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On commutators of singular integrals*

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Abstract. The commutator v.p. $\int_{-\infty}^{\infty}\frac{F(x)-F(y)}{(x-y)^2}g(y)\,dy = T(F',g)$ as well as its n-dimensional generalizations are treated through this paper. The previous known results stated that if $g\in L^p$, $F'(x)\in L^q$, with $1/p+1/q<1$, then $\|T(F',g)\|_r < C_{pq}\|F'\|_q\|g\|_p$; $1/r = 1/p+1/q$. Here it is presented the following novelty: that the restriction $1/p+1/q<1$ is not any longer necessary. We face the cases $1/p+1/q>1$, obtaining as expected the same inequality in this situation.

0. Introduction. The purpose of this paper is to extend and generalize the results proved in [1] and [3].

We shall be concerned with singular integrals of the type

$$\text{p.v.}\int_{-\infty}^{\infty}\frac{F(x)-F(y)}{(x-y)^2}g(y)\,dy = T(F',g) \tag{0.1}$$

and their n-dimensional generalizations. Here, $F(x)$ stands for a function having a derivative in the distributions sense in the class L^q; g stands for a measurable function belonging to a class L^p.

If $\frac{1}{p}+\frac{1}{q}<1$, $1<p<\infty$, $1<q<\infty$ and r is given by $\frac{1}{r}=\frac{1}{q}+\frac{1}{p}$; then $T(F',g)$ exists in L^r-norm and pointwise a.e., see [1] and [3].

Through this paper the condition $1/p+1/q<1$ is relaxed to the following one:

$$1<q\leqslant\infty,\quad 1<p\leqslant\infty,\quad \frac{1}{p}+\frac{1}{q}=\frac{1}{r},\quad r\neq\infty, \tag{0.2}$$

The condition $r\neq\infty$ means that p and q cannot be infinity simultaneously.

Under the condition (0.2) we show that the operator defined in (0.1) exists as a principal value in the metric L^r (notice that r can be less than

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one) and, moreover, it exists pointwise a.e. Therefore the novelty here is the range of values of r lying between 1 and $\frac{1}{2}$. The situation q; $1 < q < \infty$ and $p = \infty$ is particularly interesting and it is used for extending the result to the n-dimensional case.

If the conditions (0.2) are relaxed to the following ones:

(0.3) $$1 \leqslant q \leqslant \infty, \quad 1 \leqslant p \leqslant \infty, \quad r \neq \infty, \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q},$$

then the principal value (0.1) exists a.e. and instead of having a strong type inequality we get a weak type inequality; more precisely:

(0.4) $$\left|E\left(|T(F', g)| > \lambda\right)\right| < \frac{C_{pq}}{\lambda^r} \|F'\|_q \|g\|_p$$

where $1/r = 1/p + 1/q$.

The case $p = 1$, $q = 1$, $r = \frac{1}{2}$ is particularly interesting because we may take g to be a finite Borel measure and F' to be another finite Borel measure and get the existence a.e. of the corresponding principal value defined by (0.1).

The n-dimensional case is both more difficult and more technical and will not be discussed here in the introduction; nevertheless, the same type of ideas are used there.

The techniques used throughout the paper consist mainly of a suitable generalization of the Calderón–Zygmund covering lemma (see [2]), and also a specialized interpolation lemma.

Part I is concerned with the 1-dimensional result, and Part II is devoted to the n-dimensional result. I would like to mention at this opportunity, that a weaker form of theorem [A] in Part I was gotten by the author in 1971 during his stay at "Universidad de Buenos Aires". This result was exposed by the author in [5].

Finally, I would like to thank Professor E. Fabes and Professor N. M. Rivière for their valuable advice and discussions while this paper was in preparation.

I. THE ONE-DIMENSIONAL CASE

Before proving the main results of this part we shall state and prove some auxiliary lemmas.

1.1. Let $T(f, g)$ be an operator mapping $L^\infty_{\boldsymbol{R}^n} \times L^{p_0}_{\boldsymbol{R}^n}$ $(1 \leqslant p_0 < \infty)$ into the set of real measurable functions defined on $\boldsymbol{R}^n$. Suppose that the operator T satisfies also the following properties:

(1.1.1) (i) $$|T(c_1 f, c_2 g)| \leqslant |C_1| \cdot |C_2| |T(f, g)|,$$

(ii) (a) $$\Big|T\Big(\sum_1^\infty f_k, g\Big)\Big| \leqslant \sum_1^\infty |T(f_k, g)|,$$

(b) $$\Big|T\Big(f, \sum_1^\infty g_k\Big)\Big| \leqslant \sum_1^\infty |T(f, g_k)|.$$

Let us denote by S a cube and by lS the dialation of S l times about its center.

(iii) *If f is supported on S and $\int_S f\,dt = 0$, we have*

$$\int_{(lS)'} |T(f, g)|\,dx \leqslant \frac{C}{|S|} \int_S |f|\,dt \int_S |g|\,dt$$

where $(lS)'$ stands for the complement of lS; $|S|$ stands, as usual, for the measure of S and the constant C does not depend on f, g, or S.

(iv) *If $f \in L^\infty$ and $g \in L^1 \cap L^{p_0}$; g supported on S and $\int_S g\,dt = 0$; then we have:*

$$\int_{(lS)'} |T(f, g)|\,dx \leqslant C \|f\|_\infty \cdot \int_S |g|\,dt.$$

As before C does not depend on f, g, or S.

(v) *If $f \in L^\infty$ and $g \in L^{p_0}$, then:*

$$E\left(|T(f, g)| > \lambda\right) < \frac{C}{\lambda^{p_0}} \|f\|_\infty^{p_0} \|g\|_{p_0}^{p_0}.$$

Here $E\left(|T(f, g)| > \lambda\right)$ stands for the measure of the set of points where $|T(f, g)|$ exceeds λ. The constant C does not depend on f, g or λ.

The following lemmas are a generalization of Calderón–Zygmund covering lemma (see [2]).

1.2. Lemma. *If the operator T satisfies properties* (i), (ii)(a), (iii) *and* (v) *we have for all q such that $1 \leqslant q \leqslant \infty$ and r given by $\frac{1}{r} = \frac{1}{q} + \frac{1}{p_0}$ the following estimate:*

(1.2.1) $$E\left(|T(f, g)| > \lambda\right) < \frac{C}{\lambda^r} \|f\|_q^r \|g\|_{p_0}^r$$

where the constant C does not depend on λ, f or g.

Proof. We are going to define the following set G_λ. Let $\lambda > 0$ be a chosen number and a family I_k of cubes defined in the following way:

$$\mathring{I}_j \cap \mathring{I}_i = \emptyset, \quad i \neq j,$$

(1.2.2) $$\lambda^{r/q} < \frac{1}{|I_k|}\int_{I_k} f\,dt \leqslant \lambda^{r/q}2^n,$$

where $f \geqslant 0$, $\|f\|_q = 1$ and q such that $1 \leqslant q < \infty$; once we have fixed q, r is given by

(1.2.3) $$\frac{1}{r} = \frac{1}{q} + \frac{1}{p_0}.$$

The cubes are so chosen that in $\boldsymbol{R}^n - \bigcup_1^\infty I_k$ we have $f \leqslant \lambda^{r/q}$ a.e. (see [2]). The set $G_\lambda = \bigcup_1^\infty I_k$. To the set G_λ we are going to associate the set $\overset{*}{G}_\lambda$, where

(1.2.4) $$\overset{*}{G}_\lambda = \bigcup_1^\infty lI_k.$$

Clearly, we have from (1.2.2)

$$|G_\lambda| \leqslant \frac{1}{\lambda^r}\int_{\boldsymbol{R}^n} f^q\,dx, \qquad |\overset{*}{G}_\lambda| \leqslant \frac{l^n}{\lambda^r}\int_{\boldsymbol{R}^n} f^q\,dx.$$

We are going to construct the following functions

(1.2.5) $$f = \tilde{f} + \tilde{\tilde{f}} \text{ a.e.}$$

where

$$\tilde{\tilde{f}} = \sum_1^\infty \phi_k(x)\{f(x) - \mu_k),$$

μ_k is given by

$$\mu_k = \frac{1}{I_k}\int_{I_k} f\,dt;$$

$\phi_k(x)$ are the characteristic functions of the cubes I_k.

If $\phi_0(x)$ is the characteristic functions of $\boldsymbol{R}^n - \bigcup_1^\infty I_k$, then $\tilde{f}$ is given by

$$\tilde{f} = f\phi_0(x) + \sum_1^\infty \mu_k\phi_k(x).$$

Clearly we have

(1.2.6) $$0 \leqslant \tilde{f} \leqslant 2^{n+1}\lambda^{r/q} \text{ a.e.}$$

For the sake of simplicity call $f_k = \Phi_k(x)\{f(x) - \mu_k\}$. Then

(1.2.7) $$\int_{I_k} |f_k|\,dx \leqslant 2^{n+1}\lambda^{r/q}|I_k|.$$

Let us fix $g \in L^{p_0}(\boldsymbol{R}^n)$ and write $f = \tilde{f} + \tilde{\tilde{f}}$. We have the following inclusion relation:

(1.2.8) $$\{|T(f, g)| > \lambda\} \subset \{|T(\tilde{f}, g)| > \tfrac{1}{2}\lambda\} \cup \{T(\tilde{\tilde{f}}, g) > \tfrac{1}{2}\lambda\}.$$

Therefore:

(1.2.9) $$E(|T(f, g)| > \lambda) \leqslant E(|T(\tilde{f}, g)| > \tfrac{1}{2}\lambda) + E(|T(\tilde{\tilde{f}}, g)| > \tfrac{1}{2}\lambda);$$

from condition (v) and from (1.2.6) we have

(1.2.10) $$E(|T(\tilde{f}, g)| > \tfrac{1}{2}\lambda) \leqslant C2^{(n+2)p_0}\frac{\lambda^{\frac{r}{q}p_0}}{\lambda^{p_0}}\|g\|_{p_0}^{p_0}$$
$$= C2^{(n+2)p_0}\frac{1}{\lambda^r}\int_{\boldsymbol{R}^n}|g|^{p_0}\,dx \qquad \left(\frac{1}{r} = \frac{1}{q} + \frac{1}{p_0}\right).$$

Now we are going to evaluate the measure:

(1.2.11) $$|\{|T(\tilde{\tilde{f}}, g)| > \tfrac{1}{2}\lambda\} \cap \{\boldsymbol{R}^n - \overset{*}{G}_\lambda\}|.$$

Since we have $|T(\tilde{\tilde{f}}, g)| \leqslant \sum_1^\infty |T(f_k, g)|$; (this last follows from property [ii (a)]) and taking into account that $(\boldsymbol{R}^n - \overset{*}{G}_\lambda) \subset \bigcup_1^\infty (lI_k)'$ we observe:

(1.2.12) $$\int_{\boldsymbol{R}^n - \overset{*}{G}_\lambda} |T(\tilde{\tilde{f}}, g)|\,dx \leqslant \sum_1^\infty \int_{(lI_k)'} |T(f_k, g)|\,dx$$
$$\leqslant C\sum_1^\infty \frac{1}{|I_k|}\int_{I_k}|f_k|\,dx \int_{I_k}|g|\,dx \leqslant C2^{n+1}\lambda^{r/q}\sum_1^\infty \int_{I_k}|g|\,dx.$$

Observe now that $|G_\lambda| \leqslant C\lambda^{-r}$ since $\|f\|_q = 1$, this last gives

(1.2.13) $$|G_\lambda|^{1-1/p_0} \leqslant C^{1-1/p_0}\lambda^{-r(1-1/p_0)} = C'\lambda^{-r+r/p_0}.$$

Observing that

$$\int_{G_\lambda}|g|\,dx \leqslant |G_\lambda|^{1-1/p_0}\Big(\int_{\boldsymbol{R}^n}|g|^{p_0}\,dx\Big)^{1/p_0},$$

we have immediately by (1.2.12) and (1.2.13):

(1.2.14) $$\int_{\boldsymbol{R}^n - \overset{*}{G}_\lambda} |T(\tilde{\tilde{f}}, g)|\,dx \leqslant C''2^{n+1}\lambda^{r/q}\lambda^{-r}\lambda^{r/p_0}\Big(\int_{\boldsymbol{R}^n}|g|^{p_0}\,dt\Big)^{1/p_0}.$$

Applying Kolmogoroff's inequality to the right-hand member of (1.2.14), we get

(1.2.15) $$|\{|T(\tilde{\tilde{f}}, g)| > \tfrac{1}{2}\lambda\} \cap \{\boldsymbol{R}^n - \overset{*}{G}_\lambda\}| \leqslant C'''2^{n+1}\frac{1}{\lambda^r}\Big(\int |g|^{p_0}\,dx\Big)^{1/p_0}.$$

If p_0 happens to be equal to one, $p_0 = 1$, we have $1/r = 1/q+1$; $1 = r/q+r$, from (1.2.12) follows directly

$$\int_{R^n - \overset{*}{G}_\lambda} |T(\tilde{\tilde{f}}, g)|\,dx \leqslant (\text{Constant})\,\lambda^{1-r} \int_{R^n} g\,dx$$

and therefore (1.2.15) is satisfied in this case. Taking also $\|g\|_{p_0} = 1$, we see after collecting estimates that:

$$E\{|T(f,g)| > \lambda\} \leqslant \frac{\text{Constant}}{\lambda^r} \tag{1.2.16}$$

since $\|f\|_q = \|g\|_{p_0} = 1$.

Here the constant does not depend on the functions but only on the q involved. The homogenity condition (i) gives the thesis for general f and g.

1.3. Before stating the two interpolation lemmas that are going to be used later we are going to add some definitions.

$D_{k,p}$ will denote the space of distributions F on $\boldsymbol{R}^n$ having all derivatives of order k in $L^p(\boldsymbol{R}^n)$. We are going to introduce the following pseudo norm:

$$|||F|||_{p,k} = \sum_{|\alpha|=k} \|D^\alpha F\|_p. \tag{1.3.1}$$

If R_j denotes the jth Marcel Riesz Transform, then we define ΛF in the following way:

$$\Lambda F = \sum_{j=1}^{n} R_j\left(\frac{\partial}{\partial x_j} F\right) \tag{1.3.2}$$

where $\partial/\partial x_j$ denotes the corresponding partial derivative in the distributions sense. Observe that (1.3.2) is well defined for any distribution of $D_{k,p}$, $p > 1$, $p < \infty$.

In general, we define $\Lambda^k = \Lambda\Lambda^{k-1}$. From the usual arguments of singular integrals it can be shown easily (see [3])

$$C_p\,|||F|||_p \leqslant \|\Lambda^{(k)} F\|_p \leqslant C_p\,|||F|||_{p,k}$$

for $1 < p < \infty$.

Now we are going to prove a lemma which is due to Mary Weiss.

1.4. Lemma. *Suppose that $F \in D_{1,p}$ and $p > n$, n stands for the dimension; then if we define*

$$\overset{*}{D}F(x) = \sup_{y \neq x} \left|\frac{F(x)-F(y)}{|x-y|}\right|,$$

we have

(i) $$\|\overset{*}{D}F\|_p \leqslant C_p \|\operatorname{grad} F\|_p, \qquad p > n.$$

Furthermore we have the estimate

(ii) $$\left|\frac{F(x)-F(y)}{|x-y|}\right| \leqslant C_p\left(\frac{1}{|x-y|^n}\int_{|x-s|\leqslant l|x-y|} |\operatorname{grad} F|^p\,ds\right)^{1/p}, \qquad p > n,$$

where C_p depends on p and on l, $l > 1$.

Proof. It will be necessary only to prove (ii) since (i) will follow from (i) by a standard argument. If $F \in C_0^1$ we have the following representation:

$$F(y) = C_n \sum_{i=1}^{n} \int_{R^n} \frac{\partial F}{\partial s_i}\,\frac{y_i - s_i}{|y-s|^n}\,ds \tag{1.4.1}$$

where C_n depends only on the dimension.

Now, if $\varphi(y) = 1$ for $|y| \leqslant 1$, $\varphi(y) \neq 0$ and $\dfrac{\partial}{\partial y_j}\varphi \neq 0$ for y, $1 < |y| < 2$, $j = 1, 2, \ldots, n$; $\varphi(y) = 0$ if $|y| > 2$. Suppose in addition that φ is C^∞. Now consider the function $\varphi_\varepsilon = \varphi[\varepsilon^{-1}y]$; with the aid of this function we construct the following function of y:

$$\{F(x) - F(y)\}\varphi_\varepsilon(x-y) \tag{1.4.2}$$

that can be represented as

$$\begin{aligned}\{F(y)-F(x)\}\varphi_\varepsilon(x-y) \\ = C_n \sum_{i=1}^{n} \int \frac{\partial}{\partial s_i}\big[(F(s)-F(x))\varphi_\varepsilon(s-x)\big] K_i(y-s)\,ds\end{aligned} \tag{1.4.3}$$

for $|y-x| < 2\varepsilon$ where $K_i(s) = s_i/|s|^n$ if $|s| < 8\varepsilon$ and zero otherwise.

Observe that

$$\frac{\partial}{\partial s_i}\{F(s)-F(x)\}\varphi_\varepsilon(x-s) = \frac{\partial F}{\partial s_i}\varphi_\varepsilon(x-s) + \big(F(s)-F(x)\big)\frac{\partial}{\partial s_i}\varphi_\varepsilon(s-x)$$

where

$$\begin{aligned} &\left|\big(F(x)-F(x)\big)\frac{\partial}{\partial s_i}\varphi_\varepsilon(s-x)\right| = 0 && \text{if } \begin{array}{l}|s-x| > 2\varepsilon,\\ |s-x| < \varepsilon,\end{array} \\ &\left|\big(F(s)-F(x)\big)\frac{\partial}{\partial s_i}\varphi_\varepsilon(s-x)\right| \leqslant C\left|\frac{F(s)-F(x)}{|s-x|}\right| && \text{if } \varepsilon < |s-x| < 2\varepsilon.\end{aligned} \tag{1.4.4}$$

Accordingly, we have for $|y-x| < 2\varepsilon$

$$(1.4.5)\quad |\{F(y)-F(x)\}\varphi_\varepsilon(x-y)|$$
$$\leqslant C_1\Big(\int\limits_{|x-s|<2\varepsilon}|\operatorname{grad} F|^p\,ds\Big)^{1/p}\Big(\int\limits_{|s|<10\varepsilon}\Big(\frac{1}{|s|^{n-1}}\Big)^{p/p-1}ds\Big)^{(p-1)/p}+$$
$$+C_2\Big(\int\limits_{|x-s|<2\varepsilon}\Big(\frac{F(x)-F(s)}{|x-s|}\Big)^p ds\Big)^{1/p}\Big(\int\limits_{|s|<10\varepsilon}\Big(\frac{1}{|s|^{n-1}}\Big)^{p/p-1}ds\Big)^{(p-1)/p}.$$

Observe that

$$(1.4.6)\quad \int\limits_{|x-s|<2\varepsilon}\left|\frac{|F(x)-F(s)|}{|s-x|}\right|^p ds \leqslant \int\limits_{\Sigma}\int\limits_0^{2\varepsilon}\left|\frac{1}{\varrho}\int\limits_0^{\varrho}|\operatorname{grad} F(x+\alpha t)|\,dt\right|^p\varrho^{n-1}\,d\varrho\,d\sigma.$$

After an application of Hardy–Littlewood inequality (see [7], Vol. I, p. 20) to the inner integral on the right-hand side, for $p>n$ we get

$$\Big(\int\limits_{|x-s|<2\varepsilon}\left|\frac{|F(x)-F(s)|}{|x-s|}\right|^p ds\Big) \leqslant C_p\int\limits_{\Sigma}\int\limits_0^{2\varepsilon}|\operatorname{grad} F(x+\alpha\varrho)|^p\varrho^{n-1}\,d\varrho\,d\sigma,$$
$$C_p\to\infty \text{ when } p\downarrow n.$$

Taking into account that

$$\Big(\int\limits_{|s|<10\varepsilon}\Big(\frac{1}{|s|^{n-1}}\Big)^{p/p-1}ds\Big)^{\frac{p-1}{p}} = \varepsilon^{-\frac{n}{p}+1}C_{p,n} \quad\text{with}\quad C_{pn}\to\infty \text{ when } p\downarrow n,$$

we get finally for $|y-x|<\varepsilon$

$$(1.4.7)\quad |F(y)-F(x)|\leqslant \varepsilon\bar{C}_{n,p}\Big(\frac{1}{\varepsilon^n}\int\limits_{|x-s|<2\varepsilon}|\operatorname{grad} F|^p\,ds\Big)^{1/p},$$
$$\bar{C}_{n,p}\to\infty \text{ when } p\downarrow n.$$

Taking in (1.4.7) $|x-y| = 2^{-1}\varepsilon$ we get the thesis (ii). This finishes the lemma's proof.

1.5. First interpolation lemma. *Let T_a be a family of sublinear operators mapping the space D_{k,p_i} of $\boldsymbol{R}^n$ into the space $\mathscr{M}$ of measurable functions defined on $\boldsymbol{R}^n$, $i=1,2$ and $1<p_1<p_2<\infty$. Call $\overset{*}{D}_F(\lambda) = \sup\limits_{a\in A} E(|T_a(F)|>\lambda)$ and suppose that*

$$E(|T_a(F)|>\lambda) < \frac{C_i}{\lambda^{p_i}}|||F|||_{p_i,k}, \quad i=1,2,$$

where C_i does not depend on a or the particular F. Then we have the following inequality:

(i) $\int\limits_0^\infty \overset{*}{D}_F(\lambda)\lambda^{p-1}d\lambda \leqslant C_p|||F|||_{p,k}$ *for all p such that $p_1<p<p_2$ where C_p does not depend on F; it depends on p only. As usual,*

$$C_p\to\infty \quad \textit{for} \quad p\downarrow p_1 \textit{ or } p\uparrow p_2.$$

Proof. We are going to prove the lemma for $k<n$ (dimension). Consider the subset S of all distributions F for which we have

$$(1.5.1)\quad F(x) = \int\limits_{\boldsymbol{R}^n}\frac{1}{|x-y|^{n-k}}g(y)\,dy$$

with $g(y)$ in $L^p(\boldsymbol{R}^n)$ and vanishing outside of a compact set; clearly we have $|||F|||_{p,k}$ equivalent to $\|\Lambda^k F\|_p$ and consequently to $\|g\|_p$, $1<p<\infty$.

Then the operators T_a can be thought of as operating on the spaces of functions g. Therefore we have:

$$(1.5.2)\quad E(|T_a(F)|>\lambda) < \frac{\bar{C}_{p_i}}{\lambda^{p_i}}\int\limits_{\boldsymbol{R}^n}|g|^{p_i}dx.$$

Also:

$$(1.5.3)\quad \overset{*}{D}_F(\lambda)\leqslant\frac{\bar{C}_{p_i}}{\lambda^{p_i}}\int\limits_{\boldsymbol{R}^n}|g|^{p_i}dx.$$

Notice that if $F = F_1+F_2$ we have in this case:

$$(1.5.4)\quad \overset{*}{D}_F(\lambda)\leqslant\overset{*}{D}_{F_1}(\tfrac12\lambda)+\overset{*}{D}_{F_2}(\tfrac12\lambda).$$

Given $\lambda>0$, we partition $g>0$ as $g = g^\lambda+g_\lambda$, where $g^\lambda = g$ if $g(x)\leqslant\lambda$ and zero otherwise and $g_\lambda = g-g^\lambda$.

Defining

$$(1.5.5)\quad \begin{aligned} F_1 &= \int\limits_{\boldsymbol{R}^n}\frac{1}{|x-y|^{n-k}}g^\lambda(y)\,dy,\\ F_2 &= \int\limits_{\boldsymbol{R}^n}\frac{1}{|x-y|^{n-k}}g_\lambda(y)\,dy,\end{aligned}$$

we have

$$(1.5.6)\quad \overset{*}{D}_F(\lambda)\leqslant\overset{*}{D}_{F_1}(\tfrac12\lambda)+\overset{*}{D}_{F_2}(\tfrac12\lambda).$$

For p such that $p_1<p<p_2$, we have from (1.5.3)

$$\int\limits_0^\infty\overset{*}{D}_F(\lambda)\lambda^{p-1}d\lambda\leqslant\bar{C}_2\int\limits_0^\infty\frac{2^{p_2}}{\lambda^{p_2}}\Big[\int\limits_{\boldsymbol{R}^n}[g^\lambda]^{p_2}dx\Big]\lambda^{p-1}d\lambda+$$
$$+C_1\int\limits_0^\infty\frac{2^{p_1}}{\lambda^{p_1}}\Big[\int\limits_{\boldsymbol{R}^n}[g_\lambda]^{p_1}dx\Big]\lambda^{p-1}d\lambda\leqslant C_p\int\limits_{\boldsymbol{R}^n}g^p\,dx.$$

This last estimate follows after interchanging the order of integration in the sum of the two integrals.

As the reader will notice this lemma is essentially Marcinkiewicz's interpolation lemma.

If $k = 0$ we have $F = g$ and the interpolation argument is valid for $p_1 = 1$ and $p_2 = \infty$, provided that for $p_2 = \infty$ we replace the weak inequality for the strong type inequality.

This finishes the proof of the lemma.

1.6. SECOND INTERPOLATION LEMMA. *Suppose that in addition to properties* (i), (ii) (a), (iii), (v), *the operator T satisfies also* (ii) (b) *and* (iv). *Suppose further that p_0 in* (v) *is strictly greater than* 1, $p_0 > 1$. *Then under all the previous assumptions we have*

$$\text{(i)} \qquad \Big(\int_{R^n} [T(f,g)]^r dx\Big)^{1/r} \leqslant C_{p,q} \|f\|_q \|g\|_p$$

for all p such that $1 < p < p_0$, all q such that $1 < q \leqslant \infty$ and r given by $\dfrac{1}{r} = \dfrac{1}{p} + \dfrac{1}{q}$.

Proof. For $q = \infty$, it is essentially Benedek–Calderón–Panzone Theorem (see [2]). For proving this part we use only hypothesis (i), (ii) (b), (iv) and (v).

Observe that we have for $1 \leqslant p \leqslant p_0$, $\dfrac{1}{r} = \dfrac{1}{q} + \dfrac{1}{p}$, the following estimate:

$$E(|T(f,g)| > \lambda) < \frac{C_{pq}}{\lambda^r} \|f\|_q^r \|g\|_p^r. \tag{1.6.1}$$

Furthermore, the construction we did in Lemma 1.2 can be repeated in this case, since we have the case $q = \infty$, $1 \leqslant p < p_0$ as a consequence of Benedek–Calderón–Panzone's Theorem.

Let us fix $p > 1$, $p < p_0$ and consider p_1 such that $1 < p_1 < p < p_0$. Let q be greater than 1, $q > 1$ and r such that

$$\frac{1}{r} = \frac{1}{q} + \frac{1}{p}.$$

Let r_0 be determined by the condition:

$$\frac{r_0}{p_0} = \frac{r}{p} \quad \text{and} \quad q_0 \text{ such that } \frac{1}{r_0} = \frac{1}{q_0} + \frac{1}{p_0}.$$

Then, clearly, we have

$$\frac{r_0}{p_0} = \frac{r}{p}, \qquad \frac{r_0}{q_0} = \frac{r}{q}. \tag{1.6.2}$$

In the same way we determine r_1 by the condition:

$$\frac{r_1}{p_1} = \frac{r}{p}, \quad \text{and} \quad q_1 \text{ by } \frac{1}{r_1} = \frac{1}{q_1} + \frac{1}{p_1}.$$

In this case we have also

$$\frac{r_1}{p_1} = \frac{r}{p} = \frac{r_0}{p_0}, \qquad \frac{r_1}{q_1} = \frac{r}{q} = \frac{r_0}{q_0}. \tag{1.6.3}$$

Now we are going to repeat the construction of Lemma 1.2

$$G_\lambda = \bigcup_1^\infty I_k, \quad \overset{*}{G}_\lambda = \bigcup_1^\infty lI_k, \qquad \lambda^{r/q} < \frac{1}{|I_k|} \int_{I_k} f\,dt < 2^n \lambda^{r/q}, \tag{1.6.4}$$

and also the decomposition

$$f = \tilde{f} + \tilde{\tilde{f}}. \tag{1.6.5}$$

We get the estimate

$$E(|T(f,g)| > \lambda) \leqslant E(|T(\tilde{f},g)| > \tfrac{1}{2}\lambda) + E(|T(\tilde{\tilde{f}},g)| > \tfrac{1}{2}\lambda). \tag{1.6.6}$$

Estimate for $\int_0^\infty \lambda^{r-1} E(|T(\tilde{f},g)| > \tfrac{1}{2}\lambda)\,d\lambda$.

Call

$$\overset{*}{D}_g(\lambda) = \sup_{f \,|\, \|f\|_\infty \leqslant 2^{n+1}} E(|T(f,g)| > \lambda).$$

According to Lemma 1.5 and the fact that our Theorem is valid for $q = \infty$, we have

$$\int_0^\infty \overset{*}{D}_g(\lambda) \lambda^{p-1}\,d\lambda \leqslant C_p \|g\|_p^p \tag{1.6.7}$$

if $1 < p < p_0$.

On the other hand we have

$$\{|T(\tilde{f},g)| > \lambda\} = \{|T(\tilde{f}\lambda^{-r/q}, g)| > \lambda^{r/p}\}. \tag{1.6.8}$$

Consequently:

$$D(|T(\tilde{f},g)| > \lambda) \leqslant \overset{*}{D}_g(\lambda^{r/p}) \tag{1.6.9}$$

since $\|\tilde{f}\lambda^{-r/q}\|_\infty \leqslant 2^{n+1}$.

Therefore we have that our integral is dominated by

$$C\int_0^\infty \lambda^{r-1} \overset{*}{D}_g(\lambda^{r/p})\,d\lambda = \int_0^\infty \lambda^{p-1} \overset{*}{D}_g(\lambda)\,d\lambda \leqslant C_p \int_{R^n} |g|^p\,dx. \tag{1.6.10}$$

Estimate for $\int_0^\infty \lambda^{r-1}E(|T(\tilde{\tilde f},g)|>\frac12\lambda)\,d\lambda$.

Recall that

$$E(|T(\tilde{\tilde f},g)|>\tfrac12\lambda)\leqslant|\{|T(\tilde{\tilde f},g)|>\tfrac12\lambda\}\cap\{\boldsymbol{R}^n-\overset{*}{G}_\lambda\}|+|\overset{*}{G}_\lambda|.$$

We are going to estimate first the following integral:

$$\int_0^\infty\lambda^{p-1}|\overset{*}{G}_\lambda|\,d\lambda \tag{1.6.11}$$

Recall that $|\overset{*}{G}_\lambda|\leqslant l^n|G_\lambda|$ and from the very definition it follows

$$G_\lambda\subset\{M(f)>\lambda^{r/q}\}$$

where

$$M(f)=\sup_{R\supset\{x\}}\left|\frac{1}{|R|}\int_R f\,dt\right|$$

over all possible rectangles (n-dimensional rectangle) R containing the point x, and having edges parallel to the coordinate axes. Clearly, $M(f)$ is the maximal function of Jessen–Marcinkiewicz–Zygmund. Thus, we have that (1.6.11) is dominated by

$$l^n\int_0^\infty\lambda^{r-1}D_{M(f)}(\lambda^{r/q})\,d\lambda=l^n\int_0^\infty\lambda^{q-1}D_{M(f)}(\lambda)\,d\lambda. \tag{1.6.12}$$

Now, from the Jessen–Marcinkiewicz–Zygmund theorem if follows:

$$l^n\int_0^\infty\lambda^{q-1}D_{M(f)}(\lambda)\,d\lambda\leqslant C(q)\,l^n\int_{\boldsymbol{R}^n}|f|^q\,dx. \tag{1.6.13}$$

This is the desired bound for (1.6.11).

Let us return now to

$$|\{|T(\tilde{\tilde f},g)|>\tfrac12\lambda\}\cap\{\boldsymbol{R}^n-\overset{*}{G}_\lambda\}|. \tag{1.6.14}$$

In Lemma 1.2 we found a first bound for this measure, that is,

$$\frac{C}{\lambda}\int_{\boldsymbol{R}^n-\overset{*}{G}_\lambda}|T(\tilde{\tilde f},g)|\,dx\leqslant\frac{C}{\lambda}\lambda^{r/q}\int_{\overset{*}{G}_\lambda}|g|\,dt. \tag{1.6.15}$$

We shall decompose $|g|=|g|^{\lambda^{r/p}}+|g|_{\lambda^{r/p}}$ where $|g|^{\lambda^{r/p}}=|g|$ if $|g|\leqslant\lambda^{r/p}$ and zero otherwise. $|g|_{\lambda^{r/p}}$ is then $|g|-|g|^{\lambda^{r/p}}$. Then the right-hand member of (1.6.15) is dominated by

$$C\lambda^{-1}\lambda^{r/q}\lambda^{r/p}|\overset{*}{G}_\lambda|+C\lambda^{r/q-1}\int_{\overset{*}{G}_\lambda}|g|_{\lambda^{r/p}}\,dx\leqslant C|\overset{*}{G}_\lambda|+\lambda^{r/q-1}\int_{\overset{*}{G}_\lambda}|g|_{\lambda^{r/p}}\,dx. \tag{1.6.16}$$

Observe that:

$$C\int_0^\infty\lambda^{r-1}|\overset{*}{G}_\lambda|\,dx\leqslant C_q l^n\int_{\boldsymbol{R}^n}f^q\,dx. \tag{1.6.17}$$

For the integral $\lambda^{r/q-1}\int_{\overset{*}{G}_\lambda}|g|_{\lambda^{r/p}}\,dx$ we are going to replace r/q by r_1/q_1 and r/p by r_1/p_1 and, after an application of Hölder's inequality, we get

$$\lambda^{r_1/q_1-1}|\overset{*}{G}_\lambda|^{(p_1-1)/p_1}\Big(\int_{\boldsymbol{R}^n}|g|^{p_1}_{\lambda^{r_1/p_1}}\,dx\Big)^{1/p_1}. \tag{1.6.18}$$

Observe that $|\overset{*}{G}_\lambda|\leqslant l^n\frac{1}{\lambda^{r_1}}\int_{\boldsymbol{R}^n}M(f)^{q_1}_{\lambda^{r_1/q_1}}\,dx$ where $M(f)_{\lambda^{r_1/q_1}}=M(f)$ if $M(f)>\lambda^{r_1/q_1}$ and zero otherwise.

Then (1.6.18) is dominated by:

$$\lambda^{r_1/q_1-1}l^{(p_1-1)/p_1 n}\lambda^{r_1/p_1}\lambda^{-r_1}\Big[\int_{\boldsymbol{R}^n}[M(f)_{\lambda^{r_1/q_1}}]^{q_1}\,dx\Big]^{(p_1-1)/p_1}\times \tag{1.6.19}$$
$$\times\Big[\int(|g|_{\lambda^{r_1/p_1}})^{p_1}\,dx\Big]^{1/p_1}.$$

Taking into account that:

$$|AB|\leqslant\frac{p_1-1}{p_1}|A|^{p_1/(p_1-1)}+\frac{1}{p_1}|B|^{p_1},$$

we get that (1.6.19) is dominated by

$$C_1\frac{1}{\lambda^{r_1}}\int_{\boldsymbol{R}^n}M(f)^{q_1}_{\lambda^{r_1/q_1}}\,dx+C_2\frac{1}{\lambda^{r_1}}\int_{\boldsymbol{R}^n}|g|^{p_1}_{\lambda^{r_1/p_1}}\,dx. \tag{1.6.20}$$

Now we have to integrate (1.6.20) against λ^{r-1} between 0 and ∞. This gives, after interchanging the order of integration,

$$C_1\int_{\boldsymbol{R}^n}M(f)^{q_1}\Big[\int_0^{M(f)^{q_1/r_1}}\frac{\lambda^{r-1}}{\lambda^{r_1}}\,d\lambda\Big]dx+C_2\int_{\boldsymbol{R}^n}|g|^{p_1}\Big[\int_0^{|g|^{p_1/r_1}}\frac{\lambda^{r-1}}{\lambda^{r_1}}\,d\lambda\Big]dx. \tag{1.6.21}$$

Noticing that $r_1/p_1=r/p$ and $r_1/q_1=r/q$ we get (1.6.21) to be equal to

$$C_1\int_{\boldsymbol{R}^n}M(f)^q\,dx+C_2\int_{\boldsymbol{R}^n}|g|^p\,dx\leqslant\bar C_{1,q}\int_{\boldsymbol{R}^n}f^q\,dx+C_2\int_{\boldsymbol{R}^n}|g|^p\,dx. \tag{1.6.22}$$

This last estimate together with (1.6.17) and (1.6.13) finishes the proof of the lemma since we get

$$\int_{\boldsymbol{R}^n}|T(f,g)|^r\,dx\leqslant C_1\|f\|_q^q+C_2\|g\|_p^p$$

where C_1 and C_2 depend on p_1 and q. An homogenity argument gives the inequality (i) of the thesis.

1.7. Our next task is to prove a lemma which will be useful to handle the maximal operators of the pointwise convergence. We are going to assume that the operator $\overset{*}{T}(f,g)$ satisfies the following properties:

(1.7.1) *Properties* (i), (ii) (a) *and* (ii) (b) *and property* (v) *for all* p *such that* $1 \leqslant p \leqslant p_0$.

(1.7.2) $\overset{*}{T}(f,g)$ *admits the following domination:*

$$|\overset{*}{T}(f,g)| \leqslant T_1(f,g) + T_2(f,g) \quad \text{a.e.}$$

for f having the form $\sum_1^\infty f_k = f$, *and* $x \in \left(\bigcup_1^\infty lS_k\right)'$, *where* supp $f_k = S_k$ *(sphere),* $S_i \cap S_j = \emptyset$, $i \neq j$ *and* $\int_{S_k} f_k\,dx = 0$.

As before lS_k will denote a dilation of S_k about its center; l gives the size of the dilation. T_1 satisfies the following inequality:

(a) $$\int_{R^n - \bigcup_{k=1}^\infty lS_k} T_1(f,g)\,dx \leqslant C \sum_1^\infty \frac{1}{|S_k|} \int_{S_k} |g|\,dx \int_{S_k} |f_k|\,dx,$$

where the constant C is independent of f and g, T_2 satisfies the following inequality:

(b) $$T_2(f,g)(x) \leqslant C\left(\sup_k \frac{1}{|S_k|} \int_{S_k} |f_k|\,dx\right) T_3(g)(x)$$

where $E\big(T_3(g) > \lambda\big) < \dfrac{C}{\lambda^p} \displaystyle\int_{R^n} |g|^p\,dx$ for all p such that $1 \leqslant p \leqslant p_0$. The T_i are non-negative $T_i(f,g) \geqslant 0$ a.e., T_3 is sublinear.

1.8. Lemma. *Suppose that* $\overset{*}{T}(f,g)$ *satisfies the properties stated in* (1.7.1) *and* (1.7.2). *Then, if* $1 < p < p_0$ *and* $q > 1$, *we have:*

(i) $$\left(\int_{R^n} |\overset{*}{T}(f,g)|^r\,dx\right)^{1/r} \leqslant C_{p,q} \|f\|_q \|g\|_p,$$

r is given by $1/r = 1/p + 1/q$, $C_{p,q}$ *depends on* p, q *only.*

If $1 \leqslant p \leqslant p_0$ *and* $1 \leqslant q \leqslant \infty$, *we have instead*

(ii) $$E\big(|\overset{*}{T}(f,g)| > \lambda\big) < \frac{\bar{C}_{p,q}}{\lambda^r} \|f\|_q^r \|g\|_p^r$$

where as before $\bar{C}_{r,q}$ *does not depend on* f *or* g.

Proof. We are going to use the construction of Lemma 1.2 and also the proof of Lemma 1.6. From (1.7.1) we see that the thesis is true for $q = \infty$. As in Lemma 1.2 we pick p such that $1 \leqslant p \leqslant p_0$ and q such that $1 \leqslant q < \infty$, $1/r = 1/p + 1/q$. As before we construct the $\{I_k\}$, G_λ and $\overset{*}{G}_\lambda$,

$$G_\lambda = \bigcup_1^\infty I_k, \qquad \overset{*}{G}_\lambda = \bigcup_1^\infty lI_k, \qquad \lambda^{r/q} < \frac{1}{|I_k|} \int_{I_k} f\,dx \leqslant 2^n \lambda^{r/q}$$

and in $R^n - \bigcup_1^\infty I_k$, $f < \lambda^{r/q}$ a.e. Recall $f \geqslant 0$ a.e. As before we decompose f as $f = \tilde{f} + \tilde{\tilde{f}}$ and have

(1.8.1) $$E\big(|\overset{*}{T}(f,g)| > \lambda\big) \leqslant E\big(|\overset{*}{T}(\tilde{f},g)| > \tfrac{1}{2}\lambda\big) + E\big(|\overset{*}{T}(\tilde{\tilde{f}},g)| > \tfrac{1}{2}\lambda\big).$$

Recalling that $\tilde{f} \leqslant 2^{n+2}\lambda^{r/q}$ a.e., we have

(1.8.2) $$E\big(|\overset{*}{T}(\tilde{f},g)| > \tfrac{1}{2}\lambda\big) \leqslant C_{p,q,n} \frac{\lambda^{\frac{r}{q}p}}{\lambda^p} \int_{R^n} |g|^p\,dx = C_{p,q,n} \frac{1}{\lambda^r} \int_{R^n} |g|^p\,dx.$$

For handling $E\big(|\overset{*}{T}(\tilde{\tilde{f}},g)| > \tfrac{1}{2}\lambda\big)$ we are going to use the conditions (1.7.2) (a) and (b), that is, by using the operators T_1, T_2, T_3.

Keeping the notation of Lemma 1.2 we see that

$$\tilde{\tilde{f}} = \sum_1^\infty f_k, \qquad \int_{I_k} f_k\,dx = 0, \qquad \int_{I_k} |f_k|\,dx \leqslant 2^{n+2}\lambda^{r/q}|I_k|.$$

Now it will be enough to estimate:

(1.8.3) $$E\big(T_1(\tilde{\tilde{f}},g) > \tfrac{1}{4}\lambda\big) + E\big(T_2(\tilde{\tilde{f}},g) > \tfrac{1}{4}\lambda\big).$$

The first summand, according to (1.7.2) (a), is dominated by

(1.8.4) $$\frac{4}{\lambda} \int_{R^n - \overset{*}{G}_\lambda} |T_1(\tilde{\tilde{f}},g)|\,dx \leqslant \frac{C}{\lambda} \lambda^{r/q} \int_{\overset{*}{G}_\lambda} |g|\,dx$$

if this is handled as in Lemma 1.2 and one obtains the desired weak type estimate.

According to (1.7.2) (b) we have

(1.8.5) $$T_2(\tilde{\tilde{f}},g)(x) \leqslant C\lambda^{r/q} T_3(g)(x).$$

And observe that

(1.8.6) $$\{\lambda^{r/q} T_3(g) > \lambda\} = \{T_3(g) > \lambda^{r/p}\}.$$

Therefore:

(1.8.7) $$E\big(\lambda^{r/q} T_3(g) > \lambda\big) < \frac{C}{\lambda^r} \int_{R^n} |g|^p\,dx.$$

So (1.8.7), (1.8.4) and (1.8.2) give part (ii) of the thesis.

In order to obtain part (i) of the thesis we pick p, and q such that

$$1<p<p_0, \qquad 1<q<\infty$$

and r is given by

$$\frac{1}{r}=\frac{1}{p}+\frac{1}{q}.$$

As we did in Lemma 1.6 we pick p_1 such that

$$1<p_1<p$$

and r_1 such that $r_1/p_1=r/p$, then q_1 is given by $1/q_1+1/p_1=1/r_1$.

As before we estimate $E(|\overset{*}{T}(\tilde{f}, g)|>\frac{1}{2}\lambda)$. Calling

$$\overset{*}{D}_g(\lambda)=\sup_{f\,|\,\|f\|_\infty<2^{n+2}} E(|\overset{*}{T}(f, g)|>\lambda)$$

we see as in Lemma 1.6 that

$$E(|\overset{*}{T}(f, g)|>\lambda)<C\overset{*}{D}_g(c\lambda^{r/p}) \tag{1.8.8}$$

where C and c are different constants.

By using inequality (1.8.4) proceeding as we did in Lemma 1.6 we get, as in Lemma 1.6,

$$E(T_1(f, g)>\tfrac{1}{2}\lambda)\leqslant C_1\frac{1}{\lambda^{r_1}}\int\limits_{R^n} M(f)^{q_1}_{\lambda^{r_1/q_1}}dx+C_2\frac{1}{\lambda^{r_2}}\int\limits_{R^n}|g|^{p_1}_{\lambda^{r_1/p_1}}dx \tag{1.8.9}$$

so when we integrate against λ^{r-1} between 0 and ∞, we get $C_1\|f\|_q^q+C_2\|g\|_p^p$, which is the desired bound.

Observe that $T_3(g)$ is strong type (p, p) if $1<p<p_0$, and from inequality (1.8.5) and from (1.8.6) we have

$$E(|T_2(\tilde{\tilde{f}}, g)|>\tfrac{1}{4}\lambda)\leqslant CD_{T_3(g)}(c\lambda^{r/p}) \tag{1.8.10}$$

where $D_{T_3(g)}(S)$ stands for the distribution function of $T_3(g)$.

When integrating the right side of (1.8.10) against λ^{r-1} between 0 and ∞, we get the p-norm of $T_3(g)$ which is a desired bound also.

This finishes the proof of part (i).

1.9. We are going to apply now this general lemma to the theory of commutators. We shall deal in this part with the one-dimensional case.

As we said in the introduction we shall deal with the following operators:

$$\begin{aligned} T(f, g)&=\lim_{\varepsilon\to 0}\int\limits_{|x-y|>\varepsilon}\frac{F(x)-F(y)}{(x-y)^2}g(y)\,dy=\lim_{\varepsilon\to 0}T_\varepsilon(f, g),\\ \overset{*}{T}(f, g)&=\sup_{\varepsilon>0}\Big|\int\limits_{|x-y|>\varepsilon}\frac{F(x)-F(y)}{(x-y)^2}g(y)\,dy\Big| \end{aligned} \tag{1.9.1}$$

where $F'(x)=f\in L^q(-\infty, \infty)$, $g\in L^p(-\infty, \infty)$.

Now we are in conditions of stating the following theorem:

1.10. THEOREM A. *If $1\leqslant p<\infty$, $1\leqslant q\leqslant\infty$ and r is given by $1/r=1/p+1/q$, $f\in L^q(-\infty, \infty)$, $g\in L^p(-\infty, \infty)$, then we have*

(i) $$\lim_{\varepsilon\to 0}\int\limits_{|x-y|>\varepsilon}\frac{F(x)-F(y)}{(s-y)^2}g(y)\,dy \text{ exists a.e.}$$

Furthermore, we have

(ii) $$E(\overset{*}{T}(f, g)>\lambda)<\frac{C_{pq}}{\lambda^r}\|f\|_q^r\|g\|_p^r$$

where C_{pq} depends on p and q only.

If $1<q\leqslant\infty$, $1<p<\infty$ we have:

(iii) $$\Big(\int\limits_{-\infty}^{\infty}\big(\overset{*}{T}(f, g)^r\,dx\big)\Big)^{1/r}\leqslant C_{p,q}\|f\|_q\|g\|_p$$

and consequently

(iv) $$\Big(\int\limits_{-\infty}^{\infty}|T_\varepsilon(f, g)-T(f, g)|^r\,dx\Big)^{1/r}\to 0 \quad \text{as} \quad \varepsilon\to 0.$$

Proof of Theorem A. It can be readily seen that the operators $T_\varepsilon(f, g)$ satisfy the conditions (i) to (v) in paragraph 1.1. Nevertheless, it will be enough to show that $\overset{*}{T}(f, g)$ (the maximal operator) satisfies the conditions in paragraph 1.7. So that is going to be our task. First it follows from Bajsanski–Coifman Theorem (see [1]) that

$$E(\overset{*}{T}(f, g)>\lambda)<\frac{C^p}{\lambda^p}\|f\|_\infty^p\|g\|_p^p, \qquad 1<p<\infty. \tag{1.10.1}$$

(This part is on page 16, paragraph 8 of [1].)

The other properties of (1.7.1) can be readily checked, since they are trivial verifications. Now we are going to check properties (a) and (b) of (1.7.2) for a suitable choice of the opertors T_1, T_2, T_3. We are going to define the T_i in the following way:

$$\begin{aligned} T_1(f, g)(x)&=C_1\sum_{1}^{\infty}\frac{1}{|I_k|}\int\limits_{I_k}|f_k|\,dt\cdot\int\limits_{I_k}\frac{|I_k|}{|I_k|^2+(x-y)^2}|g(y)|\,dy,\\ T_2(f, g)(x)&=C_2\Big(\sup\nolimits_k\frac{1}{|I_k|}\int\limits_{I_k}|f_k|\,dt\Big)M(|g|)(x),\\ T_3(g)(x)&=M(|g|)(x), \end{aligned} \tag{1.10.2}$$

where $M(|g|)$ stands for the Hardy–Littlewood function.

Call I_k to the S_k and choose $l=5$. Defining $F_k(y)=\int\limits_{-\infty}^{y} f_k(t)\,dt$, we see that $\operatorname{supp} F_k = I_k$. Pick now $x\in(\bigcup\limits_1^\infty 5I_k)$; then we get:

$$(1.10.3)\qquad \Big|\int\limits_{|x-y|>\varepsilon}\frac{F(x)-F(y)}{(x-y)^2}g(y)\,dy\Big| \leqslant \sum_{k;\,d(I_k,x)\geqslant\varepsilon}\int\limits_{I_k}\frac{|F_k(y)|}{(x-y)^2}|g(y)|\,dy + {}$$
$$+\max_{i=1,2}\Big(\frac{1}{|I_{k_i}|}\int\limits_{I_{k_i}}|f_{k_i}|\,dx\Big)C\varepsilon\int\limits_{\frac{\varepsilon}{15}<|x-y|<15\varepsilon}|g(y)|\frac{1}{|x-y|^2}\,dy$$

where I_{k_1} and I_{k_2} are the two possible intervals containing the endpoints of $[x-\varepsilon, x+\varepsilon]$. Now $\sup\limits_y|F_k(y)|\leqslant\int\limits_{I_k}|f_k(t)|\,dt$ and also for $d(x, I_k)>2|I_k|$ we have

$$(1.10.4)\qquad \frac{1}{|x-y|^2}\leqslant \text{Constant}\,\frac{1}{|I_k|^2+|x-y|^2}.$$

If $y\in I_k$, the constant does not depend on k.

On account of these two last remarks we get

$$(1.10.5)\qquad |T_\varepsilon(f,g)(x)|\leqslant C_1\sum_1^\infty\frac{1}{|I_k|}\int\limits_{I_k}|f_k|\,dt\int\limits_{I_k}\frac{|I_k|}{|I_k|^2+(x-y)^2}|g(y)|\,dy + {}$$
$$+C_2\Big(\sup_k\frac{1}{|I_k|}\int\limits_{I_k}|f_k|\,dt\Big)M(|g|)(x).$$

Since the right-hand member of (1.10.5) does not depend on $\varepsilon>0$, we have the desired estimate for the maximal operator $\overset{*}{T}(f,g)$. This finishes the proof of Theorem A.

1.11. Let μ be a finite measure on R, let γ be another finite measure defined on R. Let us introduce the function

$$(1.11.1)\qquad \int\limits_{-\infty}^{x}d\gamma=\gamma(-\infty,x]=F(x)$$

and consider the operators

$$(1.11.2)\qquad T_\varepsilon(\gamma,\mu)(x)=\int\limits_{|x-y|>\varepsilon}\frac{F(x)-F(y)}{(x-y)^2}\,d\mu.$$

In the same way we are going to introduce the maximal operator:

$$(1.11.3)\qquad \overset{*}{T}(\gamma,\mu)(x)=\sup_{\varepsilon>0}\Big|\int\limits_{|x-y|>\varepsilon}\frac{F(x)-F(y)}{(x-y)^2}\,d\mu\Big|.$$

By $\bigvee\limits_{-\infty}^{\infty}\mu$ and $\bigvee\limits_{-\infty}^{\infty}\gamma$ we are going to denote the total variation of the measure μ and γ, respectively.

The following theorem will take care of the cases not covered in Theorem A.

1.12. THEOREM B. *Suppose that* $g\in L^\infty(-\infty,\infty)$; *then if* $f\in L^q$, $1<q<\infty$, *we have*

(i) $\|\overset{*}{T}(f,g)\|_q<C_q\|f\|_q\|g\|_\infty$, C_q *depends on* q *only. Furthermore*, $T_\varepsilon(f,g)$ *converges pointwise* a.e. *and in* q-*norm to* $T(f,g)$.

(ii) *If* $q=1$ *we have*

$$E[\overset{*}{T}(f,g)]<\frac{C}{\lambda}\|f\|_1\|g\|_\infty.$$

As before, $T_\varepsilon(f,g)$ *converges pointwise* a.e. *to* $T(f,g)$.

(iii) *If* $f\in L^1$, $g\in L^1$ *and* $\bigvee\limits_{-\infty}^{\infty}\mu<\infty$, $\bigvee\limits_{-\infty}^{\infty}\gamma<\infty$, *we have*:

$$\text{(a)}\qquad E(\overset{*}{T}(f,\mu)>\lambda)<\frac{C_q}{\lambda^r}\|f\|_q^r\Big(\bigvee_{-\infty}^{\infty}\mu\Big)^r;\qquad 1\leqslant q\leqslant\infty,\ \frac{1}{r}=\frac{1}{q}+1,$$

$$\text{(b)}\qquad E(\overset{*}{T}(\gamma,g))<\frac{C_p}{\lambda^r}\Big(\bigvee_{-\infty}^{\infty}\gamma\Big)^r\|g\|_p^r;\qquad 1\leqslant p\leqslant\infty,\ \frac{1}{r}=1+\frac{1}{p},$$

$$\text{(c)}\qquad E(\overset{*}{T}(\gamma,\mu))<\frac{C}{\lambda^{1/2}}\Big(\bigvee_{-\infty}^{\infty}\gamma\Big)^{1/2}\Big(\bigvee_{-\infty}^{\infty}\mu\Big)^{1/2}.$$

In the cases (a), (b), (c), T_ε *converges pointwise* a.e. *to* T.

Proof. We are going to prove (i) first. Take

$$(1.12.1)\qquad T_\varepsilon(f,g)=\int\limits_{|x-y|>\varepsilon}\frac{F(x)-F(y)}{(x-y)^2}g(y)\,dy$$

where $f\in L^q$, $1<q<\infty$, $g\in L^\infty$; we are going to show that $T_\varepsilon(f,g)$ has L^q-norm bounded by a fixed multiple of the L^q-norm of f times the L^∞-norm of g. Let $\bar g\in L^{\frac{q}{q-1}}$; then we have

$$(1.12.2)\qquad \Big|\int\limits_{-\infty}^{\infty}\bar g T_\varepsilon(f,g)\,dx\Big|=\Big|-\int\limits_{-\infty}^{\infty}gT_\varepsilon(f,\bar g)\,dx\Big|.$$

Observe that according to Theorem A we have

$$(1.12.3)\qquad \|T_\varepsilon(f,\bar g)\|_1<C_q\|f\|_q\|\bar g\|_{\frac{q}{q-1}}.$$

So, the right-hand member of (1.12.2) is then dominated by

$$C_q \|g\|_\infty \|f\|_q \|g\|_{\frac{q}{q-1}}. \tag{1.12.4}$$

This shows that $T_\varepsilon(f, g)$ has the desired property with constant C_q independent of $\varepsilon > 0$. Accordingly, the same is true for the limit operator $T(f, g)$, that can be defined pointwise a.e. since g belongs locally to all L^p.

Let $\varphi(s)$ be a C^∞-function supported on $[-\frac{1}{4}, \frac{1}{4}]$, such that

$$\int_{-\infty}^{\infty} \varphi(s)\, ds = 1. \tag{1.12.5}$$

Call $\varphi_\varepsilon = \varepsilon^{-1}\varphi(\varepsilon^{-1}s)$ and consider the difference:

$$[T_\varepsilon(f, g) - \varphi_\varepsilon * T(f, g)](x). \tag{1.12.6}$$

This difference can be written in the following way:

$$\int_{-\infty}^{\infty} \varphi_\varepsilon(s) \Bigg[\int\limits_{|x-y|>\varepsilon} \left\{ \frac{F(x)-F(y)}{(x-y)^2} - \frac{F(x-s)-F(y)}{[(x-s)-y]^2} \right\} g(y)\, dy \Bigg] ds + \tag{1.12.7}$$
$$+ \int_{-\infty}^{\infty} \varphi_\varepsilon(s) \Bigg[\int\limits_{|x-y|<\varepsilon} \frac{F(x-s)-F(y)}{[(x-s)-y]^2}\, g(y)\, dy \Bigg] ds.$$

The second integral exists a.e. in x, and it is defined as a principal value.

Estimate for $\displaystyle\int_{-\infty}^{\infty} \varphi_\varepsilon(s) \Bigg\{ \int\limits_{|x-y|<\varepsilon} \frac{F(x-s)-F(y)}{[(x-s)-y]^2}\, g(y)\, dy \Bigg\} ds.$

Interchanging the order of integration (which we may, provided that x be a point of existence of the principal value), we get

$$-\int_{x-\varepsilon}^{x+\varepsilon} g(y) \left(\int_{-\infty}^{\infty} \frac{F(y)-F(s)}{(y-s)^2}\, \varphi_\varepsilon(x-s)\, ds \right) dy. \tag{1.12.8}$$

If $f_1 = f$, if $|y-x| < 6\varepsilon$ and zero otherwise, defining $F_1(y) = \int_{-\infty}^{y} f_1(t)\, dt$, we see that for $y \in [x-\varepsilon, x+\varepsilon]$, $s \in [x - \frac{1}{4}\varepsilon, x + \frac{1}{4}\varepsilon]$ we have

$$\frac{F(y)-F(s)}{(y-s)^2} = \frac{F_1(y)-F_1(s)}{(y-s)^2}. \tag{1.12.9}$$

So, our integral becomes

$$-\int_{x-\varepsilon}^{x+\varepsilon} g(y) \left(\int_{-\infty}^{\infty} \frac{F_1(y)-F_1(s)}{(y-s)^2}\, \varphi_\varepsilon(x-s)\, ds \right) dy. \tag{1.12.10}$$

Pick $p > 1$ and such that $1/q + 1/p < 1$ (this is possible since $q > 1$); define r such that

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

Observe that the inner integral in (1.12.10) belongs to L^r since $\varphi_\varepsilon(x-s)$ belongs to L^p and $f_1 \in L^q$, so an application of Hölder's inequality gives:

$$C_{pq} \|g\|_\infty (2\varepsilon)^{\frac{r-1}{r}} \|f_1\|_q \cdot \|\varphi_\varepsilon\|_p \tag{1.12.11}$$
$$\leqslant 2^{\frac{r-1}{r}} C_{pq} \|g\|_\infty \varepsilon^{\frac{r-1}{r}} (\varepsilon^{-1})^{\frac{p-1}{p}} \left[\int_{-\infty}^{\infty} \varepsilon^{-1} [\varphi[\varepsilon^{-1}s]]^p\, ds \right]^{\frac{1}{p}} \left(\int_{x-6\varepsilon}^{x+6\varepsilon} |f(y)|^q\, dy \right)^{1/q}$$
$$\leqslant \bar{C}_{pq} \|g\|_\infty [M(|f|^q)(x)]^{1/q}$$

where $M(|f|^q)$ stands for the maximal function of $|f|^q$.

Estimate for $\displaystyle\int_{-\infty}^{\infty} \varphi_\varepsilon(s) \Bigg\{ \int\limits_{|x-y|>\varepsilon} \left[\frac{F(x)-F(y)}{(x-y)^2} - \frac{F(x-s)-F(y)}{[(x-s)-y]^2} \right] g(y)\, dy \Bigg\} ds.$

Observe that for $|x-y| > \varepsilon$ and $|s| < \frac{1}{4}\varepsilon$ we have

$$\left| \frac{F(x-s)-F(y)}{[(x-s)-y]^2} - \frac{F(x)-F(y)}{(x-y)^2} \right| \leqslant C\varepsilon \frac{M(f)(x)}{(x-y)^2} \tag{1.12.12}$$

so the integral turns out to be bounded by

$$C \|g\|_\infty M(f)(x). \tag{1.12.13}$$

Taking into account the bounds found for the integrals and also (1.12.6), we get the desired estimate for the maximal operator $\overset{*}{T}(f, g)$ and therefore part (i) is proved.

Proof of part (ii). We are going to proceed as we did in Lemma 1.2. Take $f \geqslant 0$ and write $f = \tilde{f} + \tilde{\tilde{f}}$, where $0 \leqslant \tilde{f} \leqslant 2\lambda$ a.e. $\tilde{\tilde{f}} = \sum_1^\infty f_k$; where f_k supported on I_k and $\int_{I_k} f_k\, dt = 0$, $I_i \cap I_k = \emptyset$, calling ϕ_k to the characteristic function of I_k.

$$\mu_k = \frac{1}{|I_k|} \int_{I_k} f\, dt \quad \text{and} \quad f_k = (f(x) - \mu_k)\phi_k(x);$$

also:

$$\lambda \leqslant \frac{1}{|I_k|} \int_{I_k} f\, dt < 2\lambda.$$

As before we have $G_\lambda = \bigcup_1^\infty I_k$ and $\overset{*}{G}_\lambda = \bigcup_1^\infty 5I_k$. Finally we define $F_k(y) = \int_{-\infty}^{y} f_k(t)\, dt$. Now, we are going to evaluate $E(\overset{*}{T}(\tilde{f}, g) > \lambda)$. Since $\tilde{f} \in L^2$, we have

$$E[\overset{*}{T}(f, g) > \lambda] < \frac{C}{\lambda^2} \|g\|_\infty^2 \int_{-\infty}^{\infty} [\tilde{f}']^2\, dt. \tag{1.12.14}$$

Since $\tilde{f}' \leqslant 4\lambda$, we have

$$(1.12.15)\quad \frac{C}{\lambda^2}\|g\|_\infty^2 \int_{-\infty}^{\infty} [\tilde{f}']^2\,dt \leqslant \frac{4C\|g\|_\infty^2}{\lambda}\int_{-\infty}^{\infty} |\tilde{f}|'\,dt \leqslant \frac{C'\|g\|_\infty^2}{\lambda}\int_{-\infty}^{\infty}|f|\,dt.$$

By a homogenity reasoning we change $\|g\|_\infty^2$ to $\|g\|_\infty$ in the inequality (1.12.15).

Our next step is to bound $|T_\varepsilon(\tilde{\tilde{f}}, g)(x)|$ for x belonging to $R^1 - \overset{*}{G}_\lambda$.

As in previous cases, we have:

$$(1.12.16)\quad |T_\varepsilon(\tilde{\tilde{f}}, g)(x)|$$
$$\leqslant \sum \int_{d(x,I_k)>\varepsilon} \frac{|F_k(y)|}{|x-y|^2}|g(y)|\,dy + \|g\|_\infty \int_{\frac{\varepsilon}{15}<|x-y|<15\varepsilon} \frac{\left|\int_x^y \tilde{\tilde{f}}\,dt\right|}{|x-y|^2}\,dy$$

and as in the previous theorem we get the bound:

$$(1.12.17)\quad \overset{*}{T}(\tilde{\tilde{f}}, g)(x)$$
$$\leqslant C\sum_1^\infty \frac{1}{|I_k|}\int_{I_k}|f|\,dt\cdot\int_{I_k}\frac{|I_k|}{|I_k|^2+(x-y)^2}|g(y)|\,dy + C_1 M(\tilde{\tilde{f}})(x)\|g\|_\infty.$$

We finish the proof by observing:

$$(1.12.18)\quad \int_{-\infty}^{\infty} C\left(\sum_1^\infty \frac{1}{|I_k|}\int_{I_k}|f_k|\,dt\int_{I_k}\frac{|I|_k}{|I_k|^2+(x-y)^2}|g(y)|\,dy\right)dx$$
$$\leqslant C_1\sum_1^\infty \frac{1}{|I_k|}\int_{I_k}|f_k|\,dt\int_{I_k}|g|\,dt \leqslant C_1\|g\|_\infty\sum_1\int_{I_k}|f_k|\,dt$$
$$\leqslant 4C_1\|g\|_\infty\int_{-\infty}^{\infty}|f|\,dt.$$

Now we are going to proceed to prove (iii).

Cases (a), (b) and (c) can be handled very similarly; so, for the sake of simplicity we are going to deal with case (c) only.

We know that if $f\in L^1$ and $g\in L^1$ from Theorem A it follows

$$(1.12.19)\quad E\left(\overset{*}{T}(f,g)>\lambda\right) < \frac{C}{\lambda^{1/2}}\|f\|_1^{1/2}\|g\|_1^{1/2}.$$

Suppose we have two regular measures γ and μ, respectively. Suppose that

$$(1.12.20)\quad \bigvee_{-\infty}^{\infty}\gamma<\infty,\qquad \bigvee_{-\infty}^{\infty}\mu<\infty.$$

Let γ_n and μ_n be sequences of absolutely continuous measures converging weakly to γ and μ, respectively. Suppose further that

$$(1.12.21)\quad \bigvee_{-\infty}^{\infty}\gamma_n \leqslant 2\bigvee_{-\infty}^{\infty}\gamma,\qquad \bigvee_{-\infty}^{\infty}\mu_n < 2\bigvee_{-\infty}^{\infty}\mu.$$

Fix N natural and arbitrary and consider $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N$ such that $\varepsilon_i>0$, $i=1,2,\ldots,N$. As a consequence of the Egoroff Theorem applied to $T_{\varepsilon_i}(\gamma_n,\mu_n)$ on I (interval) we get $T_{\varepsilon_i}(\gamma_n,\mu_n)\to T_{\varepsilon_i}(\gamma,\mu)$ equiuniformly in I except for a set I_λ of measure less than

$$\frac{1}{\lambda^{1/2}}\left(\bigvee_{-\infty}^{\infty}\gamma\right)^{1/2}\left(\bigvee_{-\infty}^{\infty}\mu\right)^{1/2}\quad\text{for fixed } \lambda>0.$$

Consequently, there exists a number $n_0(\lambda)$ such that

$$(1.12.22)\quad |T_{\varepsilon_i}(\gamma_{n_0},\mu_{n_0})-T_{\varepsilon_i}(\gamma,\mu)| < \tfrac{1}{2}\lambda \text{ in } I$$

except for the set I_λ.

Calling $\overset{*}{T}_N(\gamma,\mu) = \sup\limits_{1\leqslant i\leqslant N}|T_{\varepsilon_i}(\gamma,\mu)|$ we have that in $I-I_\lambda$ we have the inequality

$$(1.12.23)\quad E\left(\overset{*}{T}_N(\gamma,\mu)>\lambda\right)\subset E\left[\overset{*}{T}_N(\gamma_{n_0},\mu_{n_0})>\tfrac{1}{2}\lambda\right].$$

Consequently, we have on I

$$(1.12.24)\quad E\left(\overset{*}{T}_N(\gamma,\mu)>\lambda\right) < \frac{4C}{\lambda^{1/2}}\left(\bigvee_{-\infty}^{\infty}\mu\right)^{1/2}\left(\bigvee_{-\infty}^{\infty}\gamma\right)^{1/2}.$$

Notice that the constant $4C$ does not depend on I or N. Therefore by taking limit we get the desired inequality in the general case.

In order to prove the pointwise convergence, take a typical case, for example μ singular and γ arbitrary. Decompose $\mu=\mu_1+\mu_2$ where $\bigvee_{-\infty}^{\infty}\mu_2<\varepsilon^2$ and $\operatorname{supp}\mu_1\subset F/F$ closed and $|F|<\varepsilon^2$. Consider $G=R^1-F$ and x belonging to that set. Then

$$(1.12.25)\quad \lim_{\delta\to 0} T_\delta(\gamma,\mu_1)(x) = T_{\delta(x)}(\gamma,\mu_1)(x)$$

where $\delta(x)=$ distance (x,F).

On the other hand, $\overline{\lim\limits_{\delta\to 0}}|T_\delta(\gamma,\mu_2)(x)|>\varepsilon$ if x belongs to a set of measure at most

$$\frac{C}{\varepsilon^{1/2}}\left(\bigvee_{-\infty}^{\infty}\gamma\right)^{1/2}\varepsilon = C\varepsilon^{1/2}\left(\bigvee_{-\infty}^{\infty}\gamma\right)^{1/2}.$$

This finishes the proof.

II. THE n-DIMENSIONAL CASE

2.1. We are going to consider functions $K(x)$ defined on $\boldsymbol{R}^n$ that are homogeneous of degree $-n-1$ satisfying one of the two conditions:

(2.1.1) (a) $K(x)$ *is even and* $\int_{\Sigma} |K(x')|\,d\sigma < \infty$,

(b) $K(x)$ *is odd*, $\int_{\Sigma} K(x')x_j\,d\sigma = 0$, $j = 1, 2, \ldots, n$

and $\int_{\Sigma} |K|\log^+|K|\,d\sigma < \infty$.

We are going to consider the operators

$$\lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} K(x-y)\{F(x)-F(y)\}g(y)\,dy = T(\operatorname{grad} F, g) \tag{2.1.2}$$

where $K(x)$ satisfies (2.1); $\operatorname{grad} F \in L^q(\boldsymbol{R}^n)$; $g \in L^p(\boldsymbol{R}^n)$; $1 < q \leqslant \infty$, $1 < p \leqslant \infty$ and

$$0 < \frac{1}{p} + \frac{1}{q} \leqslant 1. \tag{2.1.3}$$

Condition (2.1.3) says that p and q cannot be infinity simultaneously, and also that p could be $q/(q-1)$, that is, q's conjugate. We are going to consider more generally the operator

$$\overset{*}{T}(\operatorname{grad} F, g) = \operatorname*{Sup}_{\varepsilon>0} \Big| \int_{|x-y|>\varepsilon} K(x-y)\{F(x)-F(y)\}g(y)\,dy \Big| \tag{2.1.4}$$

or:

$$\overset{*}{T}(\operatorname{grad} F, g) = \operatorname*{Sup}_{\varepsilon>0} |T_\varepsilon(\operatorname{grad} F, g)|.$$

2.2. THEOREM C. *Suppose that K satisfies conditions* 2.1, (2.1.1)(a) *or* (2.1.1) (b). *Suppose that* $\operatorname{grad} F \in L^q(\boldsymbol{R}^n)$, $g \in L^p(\boldsymbol{R}^n)$, $p > 1$, $q > 1$ *and*

$$0 < \frac{1}{p} + \frac{1}{q} \leqslant 1.$$

If r is given by $\frac{1}{r} = \frac{1}{q} + \frac{1}{p}$, *then we have*

(i) $\|\overset{*}{T}(\operatorname{grad} F, g)\|_r \leqslant C_{p,q}\|\operatorname{grad} F\|_q\|g\|_p$ *where* $C_{p,q}$ *depends on* p, q *and on* $\int_{\Sigma}|K|\,d\sigma$ *if K is even and on* $\int_{\Sigma}|K|\log^+|K|\,d\sigma$ *if K is odd.*

(ii) $T_\varepsilon(\operatorname{grad} F, g)$ *converges pointwise* a.e. *and in L^r-norm to the limit operator* $T(\operatorname{grad} F, g)$.

Remark. Here, the novelty consists in the limiting cases $p = q/(q-1)$ $1/p + 1/q = 1$, $p > 1$, $q > 1$, $r \neq \infty$, and also the case $p = \infty$, $q = r$, $q > 1$. (See [1] and [3].)

The proof of this theorem follows from the one-dimensional case by the method employed in [1] (rotation method) in the case K even. In the case K odd it is reduced to the case K even in the way it is done in [1] with only one variant, an adaptation of a lemma (Section 3, page 6 in [1]) of the maximal type. Actually it will depend only on parts (i) and (iii) of the coming lemma.

2.3. LEMMA. *Let $k(x)$ be a function belonging to $L^1(\boldsymbol{R}^n)$; suppose that it satisfies the following properties:*

(a) $|k(x)| \leqslant \dfrac{\Omega(x')}{|x|^n}$, $x \in \Sigma$; $\int \Omega(x')\,d\sigma < \infty$;

(b) *for some* $p_0 \geqslant 1$ *if* $k_\varepsilon = \varepsilon^{-n}k[\varepsilon^{-1}x]$ *we have*

$$E(\sup_{\varepsilon>0}|k_\varepsilon * g| > \lambda) < \frac{C_{p_0}}{\lambda^{p_0}} \int_{\boldsymbol{R}^n} |g|^{p_0}\,dy$$

where C_{p_0} does not depend on g.

Let F be a function defined on $\boldsymbol{R}^n$ such that $\operatorname{grad} F \in L^q$; *let g belong to L^p and consider the operator*

$$\overset{*}{T}(\operatorname{grad} F, g) = \sup_{\varepsilon>0} \Big| \int k_\varepsilon(x-y) \Big|\frac{F(x)-F(y)}{|x-y|}\Big| g(y)\,dy.$$

(i) *If* $p > p_0$; $q > 1$; $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$; $r \geqslant 1$ $\left(\frac{1}{p} + \frac{1}{q} \leqslant 1\right)$ *the following is valid*

$$\|\overset{*}{T}(\operatorname{grad} F, g)\|_r < C_{p,q}\|\operatorname{grad} F\|_q\|g\|_p.$$

(ii) *If we replace properties* (a) *and* (b) *of the hypothesis by the following one*

(a) $k(x) = k(|x|) \leqslant \dfrac{A}{(1+|x|^2)^{\frac{n+1}{2}}}$, *where A is a fixed constant,*

we have

(c) $E(\overset{*}{T}(\operatorname{grad} F, g) > \lambda) < \dfrac{C_{pq}}{\lambda^r}\|\operatorname{grad} F\|_q^r\|g\|_p$ *where* $1 \leqslant p \leqslant \infty$, $q > n$ *(dimension) and* $\frac{1}{r} = \frac{1}{q} + \frac{1}{p}$.

If $p > 1$, $q > n$, *then*

(d) $\int\limits_{R^n} \{\overset{*}{T}(\operatorname{grad} F, g)\}^r dy < C_{\chi q} \|\operatorname{grad} F\|_q \cdot \|g\|_p$.

Notice that here r might be less than 1.

(iii) *Suppose $n = 1$, calling $f = F'$, and suppose that $p > p_0$, $q > 1$. Then*

$$\Big\{\int\limits_{-\infty}^{\infty} \{\overset{*}{T}(f, g)\}^r dy\Big\}^{1/r} < C_{pq} \|f\|_q \|g\|_p; \quad \frac{1}{r} = \frac{1}{q} + \frac{1}{p}.$$

If $p \geqslant p_0$, $q \geqslant 1$ then:

$$E(\overset{*}{T} > \lambda) < \frac{\bar{C}_{pq}}{\lambda^r} \|f\|_q^r \|g\|_p^r.$$

Proof. We are going to start by the one-dimensional case. The one-dimensional case will follow as an application of Lemma 1.8, in which (as the reader will notice), if we change the hypothesis $1 < p < p_0$ by $p_0 < p < p_1$, in which we have replaced 1 by p_0 and p_0 by a $p_1 > p_0$, we have the same type of conclusions. Of course, the conclusions will be valid for the range of p such that $p_0 < p < p_1$ for the strong type, and $p_0 \leqslant p \leqslant p_1$ for the weak type.

From this remark, we see that $\overset{*}{T}(f, g)$ verifies (1.7.1) and (1.7.2) with

(2.3.1) $$T_1(f, g)(x) = C \sum_1^{\infty} \frac{1}{|I_k|} \int\limits_{I_k} |f_k|\, dt \int\limits_{I_k} \frac{|I_k|}{|I_k|^2 + (x-y)^2} |g(y)|\, dy$$

and $T_2 = T_3 = 0$.

Since (1.7.1) does not need to be checked in this case according to the hypothesis, we have to check only conditions (1.7.2).

Let f be given by $\sum\limits_1^{\infty} f_k$, where each f_k is supported on I_k, $\int\limits_{I_k} f_k\, dt = 0$, and $I_i \cap I_j = \emptyset$. So, in this case we have

(2.3.2)
$$\int\limits_{-\infty}^{\infty} k_\varepsilon(x-y) \left|\frac{F(x)-F(y)}{|x-y|}\right| g(y)\, dy = \sum_{j=1}^{\infty} \int\limits_{-\infty}^{\infty} k_\varepsilon(x-y) \frac{|F_j(y)|}{|x-y|} g(y)\, dy$$

if $x \in (\bigcup\limits_{j=1}^{\infty} 5I_j)'$ and $F_j(y) = \int\limits_{-\infty}^{y} f_j(t)\, dt$. Recall that $|k_\varepsilon(x)| < \varepsilon^{-1} C / |\varepsilon^{-1} x| = C/|x|$. Thus we get

(2.3.3) $$|T_\varepsilon(f, g)| \leqslant C \sum_{j=1}^{\infty} \int\limits_{I_j} \frac{1}{|x-y|^2} |F_j(y)|\, |g(y)|\, dy$$

and for $d(x, I_j) > 2|I_j|$ the right-hand member of (2.3.3) is dominated by (2.3.1) for a suitable value of the constant. This finishes the proof of the one-dimensional case.

Part (i) follows from the 1-dimensional by an application of the "method of rotation" which is valid for $r \geqslant 1$. We shall be concerned now with the proof of part (ii). We are going to define two avoiding sets G_λ and F_λ defined in the following way:

(2.3.4) $$G_\lambda = \{M(\operatorname{grad} F)(x) > \lambda^{r/q}\}$$

where

$$M(\operatorname{grad} F)(x) = \operatorname*{Sup}_{y} \frac{1}{|x-y|} \{|F(x) - F(y)|\}.$$

We have for G_λ the estimates

(2.3.5) $$|G_\lambda| < \frac{C_q}{\lambda^r} \int |\operatorname{grad} F|^q dy \text{ or } < D_{M(\operatorname{grad} F)}(\lambda^{r/q}).$$

Recall that, according to Lemma 1.4, $M(\operatorname{grad} F)(x)$ is strong type $q-q$ for $q > n$.

The set F_λ will be associated to the decomposition: $0 \leqslant g = \tilde{g} + \tilde{\tilde{g}}$ where $|\tilde{g}| \leqslant \lambda^{r/p}$ a.e. and

(2.3.6) $$\tilde{\tilde{g}} = \sum_{j=1}^{\infty} (g(x) - m_j)\phi_j(x)$$

where $m_j = \frac{1}{|I_j|} \int\limits_{I_j} g(y)\, dy$, ϕ_j characteristic function of I_j, $I_i \cap I_j = \emptyset$, $i \neq j$.

The I_j have been chosen such that

(2.3.7) $$\lambda^{r/p} < \frac{1}{|I_j|} \int\limits_{I_j} g\, dy \leqslant 2^n \lambda^{r/p}.$$

Also $R^n - \bigcup\limits_1^{\infty} I_j$, $0 \leqslant g \leqslant \lambda^{r/p}$ a.e. Now, we define F_λ to be

(2.3.8) $$F_\lambda = \bigcup_1^{\infty} 5I_j$$

where $5I_j$ stands for a dilation of I_j five times about its center.

Consider a point x in $R^n - \{F_\lambda \cap G_\lambda\}$ and also

(2.3.9) $$\int\limits_{R^n} K_\varepsilon(x-y) \frac{|F(x) - F(y)|}{|x-y|} g(y)\, dy.$$

The absolute value of (2.3.9) is dominated by

$$(2.3.10)\qquad AM(\operatorname{grad}F)(x)\int_{\boldsymbol{R}^n}\frac{\varepsilon}{(\varepsilon^2+|x-y|)^{(n+1)/2}}\,|\tilde g(y)|\,dy+$$

$$+AM[\operatorname{grad}F](x)\int_{\boldsymbol{R}^n}\frac{\varepsilon}{[\varepsilon^2+|x-y|^2]^{(n+1)/2}}\,|\tilde{\tilde g}(y)|\,dy.$$

The first integral of (2.3.10) is dominated by

$$(2.3.11)\qquad AC\lambda^{r/q}\lambda^{r/p};\quad \text{since}\quad |\tilde g(y)|\leqslant 2\lambda^{r/p}.$$

Therefore

$$(2.3.12)\qquad \overset{*}{T}(\operatorname{grad}F,\tilde g)(x)\leqslant C_1\lambda.$$

In order to dominate the second integral of (2.3.10) we have to take into account the following property of the kernel $\dfrac{\varepsilon}{[\varepsilon^2+|x-y|^2]^{(n+1)/2}}=\bar K_\varepsilon(x-y)$.

If $|x-y|>ls$, $l>1$ and $|y-y_1|\leqslant s$, we have

$$(2.3.13)\qquad 0<\bar C<\frac{\bar K_\varepsilon(x-y)}{\bar K_\varepsilon(x-y_1)}\leqslant\bar{\bar C},$$

where $\bar C$ and $\bar{\bar C}$ depend on l only.

On account of this previous remark we have

$$(2.3.14)\qquad |T_\varepsilon(\operatorname{grad}F,\tilde{\tilde g})(x)|\leqslant A\lambda^{r/q}\bar{\bar C}\sum_{k=1}^{\infty}\frac{\varepsilon}{[\varepsilon^2+|x-y_k|^2]^{(n+1)/2}}\int_{I_k}|g_k(y)|\,dy.$$

Recall $\int_{I_k}|g_k(y)|\,dy<2^{n+2}\lambda^{r/p}|I|_k$.

On account of this last estimate we get

(2.3.15)

$$|T_\varepsilon(\operatorname{grad}F,\tilde{\tilde g})(x)|\leqslant A\lambda^{r/q}\bar{\bar C}\bar C2^{n+2}\lambda^{r/p}\int_{\boldsymbol{R}^n}\frac{\varepsilon}{[\varepsilon^2+|x-y|^2]^{(n+1)/2}}\,dy\leqslant C_2\lambda.$$

Observe that for $|F_\lambda|$ the following bounds are valid:

$$(2.3.16)\qquad |F_\lambda|\leqslant 5^n\Big|\bigcup_1^\infty I_k\Big|\leqslant 5^nD_{M(g)}(\lambda^{r/p}).$$

Also

$$|F_\lambda|\leqslant\frac{5^n}{\lambda^r}\int_{\boldsymbol{R}^n}|g|^p\,dy.$$

Now by selecting $C>2\max(C_1,C_2)$ we get

$$(2.3.17)\qquad \{\overset{*}{T}(\operatorname{grad}F,g)>C\lambda\}\subset G_\lambda\cup F_\lambda.$$

That is,

$$(2.3.18)\qquad D_{\overset{*}{T}(\operatorname{grad}F,g)}(C\lambda)\leqslant|G_\lambda|+|F_\lambda|.$$

On account of the estimates we have gotten for $|G_\lambda|$ and $|F_\lambda|$ the thesis follows.

2.4. Remark. When analysing the case (ii) of the previous lemma, we see that premissible values of r are in the range $r>n/(n+1)$.

2.5. Remark. If we replace in part (ii) the condition $|K(x)|\leqslant A/[\varepsilon^2+|x-y|^2]^{(n+1)/2}$ by the more general one:

(2.5.1)

$$0<\bar C<\left|\frac{K(x)}{K(y)}\right|<\bar{\bar C}\quad\text{if}\quad |x-y|<s,\quad |y|>ls\text{ for some }l>1,$$

where $\bar C$, $\bar{\bar C}$ do not depend on x, y or s, but on l only. We still have is this case the same result.

2.6. We shall be concerned now with the "smooth case" of the n-dimensional commutators. Besides the integrability conditions imposed to $K(x)$ in (2.1.1) we are going to impose the following one:

$$(2.6.1)\qquad \int_{|x|>l|h|}|K(x+h)-K(x)|\,|x|\,dx<C$$

where $l>1$ and C does not depend on h. As before, we are going to assume that $\operatorname{grad}F\in L^q$, $q>n$, $g\in L^p$, $p\geqslant1$ and $1/p+1/q>0$.

2.7. Theorem D. *Under the assumptions made on K in 2.6 we have the following estimates for $\overset{*}{T}(\operatorname{grad}F,g)$:*

(i) *if $p>1$, $q>n$, $1/p+1/q>0$ we have*

$$\Big[\int_{\boldsymbol{R}^n}[\overset{*}{T}(\operatorname{grad}F,g)]^r\,dy\Big]^{1/r}\leqslant C_{p,q}\|\operatorname{grad}F\|_q\|g\|_p$$

where $1/r=1/p+1/q$; $C_{p,q}$ does not depend on F or g.

(ii) *If μ is a finite measure defined on $\boldsymbol{R}^n$ we have*

$$E\{\overset{*}{T}(\operatorname{grad}F,\mu)>\lambda\}<\frac{C_q}{\lambda^{q/(q+1)}}\,\|\operatorname{grad}F\|_q^{q/(q+1)}[V(\mu)]^{q/(q+1)}$$

where $V(\mu)$ denotes the total variation of μ on $\boldsymbol{R}^n$ and $q>n$, C_q depends on q only.

In order to prove the boundedness of the maximal operator in the convenient spaces we are going to use the type of methods introduced in Part I, combined with an adaptation of the method introduced in [6] by N. M. Rivière.

Proof. We are going to consider two special sets as before, G_λ and F_λ; G_λ is going to be defined as:

(2.7.1) $$G_\lambda = \left\{y;\ \sup_s \frac{|F(y)-F(s)|}{|y-s|} > \lambda^{r/q}\right\}.$$

Calling

$$M(\operatorname{grad} F)(y) = \sup_s \frac{|F(y)-F(s)|}{|y-s|},$$

we have according to Mary Weiss' lemma (Lemma 1.4) that $M(\operatorname{grad} F)$ is of strong type $q-q$ if $q>n$.

Therefore

(2.7.2) $$|G_\lambda| \leqslant D_{M(\operatorname{grad} F)}(\lambda^{r/q}) \text{ and also: } |G_\lambda| \leqslant \frac{C}{\lambda^r}\int_{R^n} |\operatorname{grad} F|^q dy, \qquad q>1.$$

Let us define F_λ. Consider a family of non-overlapping cubes $\{I_k\}$ such that for $g \geqslant 0$ a.e. we have

(2.7.3) $$\lambda^{r/p} < \frac{1}{|I_k|}\int_{I_k} g\,dt \leqslant 2^n \lambda^{r/p},$$

$$g \leqslant \lambda^{r/p} \quad \text{a.e. in } R^n - \bigcup_1^\infty I_k.$$

We are going to define F_λ to be $\bigcup_1^\infty lI_k$. As we did before, we decompose $g = \tilde g + \tilde{\tilde g}$ with $\tilde g < 2^n \lambda^{r/p}$ a.e. and $\tilde{\tilde g} = \sum_{k=1}^\infty (g-m_k)\phi_k$; where ϕ_k is the characteristic function of I_k and $m_k = \frac{1}{|I_k|}\int_{I_k} g\,dy$. Notice that F_λ admits the following two dominations:

(2.7.4) $$|F_\lambda| \leqslant \frac{l^n}{\lambda^r}\int_{R^n} g^p dy, \qquad |F_\lambda| \leqslant D_{A(g)}(\lambda^{r/p}),$$

where $A(g)$ is the strong-maximal function of Jessen–Marcinkiewicz–Zygmund (see [7], Vol. II, p. 305). We see, on account of the decomposition $g = \tilde g + \tilde{\tilde g}$,

(2.7.5) $$E\big(\overset{*}{T}(\operatorname{grad} F, g) > \lambda\big) < E\big(\overset{*}{T}(\operatorname{grad} F, \tilde g) > \tfrac12\lambda\big) + \\ + E\big(\overset{*}{T}(\operatorname{grad} F, \tilde{\tilde g}) > \tfrac12\lambda\big).$$

Estimate for $E\big(T(\operatorname{grad} F, \tilde g) > \tfrac12\lambda\big)$

It follows from Theorem C that

(2.7.6) $$E\big(\overset{*}{T}(\operatorname{grad} F, \tilde g) > \tfrac12\lambda\big) \leqslant 2^q \frac{C_q}{\lambda^q}\|\tilde g\|_\infty^q \int_{R^n} |\operatorname{grad} F|^q dy$$

for all $q \geqslant 1$, and recall that $\tilde g < 2^n\lambda^{r/p}$ a.e., where as always $1/r = 1/p + 1/q$. So it follows

(2.7.7) $$E\big(\overset{*}{T}(\operatorname{grad} F, \tilde g) > \tfrac12\lambda\big) < 2^{q(n+1)}\frac{C_q}{\lambda^r}\int_{R^n} |\operatorname{grad} F|^q dy.$$

By defining

$$D_{\overset{*}{T}(\operatorname{grad} F)}(\lambda) = \sup_{g/\|g\|_\infty \leqslant 2^{n+1}} D_{\overset{*}{T}(\operatorname{grad} F, g)}(\lambda),$$

we also see that

(2.7.8) $$E\big(\overset{*}{T}(\operatorname{grad} F, \tilde g) > \tfrac12\lambda\big) = E\big(\overset{*}{T}(\operatorname{grad} F, \lambda^{-r/p}\tilde g) > \tfrac12\lambda^{r/q}\big) \leqslant D_{\overset{*}{T}(\operatorname{grad} F)}(\tfrac12\lambda^{r/q}).$$

By the interpolation lemma 1.5 we get

$$\int_0^\infty D_{\overset{*}{T}(\operatorname{grad} F)}(\tfrac12\lambda^{r/q})\lambda^{r-1}d\lambda < C_q \int_{R^n} |\operatorname{grad} F|^q dy$$

for $1<q<\infty$, $1/r = 1/p + 1/q$.

This finishes the estimation of $E\big(\overset{*}{T}(\operatorname{grad} F, \tilde g) > \tfrac12\lambda\big)$.

Let us consider now the part $E\big(\overset{*}{T}(\operatorname{grad} F, \tilde{\tilde g}) > \tfrac12\lambda\big)$.

We are going to consider $x \in R^n - \{G_\lambda \cup F_\lambda\}$ and $\tilde{\tilde g} = \sum_1^\infty g_k(x)$.

(2.7.9) $$T_\varepsilon[\operatorname{grad} F, \tilde{\tilde g}](x) = \sum_1^\infty \int_{|x-y|>\varepsilon} K(x-y)\{F(x)-F(y)\}g_k(y)dy.$$

Calling S_ε to the sphere of center x and radius $\varepsilon > 0$, we shall designate by k' the family of indices for which $I_k \cap S_\varepsilon = \emptyset$ and by k'' the family of indices for which $I_k \cap \partial S_\varepsilon \neq \emptyset$ (where ∂S_ε designates the boundary of S_ε). Then clearly

(2.7.10) $$T_\varepsilon(\operatorname{grad} F, \tilde{\tilde g}) = \sum_{k'}\int_{I_k} K(x-y)[F(x)-F(y)]g_k(y)dy + \\ + \sum_{k''}\int_{|x-y|>\varepsilon} K(x-y)[F(x)-F(y)]g_k(y)dy.$$

We are going to estimate the first sum of the right-hand member. Taking into account that the g_k have mean value zero, we have (y_k is the center of I_k):

$$(2.7.11)\qquad \sum_{k'}\int_{I_k} K(x-y)\{F(x)-F(y)\}g_k(y)\,dy$$

$$= \sum_{k'} F(x)-F(y_k)\int_{I_k} K(x-y)g_k(y)\,dy + $$

$$+\sum_{k'}\int_{I_k}\{F(y_k)-F(y)\}K(x-y)g_k(y)\,dy$$

$$= \sum_{k'} F(x)-F(y_k)\int_{I_k}[K(x-y)-K(x-y_k)]g_k(y)\,dy +$$

$$+\sum_{k'}\int_{I_k}\{F(y_k)-F(y)\}K(x-y)g_k(y)\,dy.$$

Estimate for $\sum_{k'}(F(x)-F(y_k))\int_{I_k}[K(x-y)-K(x-y_k)]g_k(y)\,dy$

We have that the above sum is dominated by:

$$(2.7.12)\qquad \theta_1(x, \operatorname{grad} F, \tilde{\tilde{g}})$$

$$= \sum_1^\infty \frac{|F(x)-F(y_k)|}{|x-y_k|}\int_{I_k}|K(x-y)-K(x-y_k)|\,|x-y_k|\,|g_k|\,dy.$$

Observe that in $\boldsymbol{R}^n-[G_\lambda\cup F_\lambda]$

$$\frac{|F(x)-F(y_k)|}{|x-y_k|}\leqslant \lambda^{r/q}\ \text{a.e.}$$

Therefore:

$$(2.7.13)\qquad E\big(\theta_1(x, \operatorname{grad} F, \tilde{\tilde{g}})>\lambda\big)\leqslant \frac{1}{\lambda}\int_{\boldsymbol{R}^n-\{G_\lambda\cup F_\lambda\}}\theta_1(x, \operatorname{grad} F, \tilde{\tilde{g}})\,dx+$$

$$+|G_\lambda|+|F_\lambda|\leqslant |G_\lambda|+|F_\lambda|+\frac{C\lambda^{r/q}}{\lambda}\sum_{k=1}^\infty\int_{I_k}|g_k(y)|\,dy.$$

Observe that

$$(2.7.14)\qquad C\lambda^{r/q-1}\sum_{k=1}^\infty 2^{n+1}\lambda^{r/p}|I_k|\leqslant C2^{n+1}|F_\lambda|.$$

Consequently:

$$(2.7.15)\qquad E[\theta_1(x, \operatorname{grad} F, \tilde{\tilde{g}})>\lambda]\leqslant C[|G_\lambda|+|F_\lambda|].$$

Estimate for $\sum_{k'}\int_{I_k} K(x-y)[F(y_k)-F(y)]g_k(y)\,dy$

We are going to call δ_k to the diameter of I_k and

$$(2.7.16)\qquad \theta_2(x, \operatorname{grad} F, \tilde{\tilde{g}}) = \sum_{k=1}^\infty\int_{I_k}|K(x-y)|\,|F(y_k)-F(y)|\,|g_k(y)|\,dy.$$

Observe that according to Lemma 1.4 we have

$$(2.7.17)\qquad |F(y_k)-F(y)|\leqslant C_{q_0}|y-y_k|\Bigg(\frac{1}{|y-y_k|^n}\int_{|s-y_k|<l|y-y_k|}|\operatorname{grad} F|^{q_0}ds\Bigg)^{1/q_0}$$

for some q_0 such that $n<q_0<q$.

Observe that $|y-y_k|^{1-n/q_0}\leqslant \delta_k^{1-n/q_0}$ since $q_0>n$. Consequently, we have

$$(2.7.18)\qquad |F(y_k)-F(y)|\leqslant C_{q_0}\delta_k\Bigg[\frac{1}{|I_k|}\int_{l|I_k|}|\operatorname{grad} F|^{q_0}\,ds\Bigg]^{1/q_0}.$$

Call

$$\Lambda_0(\operatorname{grad} F) = \sup_{I\supset\{x\}}\Bigg(\frac{1}{|I|}\int_I|\operatorname{grad} F|^{q_0}\,ds\Bigg)^{1/q_0}$$

where the I are cubes with sides parallel to the coordinate axes. Clearly, $\Lambda_0(\operatorname{grad} F)$ is strong type $q-q$ for all $q>q_0$.

We are going to consider as we did before $x\in\boldsymbol{R}^n-[G_\lambda\cup F_\lambda]$.

$$(2.7.19)\qquad \theta_2(x, \operatorname{grad} F, \tilde{\tilde{g}})$$

$$\leqslant C\sum_1^\infty\Bigg(\frac{1}{|I_k|}\int_{CI_k}|\operatorname{grad} F|^{q_0}\,ds\Bigg)^{1/q_0}\cdot\delta_k\int_{|x-y|>l\cdot\delta_k}\frac{|K[(x-y)']|}{|x-y|^{n+1}}|g_k|\,dy.$$

Here $(x-y)' = (x-y)/|x-y|$.

On account of (2.7.19) we have:

$$(2.7.20)\qquad E(\theta_2(x, \operatorname{grad} F, \tilde{\tilde{g}})>\lambda]$$

$$\leqslant |G_\lambda|+|F_\lambda|+\frac{1}{\lambda}\int_{\boldsymbol{R}^n-[G_\lambda\cup F_\lambda]}\theta_2(x, \operatorname{grad} F, \tilde{\tilde{g}})\,dx$$

$$\leqslant \frac{C}{\lambda}\sum_1^\infty\Bigg(\frac{1}{|I_k|}\int_{lI_k}|\operatorname{grad} F|^{q_0}dy\Bigg)^{1/q_0}\int_{I_k}|g_k(y)|\,dy+|G_\lambda|+|F_\lambda|$$

$$\leqslant C2^{n+2}\lambda^{r/p-1}\sum_1^\infty\Bigg(\frac{1}{|I_k|}\int_{lI_k}|\operatorname{grad} F|^{q_0}dy\Bigg)^{1/q_0}|I_k|+|G_\lambda|+|F_\lambda|$$

$$\leqslant C2^{n+2}\lambda^{r/p-1}\sum_1^\infty\int_{I_k}\Lambda_0(\operatorname{grad} F)\,ds+|G_\lambda|+|F_\lambda|$$

$$\leqslant C2^{n+2}|F_\lambda|+C2^{n+2}\lambda^{r/p-1}\int_{\Lambda_0(\operatorname{grad} F)>\lambda^{r/q}}\Lambda_0(\operatorname{grad} F)\,ds+|G_\lambda|+|F_\lambda|.$$

Consider a family of cubes J_k, non-overlapping and such that

$$(2.7.21)\qquad \lambda^{r/q}<\frac{1}{|J_k|}\int_{J_k}\Lambda_0(\operatorname{grad} F)\,ds\leqslant 2^n\lambda^{r/q}$$

and in $R^n - \bigcup_1^\infty |J_k|$, $\Lambda_0(\operatorname{grad} F) \leqslant \lambda^{r/q}$ a.e. On account of the preceeding remark the integral in the last member of inequality (2.7.20) is dominated in the following way:

$$C 2^{n+2} \lambda^{r/p-1} \int\limits_{\bigcup_1^\infty J_k} \Lambda_0(\operatorname{grad} F)\, dy \leqslant C 2^{2n+4} \lambda^{r/p-1+r/q} \sum_1^\infty |J_k|. \tag{2.7.22}$$

Clearly, $\left(\bigcup_1^\infty J_k\right) \subset \{\Lambda(\Lambda_0(\operatorname{grad} F)) > \lambda^{r/q}\}$. Therefore, the right-hand member of (2.7.22) is dominated by:

$$C 2^{2n+4} D_{\Lambda[\Lambda_0(\operatorname{grad} F)]}(\lambda^{r/q}). \tag{2.7.23}$$

Collecting estimates we see that

$$E[\theta_2(x, \operatorname{grad} F, \tilde{\tilde{g}}) > \lambda] \leqslant C\{|G_\lambda| + |F_\lambda| + D_{\Lambda[\Lambda_0 \operatorname{grad} F)]}(\lambda^{r/q})\}. \tag{2.7.24}$$

Now, let us go back to the second sum of (2.7.10), that is,

$$\sum_{k''} \int\limits_{|x-y|>\varepsilon} K(x-y)\{F(x)-F(y)\} g_k(y)\, dy. \tag{2.7.25}$$

Recall that for all $k \in \{k''\}$ we have $I_k \cap \partial S_\varepsilon \neq \emptyset$. The fact that $d(x, I_k) > l\delta_k$, and $d(x, I_k) \leqslant \varepsilon$ implies that there exist constants $\bar{C}$ and $\bar{\bar{C}}$ depending on l only such that:

$$I_k \subset \{y;\ \bar{C}\varepsilon < |x-y| < \bar{\bar{C}}\varepsilon\}, \qquad k \in \{k''\}, \tag{2.7.26}$$

where $\bar{\bar{C}} > \bar{C} > 0$. We are going to define the functions $\bar{g}_k(y)$ in the following way:

$$\bar{g}_k(y) = [g_k(y)\phi_k(y)(1-\phi_\varepsilon) - \overline{m}_k]\phi_k(y) \tag{2.7.27}$$

where $\phi_k(y)$ is the characteristic function of I_k, ϕ_ε is the characteristic function of S_ε, $\overline{m}_k$ is the mean value of $g_k(y)\Phi_k(y)(1-\phi_\varepsilon)$ over I_k.

The $\bar{g}_k(y)$ satisfy the following properties:

$$\begin{aligned} &\frac{1}{|I_k|}\int\limits_{I_k} |g_k(y)|\, dy \leqslant 2^{n+3}\lambda^{r/p}, \\ &\overline{m}_k < 2^{n+2}\lambda^{r/p}, \\ &|\bar{g}_k(y)| \leqslant |g_k(y)| + 2^{n+2}\lambda^{r/p}\phi_k(y), \end{aligned} \tag{2.7.28}$$

and finally

$$\int\limits_{I_k} \bar{g}_k(y)\, dy = 0.$$

Consider the following sum:

$$\sum_{k''} \int\limits_{|x-y|>\varepsilon} K(x-y)\{F(x)-F(y)\} g_k(y)\, dy = \sum_{k''} \int\limits_{I_k} K(x-y)\{F(x)-F(y)\} g_k(y)\phi_k(y)(1-\phi_\varepsilon)\, dy. \tag{2.7.29}$$

The right-hand member of (2.7.29) can be written in the following way:

$$\sum_{k''} \int\limits_{I_k} K(x-y)\{F(x)-F(y)\}\bar{g}_k(y)\, dy + \sum_{k''} \overline{m}_k \int\limits_{I_k} K(x-y)\{F(x)-F(y)\}\, dy \tag{2.7.30}$$

for $x \in R^n - [G_\lambda \cup F_\lambda]$ we see that the first term of (2.7.30) is dominated by

$$C_1\theta_1(x, \operatorname{grad} F, \overset{*}{g}) + C_2\theta_2(x, \operatorname{grad} F, \overset{*}{g}) \tag{2.7.31}$$

where $\overset{*}{g} = \sum_1^\infty |g_k| + \lambda^{r/p}\phi_k(x)$. The functions θ_1, θ_2 and $\overset{*}{g}$ do not depend on $\varepsilon > 0$. Given the form of $\overset{*}{g}$, we have for $\theta_i(x, \operatorname{grad} F, \overset{*}{g})$ the same type of estimates we found for $\theta_i(x, \operatorname{grad} F, \tilde{\tilde{g}})$.

The second term of (2.7.30) is dominated by

$$CM(\operatorname{grad} F)(x)\lambda^{r/p}\varepsilon \int\limits_{\bar{C}\varepsilon < |x-y| < \bar{\bar{C}}\varepsilon} |K(x-y)|\, dy \leqslant C'\lambda \quad \text{for} \quad x \in R^n - \{G_\lambda \cup F_\lambda\} \tag{2.7.32}$$

or better the estimate

$$C\lambda^{r/p} M(\operatorname{grad} F)(x). \tag{2.7.33}$$

Observe that

$$\{M(\operatorname{grad} F)\lambda^{r/p} > \lambda\} = \{M(\operatorname{grad} F) > \lambda^{r/q}\}. \tag{2.7.34}$$

Therefore

$$E\{M(\operatorname{grad} F) > \lambda^{r/q}\} < D_{\operatorname{grad} F}(\lambda^{r/q}), \qquad q > n. \tag{2.7.35}$$

Since $\theta_i(x, \operatorname{grad} F, \tilde{\tilde{g}})$ do not depend on $\varepsilon > 0$, we have

$$\overset{*}{T}(\operatorname{grad} F, g)(x) \leqslant C_1 \sum_{i=1}^2 \theta_i(x, \operatorname{grad} F, \tilde{\tilde{g}}) + \theta_i(x, \operatorname{grad} F, \overset{*}{g}) + C_2\lambda^{r/p} M[\operatorname{grad} F](x). \tag{2.7.36}$$

On account of the estimates we have gotten for the measures:

$$\begin{aligned} &E\{\theta_i(x, \operatorname{grad} F, \tilde{\tilde{g}}) > \lambda\}, \quad i = 1, 2, \\ &E\{\theta_i(x, \operatorname{grad} F, \overset{*}{g}) > \lambda\}, \quad i = 1, 2, \\ &E\{\lambda^{r/p} M[\operatorname{grad} F] > \lambda\}, \end{aligned} \tag{2.7.37}$$

we get parts (i) and (ii) of the thesis (for the case of absolutely continuous measures in (ii)). The general case in (ii) is obtained in the same way as we did in Theorem B and it is not necessary to repeat the argument.

2.8. Remark. As a corollary we have that $T_s(\operatorname{grad} F, g)$ converges a.e. to $T(\operatorname{grad} F, g)$ also in the metric L^r under the assumptions of hypothesis i).

In the case (ii) $T_s(\operatorname{grad} F, \mu)$ converges a.e.

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Fractional integration on the space H^1 and its dual

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Abstract. The theory of fractional integration and generalized Sobolev spaces, modeled on the L^p spaces, presents certain gaps in the "extreme cases" $p = 1$ and $p = \infty$. We show in this paper that the standard results for $1 < p < \infty$ have direct analogues at the extreme cases if we replace L^1 with a smaller space H^1, and L^∞ with a larger space $H^\infty = $ B.M.O. which is the dual of H^1. We also characterize a new Banach space whose dual is H^1, establishing thus the non-reflexivity of H^1 and H^∞.

Introduction. The theory of fractional integration and generalized Sobolev spaces, modeled on the Lebesgue spaces L^p, presents certain gaps in the "extreme cases" $p = 1$ and $p = \infty$. The purpose of this paper is to show that most of the results obtained when $1 < p < \infty$ have direct analogues at the extreme cases if we replace L^1 with a smaller space H^1, and L^∞ with a larger space H^∞ which is the dual of H^1.

In the first part of the paper we describe some basic results in fractional integration using certain modified Riesz potentials which, though essentially equivalent to the widely used Bessel potentials, are somewhat simpler and better suited for other applications. Short proofs have been included for the sake of completeness as well as for easy reference when the new results are presented in the second part. Some of these new results, though usually not surprising, may also prove to be of independent interest. I wish to thank Charles Fefferman for informing me of the result described in Remark 2.6, for pointing out an easy proof of Theorem 2.11, and for other useful hints that came up in our conversations.

Our notation is quite standard. Points in Euclidean space E^n, $n \geqslant 1$, are usually denoted by x; those in the dual space by ζ. Moreover, $\langle x, \zeta\rangle = x_1\zeta_1 + \ldots + x_n\zeta_n$, $|x|^2 = \langle x, x\rangle$, and $|E|$ denotes the Lebesgue measure of a set $E \subset E^n$. For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$ where $D_j = (2\pi i)^{-1}(\partial/\partial x_j)$ and these derivatives are usually taken in the sense of distributions. As usual, $\mathcal{D}$ denotes the class of smooth (C^∞) functions with compact support in E^n, $\mathcal{D}(U)$ the class of smooth functions with compact support in the open set U, $\mathcal{S}$ denotes the class of smooth rapidly decreasing functions and (its dual space) $\mathcal{S}'$ the class of all tempered distributions.