a map $f'$: $Z\to A$ such that $f\simeq i_{A\subset X}f'$. Setting $g=af'$: $Z\to B$, we get $i_{B\subset X}g=i_{B\subset X}af'\simeq i_{A\subset X}f'\simeq f$. Thus $B$ is an outer homotopic $k$-skeleton of $X$ and this completes the proof.

LEMMA 4.4. *Let $A$ be a closed subset of a metrizable (compact metrizable) space $X$ with $\dim A\leqslant k$. If for every metrizable (compact metrizable) space $Z$ with $\dim Z\leqslant k$ and for each map $f$: $Z\to X$ there is a $g$: $Z\to A$ such that $i_{A\subset X}g\simeq f$, then $A$ is an outer homotopic $k$-skeleton of $X$.*

Proof. Let $Z$ be any topological space with $\dim Z\leqslant k$. If $f$: $Z\to X$, then by Lemma 2.2 of [14] there is a metrizable space $T$ and maps $f'$: $Z\to T$, $f''$: $T\to X$ such that $\dim T\leqslant k$ and $f''f'=f$. So in the first case there is a $g'$: $T\to A$ such that $i_{A\subset X}g'\simeq f''$ and consequently $i_{A\subset X}(g'f')\simeq f''f'=f$. Thus $g=g'f'$: $Z\to A$ satisfies the required condition and it remains only to consider the second case. If $X$ is compact, then $f''$ has an extension

$\tilde{f}$: $\beta T \to X$, where $\beta T$ denotes the Čech-Stone compactification of $T$. It is well known that $\dim \beta T = \dim T \leqslant k$. Let $h$: $\beta T \to M$ and $s$: $M \to X$ be maps, where $M$ is a metrizable space with $\dim M \leqslant \dim \beta T$, such that $sh = \tilde{f}$ and $h$ is onto (for existence see [14], Lemma 2.2). Hence $M$ is a compact metrizable space and there exists a map $s'$: $M \to A$ such that $i_{A \subset X} s' \simeq s$. Setting $g = s' h i_{T \subset \beta T} f'$: $Z \to A$, we get

$$i_{A \subset X} g = i_{A \subset X} s' h i_{T \subset \beta T} f' \simeq s h i_{T \subset \beta T} f' = \tilde{f} i_{T \subset \beta T} f' = f'' f' = f\,,$$

i.e., $g$ satisfies the required conditions. Thus $A$ is the outer homotopic $k$-skeleton of $X$ and the proof is concluded.

Let us recall that a compact metric space $X$ satisfies condition $(\Delta)$ (we write $X \in (\Delta)$) provided for every point $x \in X$ and for every neighborhood $U$ of $x$ there is a neighborhood $V$ of $x$ such that each compact subset $A \subset V$ is contractible to a point in a subset of $U$ having dimension less than or equal to $\dim A + 1$ (comp. [1], p. 163).

K. Borsuk proved in [3] that every ANR-set $X \in (\Delta)$ has a homotopic $k$-skeleton for every $k = 0, 1, \ldots$ Here we prove the stronger result.

THEOREM 4.5. *If $X \in (\Delta)$ and $X \in \mathrm{Ob}\,\underline{W}$, then for each $k = 0, 1, \ldots$ $X$ has an outer homotopic $k$-skeleton.*

Proof. Since $X$ is a compact Hausdorff space and $X \in \mathrm{Ob}\,\underline{W}$, there exist a finite CW complex $W$ and maps $h$: $W \to X$, $g$: $X \to W$ such that $hg \simeq \mathrm{id}_X$. Since the subset of $X^{W^k}$ consisting of all maps $f \in X^{W^k}$ which satisfy the condition $\dim f(W^k) \leqslant \dim W^k$ is dense in the space $X^{W^k}$ (see [1], p. 164), there exists a sequence of maps $f_n$: $W^k \to X$ converging to $hi_{W^k \subset W}$ in a compact-open topology on $X^{W^k}$ such that $\dim f_n(W^k) \leqslant k$ for each $n = 0, 1, \ldots$ Therefore the sequence of maps $gf_n$: $W^k \to W$ converges to $ghi^{W^k \subset W}$ in a compact-open topology on $W^{W^k}$, and it follows from $W \in \mathrm{ANR}$ that $gf_m \simeq ghi_{W^k \subset W}$ for some $m$. Hence $f_m \simeq ghf_m \simeq hghi_{W^k \subset W} \simeq hi_{W^k \subset W}$. We show that $A = f_m(W^k)$ is an outer homotopic $k$-skeleton of $X$. Evidently $\dim A = \dim f_m(W^k) \leqslant k$. Let $f$: $Z \to X$ be a map with $\dim Z \leqslant k$. Now $gf$: $Z \to W$ and by Lemma 2.1 there is an $s$: $Z \to W^k$ such that $i_{W^k \subset W} s \simeq gf$. If we set $f' = f_m s$: $Z \to X$, then we get $f'(Z) = f_m s(Z) \subset f_m(W^k) = A$ and $f_m s \simeq hi_{W^k \subset W} s \simeq hgf \simeq f$. Thus $A$ is an outer homotopic $k$-skeleton of $X$ and this completes the proof.

Let us prove the following

THEOREM 4.6. *Let $A$ be an outer homotopic $k$-skeleton of $X \in \mathrm{Ob}\,\underline{W}$ and let $Z$ be a closed subset of a normal space $Y$, with $\mathrm{Sd}\,Z \leqslant k$. If at least one of the spaces $X$ and $Y$ is a compact Hausdorff space, then for each map $f$: $Y \to X$ there is a $g$: $Y \to X$ homotopic to $f$ with $g(Z) \subset A$.*

Proof. Let $W$ be a $CW$ complex for which there exist maps $h$: $W \to X$, $s$: $X \to W$ with $hs \simeq \mathrm{id}_X$. Since at least one of the spaces $X$ and $Y$ is a compact Hausdorff space, there is a finite subcomplex $V$ of $W$ containing $sf(Y)$. By $\beta$: $Y \to V$ we denote a map given by the formula $\beta(y) = sf(y)$ for $y \in Y$. By Lemma 2.1 there exists a map $\alpha$: $Z \to V^k$ such that $i_{V^k \subset V}\alpha$ is homotopic to $\beta i_{Z \subset Y}$. Hence in view of $V \in \mathrm{ANR}$ there exists an extension $\tilde{\alpha}$: $Y \to V$ of a map $i_{V^k \subset V}\alpha$ with $\tilde{\alpha} \simeq \beta$. Since $A$ is the outer homotopic $k$-skeleton of $X$, there is a map $h'$: $V^k \to A$ such that $i_{A \subset X} h' \simeq hi_{V^k \subset V}$. Now, by the homotopy extension property of the pair $(W, V^k)$ (see [10], Theorem 7.2, p. 68) with respect to any space, there exists an extension $\tilde{h}$: $W \to X$ of the map $i_{A \subset X} h'$: $V^k \to X$ homotopic to $h$: $W \to X$. Setting $g = \tilde{h} i_{V \subset W} \tilde{\alpha}$: $Y \to X$, we get $g = \tilde{h} i_{V \subset W} \tilde{\alpha} \simeq h i_{V \subset W} \beta = \tilde{h} sf \simeq hsf \simeq f$ and $g(Z) = \tilde{h} i_{V \subset W} \tilde{\alpha}(Z) \subset \tilde{h}(V^k) = h'(V^k) \subset A$, i.e., $g$ satisfies the required conditions. Thus the proof is concluded.



An immediate consequence of Theorem 4.6 and Lemma 4.3 is the following

COROLLARY 4.7 ([8], Theorem (3.1)). *Let $Y$ be a compact ANR-set satisfying condition $(\Delta)$ and let $Y^n \in \mathrm{ANR}$ be a homotopic $n$-skeleton of $Y$. If $\mathrm{Fd}(X) \leqslant n$, then for every map $f$: $X \to Y$ there exists map $g$: $X \to Y$ homotopic to $f$ and such that $g(X) \subset Y^n$.*

THEOREM 4.8. *Let $\{(X_\alpha, X_{0\alpha}), p_\alpha^{\alpha'}, A\}$ be an inverse system of compact Hausdorff pairs with $(X, X_0)$ as its inverse limit. If $X, X_0 \in \mathrm{Ob}\,\underline{W}$ and $X_{0\alpha}$ is an outer homotopic $k$-skeleton of $X_\alpha$ for each $\alpha \in A$, then $X_0$ is the outer homotopic $k$-skeleton of $X$.*

Proof. Since the inverse systems $\{X_\alpha, [p_\alpha^{\alpha'}], A\}$ and $\{X_{0\alpha}, [p_{0\alpha}^{0\alpha'}], A\}$ are associated with $X$ and $X_0$, respectively ($p_{0\alpha}^{0\alpha'}$: $X_{0\alpha'} \to X_{0\alpha}$ is the map defined by $p_\alpha^{\alpha'}$: $X_{\alpha'} \to X_\alpha$), there exist an $\alpha \in A$ and maps $f_\alpha$: $X_\alpha \to X$, $g_\alpha$: $X_{0\alpha} \to X_0$ such that $f_\alpha p_\alpha \simeq \mathrm{id}_X$ and $g_\alpha p_{0\alpha} \simeq \mathrm{id}_{X_0}$. We have $f_\alpha i_{X_{0\alpha} \subset X_\alpha} p_{0\alpha} = f_\alpha p_\alpha i_{X_0 \subset X} \simeq \mathrm{id}_X i_{X_0 \subset X} \simeq i_{X_0 \subset X} g_\alpha p_{0\alpha}$ and by Condition (1.3) there is an $\beta \in A$, $\alpha \leqslant \beta$ such that $f_\alpha i_{X_{0\alpha} \subset X_\alpha} p_{0\alpha}^{0\beta} \simeq i_{X_0 \subset X} g_\alpha p_{0\alpha}^{0\beta}$. Let $f$: $Z \to X$ be a continuous map with $\dim Z \leqslant k$. Then there is a map $r$: $Z \to X_{0\beta}$ such that $i_{X_{0\beta} \subset X_\beta} r \simeq p_\beta f$. Setting $g = g_\alpha p_{0\alpha}^{0\beta} r$: $Z \to X_0$, we get

$$\begin{aligned} i_{X_0 \subset X} g = i_{X_0 \subset X} g_\alpha p_{0\alpha}^{0\beta} r &\simeq f_\alpha i_{X_{0\alpha} \subset X_\alpha} p_{0\alpha}^{0\beta} r = f_\alpha p_\alpha^\beta i_{X_{0\beta} \subset X_\beta} r \\ &\simeq f_\alpha p_\alpha^\beta p_\beta f = f_\alpha p_\alpha f \simeq \mathrm{id}_X f = f\,. \end{aligned}$$

Thus $g$ satisfies the required condition and this shows that $X_0$ is actually an outer homotopic $k$-skeleton of $X$.

## 5. A generalization of cohomotopy groups.

Let $k$ be a natural number. By $\underline{HO}_k$ we denote a full subcategory of $\underline{H}$ whose objects are binormal spaces having outer homotopic $k$-skeletons. Every space $X \in \mathrm{Ob}\,\underline{HO}_k$ we consider with a fixed outer homotopic $k$-skeleton $X_0$. By Lemma 4.1 we obtain the following generalization of Corollary 3 in [9].

COROLLARY 5.1. *For each natural number $n$ with $k < 2n-1$ there is a contravariant functor $\pi_k^n$ from $\underline{HO}_k$ to the category $G$ of Abelian groups such* *that* $\pi_k^n(X) = \pi^n(X_0 \subset X)$ *for every* $X \in \mathrm{Ob}\,\underline{HO}_k$ *and* $\pi_k^n([f]) = f^*$ *for* $f\colon X \to Y$, $X, Y \in \mathrm{Ob}\,\underline{HO}_k$.

It is clear that we can consider $\underline{CW}$ as a subcategory of $\mathrm{Inv}\,\underline{CW}$. We show that by this assumption one can extend the functor $\pi_k^n$ considered only on the category $\underline{CW}$ to the functor $\pi_k^n\colon \mathrm{Inv}\,\underline{CW} \to \underline{G}$.

Let $\underline{X} = \{X_a, p_a^{a'}, A\} \in \mathrm{Ob}\,\mathrm{Inv}\,\underline{CW}$. By Corollary 5.1 $\{\pi_k^n(X_a), p_a^{a'*}, A\}$ is a direct system of Abelian groups. Let us put $\pi_k^n(\underline{X}) = \varinjlim\{\pi_k^n(X_a), p_a^{a'*}, A\}$. Now, if $\underline{f}\colon \underline{X} \to \underline{Y} = \{Y_\beta, p_\beta^{\beta'}, B\}$ is a map of systems, then $\pi_k^n([\underline{f}])\colon \pi_k^n(\underline{Y}) \to \pi_k^n(\underline{X})$ (by $[\underline{f}]$ we denote the homotopy class of $\underline{f}$) is defined by the formula

$$\pi_k^n([\underline{f}])(b) = \{\pi_k^n([f_\beta])(b_\beta)\},$$

where $b_\beta$ is a representative of $b$, i.e., $b = \{b_\beta\}$. It is easy to see that $\pi_k^n([\underline{f}])$ is well defined and $\pi_k^n$ is a contravariant functor from $\mathrm{Inv}\,\underline{CW}$ to $\underline{G}$.

Now, let us observe that Theorem 1.6 establishes (in view of Theorem 1.4) the existence of a covariant functor $F$ from $\underline{S}$ to $\mathrm{Inv}\,\underline{CW}$ such that $F(X)$ is associated with $X$. The functor $F$ will be called the *association functor*.

LEMMA 5.2. *Every two association functors* $F, G\colon \underline{S} \to \mathrm{Inv}\,\underline{CW}$ *are natural equivalent.*

Proof. Let $h_X \in \mathrm{Inv}\,\underline{CW}(F(X), G(X))$ correspond to the identity shaping $\underline{1}_X\colon X \to X$ (see Theorem 1.6). If $\underline{f}\colon X \to Y$ is a shaping, then by Theorem 1.6 $G(\underline{f})h_X$ and $h_Y F(\underline{f})$ corresponds to $\underline{f}\underline{1}_X = \underline{f}$ and $\underline{1}_Y \underline{f} = \underline{f}$, respectively. Consequently $G(\underline{f})h_X = h_Y F(\underline{f})$, i.e., the diagram

$$\begin{tikzcd}
F(X) \arrow[r, "h_X"] \arrow[d, "F(\underline{f})"'] & G(X) \arrow[d, "G(\underline{f})"] \\
F(Y) \arrow[r, "h_Y"'] & G(Y)
\end{tikzcd}$$

is commutative for each shaping $\underline{f}\colon X \to Y$. Evidently, if

$$h_X' \in \mathrm{Inv}\,\underline{CW}(G(X), F(X))$$

corresponds to $\underline{1}_X\colon X \to X$, then $h_X' h_X = 1_{F(X)}$ and $h_X h_X' = 1_{G(X)}$. Therefore $h_X$ is an isomorphism and this completes the proof.

Thus, taking the composition of functors $F$ and $\pi_k^n$, we obtain the contravariant functor $\pi_k^n F\colon \underline{S} \to \underline{G}$. Evidently for each two association functors $F$ and $G$ the functors $\pi_k^n F$ and $\pi_k^n G$ are natural equivalent.

THEOREM 5.3. *If* $k < 2n-1$ *and* $\mathrm{Sh}\,X = \mathrm{Sh}\,Y$, *then* $\pi_k^n F(X)$ *is isomorphic to* $\pi_k^n F(Y)$.

Proof. $\mathrm{Sh}\,X = \mathrm{Sh}\,Y$ means that $X$ is isomorphic to $Y$ in the category $\underline{S}$ and consequently $\pi_k^n F(X)$ is isomorphic to $\pi_k^n F(Y)$ in the category $\underline{G}$.



By $\underline{S}_k$ we denote the full subcategory of $\underline{S}$ whose objects are spaces $X$ with $\mathrm{Sd}\,X \leqslant k$.

THEOREM 5.4. *Let* $k < 2n-1$. *If we consider* $\pi_k^n F$ *and the* $n$-*th cohomotopy functor only on* $\underline{S}_k$, *then they are natural equivalent.*

Proof. By Theorems 1.4 and 2.2 there is an association functor $H$ such that for each $X \in \mathrm{Ob}\,\underline{S}_k$ if $H(X) = \{X_a, [p_a^{a'}], A\}$, then $\dim X_a \leqslant k$ for every $a \in A$. Now, by Theorem 3.5 the natural transformation

$$\Phi_X\colon \pi^n(X) \to \varinjlim\{\pi^n(X_a), p_a^{a'\#}, A\} = \pi_k^n H(X)$$

is an isomorphism. Let $X, Y \in \mathrm{Ob}\,\underline{S}_k$ and $\underline{f}\colon X \to Y$ be a shaping. By $\underline{g}$ we denote an arbitrary map from $H(X)$ to $H(Y)$ such that $[\underline{g}] = H(\underline{f})$. Let $\eta\colon Y \to S$ and $\Phi_Y([\eta]) = \{[\eta_\beta]\}$ (i.e., $[\eta] = [\eta_\beta][q_\beta]$). Then

$$\pi_k^n([\underline{g}])\Phi_Y([\eta]) = \pi_k^n([\underline{g}])\{[\eta_\beta]\} = \{[\eta_\beta][g_\beta]\}$$

and it follows by $[\underline{g}] = H(\underline{f})$ that $[\eta_\beta g_\beta p_{\sigma(\beta)}] = \underline{f}^S([\eta])$. Consequently

$$\{[\eta_\beta g_\beta]\} = \Phi_X(\underline{f}^S([\eta])) = \Phi_X(\underline{f}^\#([\eta])).$$

Thus the diagram

$$\begin{tikzcd}
\pi^n(X) \arrow[r] & \pi_k^n H(X) \\
\pi^n(Y) \arrow[r] \arrow[u, "\underline{f}^\#"] & \pi_k^n H(Y) \arrow[u, "\pi_k^n H(\underline{f})"']
\end{tikzcd}$$

for each shaping $\underline{f}\colon X \to Y$. Now $H$ and $F$ are natural equivalent and this completes the proof.

As an immediate consequence of Theorem 5.4 we have the following

COROLLARY 5.5. *If* $\mathrm{Sd}\,X \leqslant k < 2n-1$, *then the groups* $\pi_k^n F(X)$ *and* $\pi^n(X)$ *are isomorphic.*

Thus the groups $\pi_k^n F(X)$ are actually the generalization of groups $\pi^n(X)$.

We say that an association functor $H$ is a *special association functor* provided $H(W) = W$ for each $W \in \mathrm{Ob}\,\underline{CW}$ (we recall that $\underline{CW}$ is considered as a full subcategory of $\mathrm{Inv}\,\underline{CW}$). Evidently, if $H$ is a special association functor and $\eta \in \underline{CW}(W, V)$, then $\eta^*\colon \pi^n(V^k \subset V) \to \pi^n(W^k \subset W)$ is equal to $\pi_k^n H(\eta)\colon \pi_k^n H(V) \to \pi_k^n H(W)$. Let us prove the following

LEMMA 5.6. *Let* $H$ *be a special association functor. If* $H(X) = \{X_a, [p_a^{a'}], A\}$, *then for each* $\beta \in A$ *the natural projection* $r_\beta\colon \pi^n(X_\beta^k \subset X_\beta) \to \varinjlim\{\pi^n(X_a^k \subset X_a), p_a^{a'*}, A\} = \pi_k^n H(X)$ *is equal to* $\pi_k^n H(S[p_\beta])\colon \pi_k^n H(X_\beta) \to \pi_k^n H(X)$.

Proof. Let $\underline{p}_\beta$: $\{X_a, [p_a^{a'}], A\} \to X_\beta$ be a map of systems which consist only on the homotopy class $[\mathrm{id}_{X_\beta}]$. It is clear that $[\underline{p}_\beta] = H(S[p_\beta])$. Therefore

$$\pi_k^n H(S[p_\beta])(c) = \pi_k^n([\underline{p}_\beta])(c) = \{\mathrm{id}_{X_\beta}^*(c)\} = \{c\} = r_\beta(c)$$

for each $c \in \pi_k^n H(X_\beta) = \pi^n(X_\beta^k \subset X_\beta)$ and this completes the proof.

Let $\{X_a, [p_a^{a'}], A\}$ be an inverse system in $\underline{H}$ associated with $X$. If $F$ is an association functor, then each natural projection $p_a$: $X \to X_a$ induces a homomorphism $\pi_k^n F(S[p_a])$: $\pi_k^n F(X_a) \to \pi_k^n F(X)$. Moreover, $a \leqslant \beta$ implies $\pi_k^n F(S[p_a]) = \pi_k^n F(S[p_\beta]) \to \pi_k^n F(S[p_a^\beta])$. Therefore the homomorphisms $\pi_k^n F(S[p_a])$ induce a homomorphism

$$\psi_F\colon\ \varinjlim\{\pi_k^n F(X_a),\ \pi_k^n F(S[p_a^{a'}]),\ A\} \to \pi_k^n F(X)\,.$$

THEOREM 5.7. *$\psi_F$: $\varinjlim\{\pi_k^n F(X_a), \pi_k^n F(S[p_a^{a'}]), A\} \to \pi_k^n F(X)$ is an isomorphism for each association functor $F$.*

Proof. First we show that $\psi_H$ is an isomorphism for every special association functor $H$.

Let $H(X) = \{W_\beta, [q_\beta^{\beta'}], B\}$ and let $a \in \pi_k^n H(X)$. Then there exists a $\beta \in B$ such that $a = \{a_\beta\}$ for some $a_\beta \in \pi^n(W_\beta^k \subset W_\beta)$ and by Lemma 5.6 $a = \pi_k^n H(S[q_\beta])(a_\beta)$. Now, by Condition (1.2) there exist on $a \in A$ and a map $f$: $X_a \to W_a$ such that $[fp_a] = [q_a]$. Therefore

$$\begin{aligned} a &= \pi_k^n H(S[q_\beta])(a_\beta) = \pi_k^n H(S[fp_a])(a_\beta) \\ &= \pi_k^n H(S[p_a])\pi_k^n H(S[f])(a_\beta) = \psi_H\{\pi_k^n H(S[f])(a_\beta)\}\,. \end{aligned}$$

Thus we infer that $\psi_H$ is an epimorphism. It remains only to show that $\psi_H$ is a monomorphism. So let us suppose that $\pi_k^n H(S[p_a])(c) = 0$ for some element $c \in \pi_k^n H(X_a)$. We shall prove that $\{c\} = 0$, i.e., there exists an $a' \geqslant a$ such that $\pi_k^n H(S[p_a^{a'}])(c) = 0$. Let $H(X_a) = \{V_\omega, [r_\omega^{\omega'}], C\}$. By Lemma 5.6 there exists $d \in \pi_k^n H(V_\omega)$ for some $\omega \in C$ such that $\pi_k^n H(S[r_\omega])(d) = c$. Now, $r_\omega p_\omega$: $X \to V_\omega$ is a continuous map and by Condition (1.2) there exist a $\beta' \in B$ and a map $g$: $W_{\beta'} \to V_\omega$ such that $[gq_{\beta'}] = [r_\omega p_a]$. Hence

$$\begin{aligned} \pi_k^n H(S[q_{\beta'}])\pi_k^n H(S[g])(d) &= \pi_k^n H(S[gq_{\beta'}])(d) = \pi_k^n H(S[r_\omega p_a])(d) \\ &= \pi_k^n H(S[p_a])\pi_k^n H(S[r_\omega])(d) = \pi_k^n H(S[p_a])(c) = 0\,, \end{aligned}$$

and since

$$\pi_k^n H(S[q_{\beta'}])\colon\ \pi^n(W_{\beta'}^k \subset W_{\beta'}) \to \pi_k^n H(X)$$

is a natural projection (see Lemma 5.6), there is a $\beta'' \in B$, $\beta' \leqslant \beta''$, such that

$$0 = (q_{\beta'}^{\beta''})^*\pi_k^n H(S[g])(d) = \pi_k^n H(S[q_{\beta'}^{\beta''}])\pi_k^n H(S[g])(d) = \pi_k^n H(S[gq_{\beta'}^{\beta''}])(d)\,.$$

By Condition (1.2) there exist a $\chi \in A$ and a map $g_\chi$: $X_\chi \to W_{\beta''}$ such that $[g_\chi p_\chi] = [q_{\beta''}]$. Therefore $[gq_{\beta'}^{\beta''}g_\chi p_\chi] = [gq_{\beta'}^{\beta''}q_{\beta''}] = [gq_{\beta'}] = [r_\omega p_a]$ and by



Condition (1.3) there is an $a' \in A$, $a, \chi \leqslant a'$, such that $[gq_{\beta'}^{\beta''}g_\chi p_\chi^{a'}] = [r_\omega p_a^{a'}]$. Hence

$$\begin{aligned} \pi_k^n H(S[p_a^{a'}])(c) &= \pi_k^n H(S[p_a^{a'}])\pi_k^n H(S[r_\omega])(d) = \pi_k^n H(S[r_\omega p_a^{a'}])(d) \\ &= \pi_k^n H(S[gq_{\beta'}^{\beta''}g_\chi p_\chi^{a'}])(d) = \pi_k^n H(S[g_\chi p_\chi^{a'}])\pi_k^n H(S[gq_{\beta'}^{\beta''}])(d) = 0 \end{aligned}$$

(because $\pi_k^n H(S[gq_{\beta'}^{\beta''}])(d) = 0$). Thus $\psi_H$ is a monomorphism.

Let $F$ be an arbitrary association functor. Then by Lemma 5.2 the functors $F$ and $H$ are natural equivalent, i.e., there exist isomorphisms $h_\chi$: $F(X) \to H(X)$ such that the diagram

$$\begin{tikzcd} F(X) \arrow[r, "h_X"] \arrow[d, "F(f)"'] & H(X) \arrow[d, "H(f)"] \\ F(Y) \arrow[r, "h_Y"'] & H(Y) \end{tikzcd}$$

is commutative for each shaping $f$: $X \to Y$. Therefore we can define a homomorphism

$$\varphi_{H,F}\colon\ \varinjlim\{\pi_k^n H(X_a),\ \pi_k^n H(S[p_a^{a'}]),\ A\} \to \varinjlim\{\pi_k^n F(X_a),\ \pi_k^n F(S[p_a^{a'}]),\ A\}$$

by the formula $\varphi_{H,F}(a) = \{\pi_k^n(h_{X_a})(a_a)\}$, where $a_a$ is a representative of $a$. It is easy to see that $\varphi_{H,F}$ is an isomorphism and $\psi_F \cdot \varphi_{H,F} = \pi_k^n(h_X)\psi_H$. Since $\psi_H$, $\pi_k^n(h_X)$, $\varphi_{H,F}$ are isomorphisms, we infer that $\psi_F$ is also an isomorphism. Thus the proof is concluded.

## 6. Relations between $\pi_k^n(X)$ and $\pi_k^n F(X)$.

COROLLARY 6.1. *If $X \in \mathrm{Ob}\,\underline{HO}_k$ and $X \in \mathrm{Ob}\,\underline{W}$, then the groups $\pi_k^n(X)$ and $\pi_k^n F(X)$ are isomorphic.*

Proof. Let $W$ be a $CW$ complex having the same homotopy type as $X$. Then the groups $\pi_k^n F(X)$ and $\pi^n(W^k \subset W)$ are isomorphic (because $\pi^n(W^k \subset W) = \pi_k^n H(X)$ for each special association functor $H$). On the other hand, by Corollary 5.1 the groups $\pi_k^n(X)$ and $\pi^n(W^k \subset W)$ are isomorphic and therefore the groups $\pi_k^n(X)$ and $\pi_k^n F(X)$ are isomorphic. Thus the proof is concluded.

Now, we show that if a binormal space $X$ has a homotopic $k$-skeleton $X_0$ and $k \leqslant 2n-1$, then the group $\pi_k^n F(X)$ is "greater" in a certain sense than Borsuk's generalized cohomotopy group $\pi_k^n(X) = \pi^n(X_0 \subset X)$.

THEOREM 6.2. *If $X_0$ is a homotopic $k$-skeleton of a binormal space $X$ and $k \leqslant 2n-1$, then there is an epimorphism $h$: $\pi_k^n F(X) \to \pi^n(X_0 \subset X)$ for every association functor $F$.*

Proof. Let $F(X) = \{X_a, [p_a^{a'}], A\}$. Since $X_a^k$ is the outer homotopic $k$-skeleton of a $CW$ complex $X_a$, there exist maps $\varphi_a$: $X_0 \to X_a^k$ such that $i_{X_a^k \subset X_a}\varphi_a \simeq p_a i_{X_0 \subset X}$. Hence the homomorphisms $p_a^*$: $\pi^n(X_a^k \subset X_a) \to \pi^n(X_0 \subset X)$

are defined. Moreover, $\alpha \leqslant \beta$ implies $p_\alpha^* = p_\beta^* p_\alpha^{\beta *}$ and therefore the homomorphisms $p_\alpha^*$ induce a homomorphism $h\colon \varinjlim \{\pi^n(X_\alpha^k \subset X_\alpha), p_\alpha^{\alpha' *}, A\} = \pi_k^n F(X) \to \pi^n(X_0 \subset X)$.

Let $s \in S^{X_0 \subset X}$ and let $\tilde{s} \in S^X$ be an extension of $s$. Then there exist an $\alpha \in A$ and a map $s_\alpha\colon X_\alpha \to S$ such that $s_\alpha p_\alpha \simeq \tilde{s}$ (by Condition (1.2)). Setting $g = s_\alpha i_{X_\alpha^k \subset X_\alpha}\colon X_\alpha^k \to S$, we get $[g] \in [S^{X_\alpha^k \subset X_\alpha}] \subset \pi^n(X_\alpha^k \subset X_\alpha)$ and

$$p_\alpha^*([g]) = \varphi_\alpha^{\#}([g]) = [g\varphi_\alpha] = [s_\alpha i_{X_\alpha^k \subset X_\alpha} \varphi_\alpha] = [s_\alpha p_\alpha i_{X_0 \subset X}] = [\tilde{s} i_{X_0 \subset X}] = [s].$$

Thus we infer $h(\pi_k^n F(X)) \supset [S^{X_0 \subset X}]$, and since $[S^{X_0 \subset X}]$ generates the group $\pi^n(X_0 \subset X)$, $h$ is an epimorphism.

THEOREM 6.3. *Let $\underline{X} = \{(X_\alpha, X_{0\alpha}), p_\alpha^{\alpha'}, A\}$ be an inverse system such that each $X_\alpha$ is a finite CW complex and $X_{0\alpha} = X_\alpha^k$. If $(X, X_0)$ is the inverse limit of $\underline{X}$ and $X$ has a homotopic $k$-skeleton $X_1$, then the group $\pi_k^n F(X)$ is isomorphic to the group $\pi^n(X_1 \subset X) = \pi_k^n(X)$.*

Proof. Since the inverse system $\{X_\alpha, [p_\alpha^{\alpha'}], A\}$ in $\underline{W}$ is associated with $X$, we infer that the group $\pi_k^n F(X)$ is isomorphic to the group $\varinjlim \{\pi^n(X_{0\alpha} \subset X_\alpha), p_\alpha^{\alpha' *}, A\}$. On the other hand, by Theorem 4.2 we know that

$$\Phi(\pi^n(X_0 \subset X)) = \varinjlim \{\pi^n(X_{0\alpha} \subset X_\alpha), p_\alpha^{\alpha' *}, A\}$$

where the isomorphism $\Phi\colon \pi^n(X_0) \to \varinjlim \{\pi^n(X_{0\alpha}), p_\alpha^{\alpha' \#}, A\}$ is a natural transformation. Let

$$h\colon \varinjlim \{\pi^n(X_{0\alpha} \subset X_\alpha),\ p_\alpha^{\alpha' *},\ A\} \to \pi^n(X_1 \subset X)$$

be a homomorphism which is described in the proof of Theorem 6.2. The homomorphism $g\colon \pi^n(X_0 \subset X) \to \pi^n(X_1 \subset X)$ is given by the formula $g(a) = h(\Phi(a))$ for $a \in \pi^n(X_0 \subset X)$. Since $X_1$ is a homotopic $k$-skeleton of $X$, there is a map $\eta\colon X_0 \to X_1$ such that $i_{X_0 \subset X} \simeq i_{X_1 \subset X}\eta$. Therefore there exists a homomorphism $\mathrm{id}_X^*\colon \pi^n(X_1 \subset X) \to \pi^n(X_0 \subset X)$. We shall show that $\mathrm{id}_X^* g = 1_{\pi^n(X_0 \subset X)}$ and $g\,\mathrm{id}_X^* = 1_{\pi^n(X_1 \subset X)}$.

Let $s \in S^{X_0 \subset X}$ and let $\tilde{s} \in S$ be an extension of $s$. Then $g([s]) = [s_\alpha i_{X_{0\alpha} \subset X_\alpha} \varphi_\alpha]$, where $s_\alpha\colon X_\alpha \to S$ and $\varphi_\alpha\colon X_1 \to X_{0\alpha}$ are maps such that $s_\alpha p_\alpha \simeq \tilde{s}$ and $i_{X_{0\alpha} \subset X_\alpha} \varphi_\alpha \simeq p_\alpha i_{X_1 \subset X}$. Since $s_\alpha i_{X_{0\alpha} \subset X_\alpha} \varphi_\alpha \simeq s_\alpha p_\alpha i_{X_1 \subset X} \simeq \tilde{s} i_{X_1 \subset X}$, we obtain $\mathrm{id}_X^* g([s]) = \mathrm{id}_X^*([\tilde{s} i_{X_1 \subset X}]) = [\tilde{s} i_{X_1 \subset X}\eta] = [\tilde{s} i_{X_0 \subset X}] = [s]$. Thus $\mathrm{id}_X^* g = 1_{\pi^n(X_0 \subset X)}$ (because the set $[S^{X_0 \subset X}]$ generates the group $\pi^n(X_0 \subset X)$).

Let $u \in S^{X_1 \subset X}$ and let $\tilde{u} \in S^X$ be an extension of $u$. Then $g\,\mathrm{id}_X^*([u]) = g([u\eta]) = g([\tilde{u} i_{X_1 \subset X}\eta]) = g([\tilde{u} i_{X_0 \subset X}]) = [u_\beta i_{X_{0\beta} \subset X_\beta} \varphi_\beta]$, where $u_\beta\colon X_\beta \to S$ and $\varphi_\beta\colon X_1 \to X_{0\beta}$ are maps such that $u_\beta p_\beta \simeq \tilde{u}$ and $i_{X_{0\beta} \subset X_\beta} \varphi_\beta \simeq p_\beta i_{X_1 \subset X}$. Hence $u_\beta i_{X_{0\beta} \subset X_\beta} \varphi_\beta \simeq u_\beta p_\beta i_{X_1 \subset X} \simeq \tilde{u} i_{X_1 \subset X} = u$, i.e., $g\,\mathrm{id}_X^*([u]) = [u]$. Thus $g\,\mathrm{id}_X^* = 1_{\pi^n(X_1 \subset X)}$ and this completes the proof (because $\pi_k^n F(X)$ is isomorphic to the group $\pi^n(X_0 \subset X)$).



**7. An example.** In this section we give an example of ANR-set $X$ such that:

(7.1) $X$ has a homotopic 2-skeleton which is not an outer homotopic 2-skeleton.

(7.2) $X$ has the homotopy type of a 2-dimensional sphere $S$.

(7.3) The groups $\pi_2^2(X)$ and $\pi_2^2(S)$ are not isomorphic.

(7.4) This example gives a negative answer to Problems (9.1), (9.3) and (9.5) of [6].

Let $L$ be a simple arc lying in a 2-dimensional sphere $S$ and let $f$ be a continuous map of $L$ onto a 3-dimensional cube $Q^3$, disjoint with $S$. If we match every point $x \in L$ with point $f(x) \in Q^3$, we get from the set $S \cup Q^3$ a 3-dimensional space $X = S \cup_f Q^3$. This space is an ANR-set and every subset $X_2' = \{a\}$ consisting of only one point is a homotopic 2-skeleton of $X$ (see [3], p. 613). It is clear that $f\colon L \to Q^3$ is homotopic to a constant map $f'\colon L \to Q^3$. By Corollary 2.4 of [10] (p. 122) the spaces $X = S \cup_f Q^3$ and $S \cup_{f'} Q^3$ have the same homotopy type. It is easy to see that $S \cup_{f'} Q^3$ has a homotopy type of $S$ and consequently $X$ has a homotopy type of $S$. Since $\pi_2^2(X) = \pi^2(\{a\} \subset X) = 0$ and $\pi_2^2(S) = \pi^2(S \subset S) = Z$, where $Z$ denotes the group of integers, we infer that the groups $\pi_2^2(X)$ and $\pi_2^2(S)$ are not isomorphic. This gives a negative answer to Problem (9.5) of [6]. We have $\pi^2(X) = \pi^2(S) = Z$ and this implies that the groups $\pi^2(X)$ and $\pi_2^2(X)$ are not isomorphic. Thus we obtain a negative answer to Problem (9.3) of [6] (in view of $\mathrm{Fd}(X) = \mathrm{Fd}(S) = 2$) and consequently to Problem (9.1) of [6] (see [6], p. 91). Now, let us observe that $\pi_2^2 H(X) = \pi_2^2 H(S) = \pi^2(S) = Z$ ($H$ is a special association functor). Therefore $\{a\}$ is not an outer homotopic 2-skeleton of $X$ (in view of Corollary 6.1).

An immediate consequence of Theorem 6.3 is the following

COROLLARY 7.5. *There is no inverse system $\{X_\alpha, p_\alpha^{\alpha'}, A\}$ of finite CW complexes with $X$ as its inverse limit and such that $p_\alpha^{\alpha'}(X_{\alpha'}^2) \subset X_\alpha^2$ for each $\alpha, \alpha' \in A$, $\alpha \leqslant \alpha'$.*

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DEPARTMENT OF MATHEMATICS AND MECHANICS, WARSAW UNIVERSITY
WYDZIAŁ MATEMATYKI I MECHANIKI UNIWERSYTETU WARSZAWSKIEGO

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# Homeotopy groups of compact 2-manifolds

by

**David J. Sprows** (Villanova, Penn.)

**Abstract.** Let $X$ be a 2-manifold and let $H(X)$ denote the homeotopy group of $X$. Several results have been obtained concerning $H(X)$ in the case $X$ is of the form $M-F_n$ where $M$ is a closed 2-manifold and $F_n$ is a set of $n$ distinct points in $M$. In this paper it is shown that these results give rise immediately to corresponding results for compact 2-manifolds. In particular, it is shown that if $Y$ is the compact 2-manifold obtained by removing the interiors of $n$ disjoint closed discs from some closed 2-manifold $M$, then $H(Y)$ is isomorphic to $H(M-F_n)$.

**1. Introduction.** Let $X$ be a 2-manifold (connected, triangulated) and let $H(X)$ denote the homeotopy group (or mapping class group) of $X$, i.e. $H(X)$ is the group of all isotopy classes in the space of all homeomorphisms of $X$ onto $X$. W. Magnus [4] and, more recently, J. Birman [1] have obtained several results concerning $H(X)$ in the case $X$ is of the form $M-F_n$ where $M$ is a closed 2-manifold and $F_n$ is a set of $n$ distinct points in $M$. In this paper we show that these results give rise immediately to corresponding results for compact 2-manifolds. In particular, we show that if $Y$ is the compact 2-manifold obtained by removing the interiors of $n$ disjoint discs from some closed 2-manifold $M$, then $H(Y)$ is isomorphic to $H(M-F_n)$.

**2. Notation.** Let $X$ be a 2-manifold and $F$ a finite subset of $\mathrm{Int}(X)$. The homeotopy group $H(X)$ can be defined as the quotient group $G(X)/G_0(X)$ where $G(X)$ is the group of all homeomorphisms of $X$ onto $X$ and $G_0(X)$ is the normal subgroup of $G(X)$ consisting of those homeomorphisms $g$ in $G(X)$ which are isotopic to the identity (denoted $g \simeq 1_X$). Similarly, we can define $H(X,F)$ to be the quotient group $G(X,F)/G_0(X,F)$ where $G(X,F)$ is the subgroup of $G(X)$ consisting of those $g$ in $G(X)$ which map $F$ onto $F$ and $G_0(X,F)$ is the normal subgroup of $G(X,F)$ consisting of those homeomorphisms $h$ in $G(X,F)$ which are isotopic to the identity by an isotopy which keeps $F$ pointwise fixed (denoted $h \simeq 1_X(\mathrm{rel}F)$).

Let $M$ be a closed 2-manifold. Let $D_i$ for $1 \leqslant i \leqslant n$ denote a family of disjoint closed discs in $M$ with $P_i$ a point in $\mathrm{Int}(D_i)$ for each $i$ between 1