we get parts (i) and (ii) of the thesis (for the case of absolutely continuous measures in (ii)). The general case in (ii) is obtained in the same way as we did in Theorem B and it is not necessary to repeat the argument.

2.8. Remark. As a corollary we have that $T_s(\operatorname{grad} F, g)$ converges a.e. to $T(\operatorname{grad} F, g)$ also in the metric L^r under the assumptions of hypothesis i).

In the case (ii) $T_s(\operatorname{grad} F, \mu)$ converges a.e.

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UNIVERSITY OF MINNESOTA

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Fractional integration on the space H^1 and its dual

by

UMBERTO NERI (College Park, Maryland)

Abstract. The theory of fractional integration and generalized Sobolev spaces, modeled on the L^p spaces, presents certain gaps in the "extreme cases" $p = 1$ and $p = \infty$. We show in this paper that the standard results for $1 < p < \infty$ have direct analogues at the extreme cases if we replace L^1 with a smaller space H^1, and L^∞ with a larger space $H^\infty = $ B.M.O. which is the dual of H^1. We also characterize a new Banach space whose dual is H^1, establishing thus the non-reflexivity of H^1 and H^∞.

Introduction. The theory of fractional integration and generalized Sobolev spaces, modeled on the Lebesgue spaces L^p, presents certain gaps in the "extreme cases" $p = 1$ and $p = \infty$. The purpose of this paper is to show that most of the results obtained when $1 < p < \infty$ have direct analogues at the extreme cases if we replace L^1 with a smaller space H^1, and L^∞ with a larger space H^∞ which is the dual of H^1.

In the first part of the paper we describe some basic results in fractional integration using certain modified Riesz potentials which, though essentially equivalent to the widely used Bessel potentials, are somewhat simpler and better suited for other applications. Short proofs have been included for the sake of completeness as well as for easy reference when the new results are presented in the second part. Some of these new results, though usually not surprising, may also prove to be of independent interest. I wish to thank Charles Fefferman for informing me of the result described in Remark 2.6, for pointing out an easy proof of Theorem 2.11, and for other useful hints that came up in our conversations.

Our notation is quite standard. Points in Euclidean space E^n, $n \geqslant 1$, are usually denoted by x; those in the dual space by ζ. Moreover, $\langle x, \zeta\rangle = x_1\zeta_1 + \ldots + x_n\zeta_n$, $|x|^2 = \langle x, x\rangle$, and $|E|$ denotes the Lebesgue measure of a set $E \subset E^n$. For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$ where $D_j = (2\pi i)^{-1}(\partial/\partial x_j)$ and these derivatives are usually taken in the sense of distributions. As usual, $\mathscr{D}$ denotes the class of smooth (C^∞) functions with compact support in E^n, $\mathscr{D}(U)$ the class of smooth functions with compact support in the open set U, $\mathscr{S}$ denotes the class of smooth rapidly decreasing functions and (its dual space) $\mathscr{S}'$ the class of all tempered distributions.

The Fourier transform of a function (or tempered distribution) f is denoted by $\hat{f}$, and is taken with $\exp(-2\pi i\langle x, \zeta\rangle)$. The convolution of f and g is denoted by $f*g$.

The Riesz transforms R_j are formally defined by $(R_jf)^\wedge = (\zeta_j/|\zeta|)\hat{f}$, $j = 1, \ldots, n$. For $f \in L^p$, $1 \leqslant p < \infty$, R_j is given by a principal value convolution, with a kernel which is singular in the sense of Calderón–Zygmund. For $f \in L^\infty$, a slightly different integral representation is required (see [2]). The same remarks apply to the Hilbert transform on E^1 which, formally, is given by $(Hf)^\wedge = -i\,\mathrm{sgn}(\zeta)\hat{f}$. For $1 \leqslant p \leqslant \infty$ and any $k \in \boldsymbol{N} = \{\text{natural numbers}\}$, we denote by $L_k^p = L_k^p(E^n)$ the Sobolev space $\{f \in L^p\colon D^\alpha f \in L^p \text{ for all } |\alpha| \leqslant k\}$, equipped with the norm $\|f\|_{p,k} = \|f\|_p + \sum_{|\alpha|\leqslant k}\|D^\alpha f\|_p$ where $\|f\|_p$ is the L^p norm of f. Finally, for brevity, we let $E_0^n = E^n - \{0\}$.

§ 1. Review of fractional integration. Let $d \in C^\infty(E^n)$ be a strictly positive radial function such that $d(\zeta) = |\zeta|$ for $|\zeta| \geqslant 1$. For any real s we define on $\mathscr{S}'$ the operator I^s by the formula

$$(I^sf)^\wedge = d(\zeta)^{-s}\hat{f}.$$

On $\mathscr{S}'$, I^s has 2-sided inverse I^{-s}, where $(I^{-s}f)^\wedge = d(\zeta)^s\hat{f}$, and I^0 is the identity. Hence, for all real s, $\{I^s\}$ is a commutative 1-parameter group of translation invariant operators on $\mathscr{S}'$.

For any $s > 0$, $I^sf = g_s*f$ is a convolution with an integrable, radial function $g_s \in C^\infty(E_0^n)$ which is rapidly decreasing at ∞. More precisely (see [5], formula (11)),

$$g_s(|x|) = \begin{cases} c_s|x|^{s-n} + c_s'|x|^{s-n}\log|x|, & \text{near } 0, \\ \varphi_s \in \mathscr{S}, & \text{away from } 0, \end{cases} \tag{1.0}$$

where c_s and c_s' are constants, with $c_s' = 0$ if $0 < s < n$ or if s is not an integer. In particular, for any $s > 0$ and $f \in L^p$, I^sf is defined and $\|I^sf\|_p \leqslant C_s\|f\|_p$, $1 \leqslant p \leqslant \infty$. It follows from this and a result in [4] (p. 184) that for any $s > 0$ and any $k \in \boldsymbol{N}$, I^s maps L_k^p into itself continuously, with norm $\leqslant C_s$, if $p < \infty$. For $s = 1$, we have ([4], p. 187–189) the following

LEMMA 1.0. *If $1 < p < \infty$ and $k \geqslant 0$, the operators $I^1\colon L_k^p \to L_{k+1}^p$ and $I^{-1}\colon L_{k+1}^p \to L_k^p$ are continuous.*

For $n \geqslant 2$ this lemma follows from the formulas

$$D_jI^1 = R_j - R_jK_1 + K_2, \tag{1.1}$$

$$I^{-1} = \Lambda - \Lambda K_3 + K_4, \tag{1.1'}$$

where $K_1, \ldots, K_4$ are convolution operators with kernels in $\mathscr{S}$, and where $\Lambda = \sum_1^n R_jD_j$ satisfies $(\Lambda f)^\wedge = |\zeta|\hat{f}$ for all $f \in \mathscr{S}$. For $n = 1$, the same arguments yield

$$(2\pi)^{-1}\frac{d}{dx} = H - HK_1 + K_2$$

and again (1.1′) where now $\Lambda = (2\pi)^{-1}H\dfrac{d}{dx}$ and H is the Hilbert transform. Note how (1.1) shows that Lemma 1.0 fails for $p = 1$ or ∞.

As in [4], we introduce the spaces $\mathscr{L}_s^p = \mathscr{L}_s^p(E^n)$ as the images $\mathscr{L}_s^p = I^s(L^p)$, $1 \leqslant p \leqslant \infty$ and s real, equipped with the norms $\|f\|_{p,s} = \|I^{-s}f\|_p$ which make them Banach spaces (of tempered distributions if $s < 0$) isometrically isomorphic to L^p. As remarked in [5], for $1 < p < \infty$, these spaces coincide with the spaces L_s^p of [1] and hence for $s = \alpha > 0$ with the potential spaces $\mathscr{L}_\alpha^p$ in [6]. From this definition and Lemma 1.0 it is evident that

$$\mathscr{L}_k^p = L_k^p, \quad \text{if} \quad 1 < p < \infty \text{ and } k \in \boldsymbol{N}. \tag{1.2}$$

(Compare with [6], Theorem 3, p. 135.) Again, (1.2) fails for $p = 1$ or ∞, except when $n = 1$ and k is even ([6], p. 160).

Let us verify some rather well-known properties of the $\mathscr{L}_s^p$.

THEOREM 1.1. *Let $1 \leqslant p \leqslant \infty$, and let r, s, t be real. Then:*

(a) *$I^s\colon \mathscr{L}_r^p \to \mathscr{L}_{r+s}^p$ is an isometric isomorphism.*

(b) *If $s < t$, $\mathscr{L}_t^p \subset \mathscr{L}_s^p$ and the inclusion is continuous.*

(c) *If $p < \infty$, both $\mathscr{S}$ and $\mathscr{L}_\infty^p = \bigcap_s \mathscr{L}_s^p$ are dense in $\mathscr{L}_s^p$.*

(d) *If $p < \infty$ and $q = p/(p-1)$, $\mathscr{L}_{-s}^q$ is the dual of $\mathscr{L}_s^p$ with dual pairing*

$$\langle f, g\rangle = \int fg\,dx, \quad \textit{for all } f \in \mathscr{L}_\infty^p \textit{ and } g \in \mathscr{L}_\infty^q,$$

which can be extended to a continuous bilinear form on $\mathscr{L}_s^p \times \mathscr{L}_{-s}^q$.

Proof. (a) follows directly from the group property $I^sI^r = I^{r+s}$.

For (b) note that, since $t - s > 0$, the operator I^{t-s} is a convolution with an integrable kernel. Hence, if $f \in \mathscr{L}_t^p$, so $f = I^tg$ with $g \in L^p$, we have $\|f\|_{p,s} = \|I^{t-s}g\|_p \leqslant C\|g\|_p = C\|f\|_{x,t}$.

(c) If $p < \infty$, $\mathscr{S}$ is dense in L^p. For any real s, since $I^s\colon \mathscr{S} \to \mathscr{S}$ is one-one and onto, and $\mathscr{L}_s^p = I^s(L^p)$ is a continuous image, it follows that $\mathscr{S}$ is dense in $\mathscr{L}_s^p$. Hence, by (b), $\mathscr{L}_t^p$ is dense in $\mathscr{L}_s^p$ for $p < \infty$ and all $t > s$. Therefore, $\mathscr{L}_\infty^p$ is dense in $\mathscr{L}_s^p$.

(d) We may suppose that $s > 0$, so $I^sf = g_s*f$ with $g_s \in L^1$ and radial. Hence, for any $f \in \mathscr{L}_\infty^p$ and $g \in \mathscr{L}_\infty^q$ we have that $I^sg \in \mathscr{L}_\infty^q$ and, interchanging the order of integration, we obtain $\langle f, I^sg\rangle = \langle I^sf, g\rangle$. Since $I^{-s}f \in \mathscr{L}_\infty^p$, it follows that $\langle f, g\rangle = \langle I^sI^{-s}f, g\rangle = \langle I^{-s}f, I^sg\rangle$. Thus, by Hölder's inequality,

$$|\langle f, g\rangle| \leqslant \|I^{-s}f\|_p\|I^sg\|_q = \|f\|_{p,s}\|g\|_{q,-s}. \tag{1.3}$$

Since $\mathscr{L}^p_\infty$ is dense in $\mathscr{L}^p_s$, we can extend $\langle f, g\rangle$ by continuity to $\mathscr{L}^p_s \times \mathscr{L}^q_\infty$ and then (by the Hahn–Banach Theorem if $p = 1$, $q = \infty$) to $\mathscr{L}^p_s \times \mathscr{L}^q_{-s}$, so that (1.3) still holds there. For this extended form, if $f \in \mathscr{L}^p_s$ and $g \in \mathscr{L}^q_{-s}$, so $I^{-s}f = u \in L^p$ and $I^s g \in L^q$, we have

$$\langle I^{-s}f, I^s g\rangle = \langle u, I^s g\rangle = \int u(I^s g)\,dx.$$

Interchanging the order of integration, we see that $\langle u, I^s g\rangle = \langle I^s u, g\rangle = \langle f, g\rangle$. Thus, $\langle I^{-s}f, I^s g\rangle = \langle f, g\rangle$ for all $f \in \mathscr{L}^p_s$ and $g \in \mathscr{L}^q_{-s}$. Let now $\lambda(f)$ be a continuous linear functional on $\mathscr{L}^p_s$. Since $f = I^s u$ where $u \in L^p$, $1 \leqslant p < \infty$, $\lambda(f) = \lambda(I^s u)$ is a bounded functional on L^p and hence $\lambda(f) = \langle u, \tilde{g}\rangle$ where $\tilde{g} \in L^q$. Letting $g = I^{-s}\tilde{g}$, we conclude that $\lambda(f) = \langle u, I^s g\rangle = \langle I^{-s}f, I^s g\rangle = \langle f, g\rangle$. ∎

If $p < \infty$, $\mathscr{D}$ is dense in L^p_k. Thus, (1.2) and Theorem 1.1 readily imply that $\mathscr{D}$ is dense in every $\mathscr{L}^p_s$, if $1 < p < \infty$. We have also that, for all real s,

(1.4) $\qquad D_j\colon \mathscr{L}^p_s \to \mathscr{L}^p_{s-1}$ is continuous, if $1 < p < \infty$.

In fact, $D_j I^s = I^s D_j$ so that $D_j = I^{s-1}(D_j I^1) I^{-s}$. Hence, by (1.1), the conclusion follows from the continuity of R_j on L^p, $1 < p < \infty$. Note how formula (1.1) points out that, for $p = 1$ or ∞, analogues of (1.2) and (1.4) should hold if we replace L^1 and L^∞ by suitable spaces on which the R_j are continuous (see § 2).

As is well known, if $p > q$ there are no non-zero translation invariant bounded linear operators $T\colon L^p \to L^q$, but if $p \leqslant q$ there are many.

THEOREM 1.2. *Let $T\colon L^p \to L^q$ be a translation invariant bounded linear operator, where $1 \leqslant p \leqslant q < \infty$, and let $s > 0$. If $\|T\|$ is the norm of $T\colon L^p \to L^q$ then:*

(a) *$T\colon \mathscr{L}^p_s \to \mathscr{L}^s_p$ is bounded, with norm $\leqslant \|T\|$.*

(b) *If $p > 1$, T has a unique bounded extension $\tilde{T}\colon \mathscr{L}^p_{-s} \to \mathscr{L}^q_{-s}$, which commutes with translations and differentiations and $\|\tilde{T}\| = \|T\|$.*

Proof. By a familiar result (e.g. [4], p. 209) there is an $f \in \mathscr{S}'$ such that $Tu = f * u$ for all $u \in \mathscr{S}$. This readily implies that $TI^s = I^s T$ on $\mathscr{S}$. If $u \in \mathscr{S}$, $\|Tu\|_{q,s} = \|TI^s(I^{-s}u)\|_{q,s} = \|T(I^{-s}u)\|_q \leqslant \|T\|\,\|I^{-s}u\|_p = \|T\|\,\|u\|_{p,s}$. Since $\mathscr{S}$ is dense in $\mathscr{L}^p_s$, (a) is established.

Let $p' = p/(p-1)$, $q' = q/(q-1)$, and $T'\colon L^{q'} \to L^{p'}$ denote the transpose of T, that is, $\langle Tu, v\rangle = \langle u, T'v\rangle$. Then, T' is again translation invariant and, like T, commutes with differentiations. Hence by part (a), $T'\colon \mathscr{L}^{q'}_s \to \mathscr{L}^{p'}_s$ is bounded with norm $\|T'\|_s \leqslant \|T'\| = \|T\|$. If $\tilde{T}$ is the transpose of the restriction of T' to $\mathscr{L}^{q'}_s$, then, by part (d) of Theorem 1.1, $\tilde{T}\colon \mathscr{L}^p_{-s} \to \mathscr{L}^q_{-s}$. Moreover, $\|\tilde{T}\| = \|T'\|_s \leqslant \|T\|$. Again, $\tilde{T}$ commutes with translations. Let us verify that it commutes also with differentiations, and gives the desired extension of T.

For all $u \in \mathscr{L}^p_{1-s}$ and $v \in \mathscr{L}^{q'}_s$, which is dense in $\mathscr{L}^{q'}_{s-1}$, $\langle \tilde{T}D_j u, v\rangle = (-1)\langle u, D_j T'v\rangle = (-1)\langle u, T'D_j v\rangle = \langle D_j \tilde{T}u, v\rangle$. Moreover, for all $u \in L^p$ and $v \in \mathscr{L}^{q'}_s$, which is dense in $L^{q'}$, $\langle \tilde{T}u, v\rangle = \langle u, T'v\rangle = \langle Tu, v\rangle$. Hence $\tilde{T} = T$ on L^p. Finally, since L^p is dense in $\mathscr{L}^p_{-s}$ and $\tilde{T}\colon \mathscr{L}^p_{-s} \to \mathscr{L}^q_{-s}$ is continuous, it follows that this bounded extension is unique and that $\|\tilde{T}\| = \|T\|$. ∎

The following description of L^p_k and its dual is often useful.

THEOREM 1.3. *Let $1 < p < \infty$ and $k \in N$. Then:*

(a) *L^p_k coincides with the class of all $f \in L^p$ such that $D^\alpha f \in L^p$ for all α with $|\alpha| = k$, and the norm $|f|_{p,k} = \|f\|_p + \sum_{|\alpha|=k} \|D^\alpha f\|_p$ is equivalent to $\|f\|_{p,k}$.*

(b) *$L^p_{-k} = \mathscr{L}^p_{-k}$ coincides with the class of all $g \in \mathscr{S}'$ of the form $g = g_0 + \sum_{|\alpha|=k} D^\alpha g_\alpha$, where g_0 and g_α are in L^p, and the norm $\|g_0\|_p + \sum_{|\alpha|=k} \|g_\alpha\|_p$ is equivalent to $\|g\|_{p,-k}$.*

Proof. Let us denote by C^p_k the Banach space of all $f \in L^p$ such that $D^\alpha f \in L^p$ for all $|\alpha| = k$, with norm $|f|_{p,k}$. Evidently, $L^p_k \subset C^p_k$ and the inclusion is continuous. Let us show that if $f \in C^p_k$ and $0 < |\alpha| < k$ then $D^\alpha f \in L^p$, $1 < p < \infty$.

We set $m = k - |\alpha|$, $D^\alpha = I^m(I^{-m}D^\alpha)$ and $\Lambda = \sum_1^n R_j D_j$. By (1.1′) we have that

$$I^{-m} = \sum_{|\beta|=m} c_\beta R^\beta D^\beta + S_m,$$

where the c_β are constants, $R^\beta = R_1^{\beta_1}\ldots R_n^{\beta_n}$ is a product of powers of the R_j, and the operator S_m has the property that $D^\alpha S_m$ and $S_m D^\alpha$ are bounded on L^p, $1 < p < \infty$, for all α. Now, with $0 < |\alpha| < k$ and $f \in C^p_k$, we see that

$$I^{-m}D^\alpha f = \sum_{|\beta|=m} c_\beta R^\beta D^{\beta+\alpha} f + S_m D^\alpha f$$

and $D^{\beta+\alpha} f \in L^p$ since $|\beta + \alpha| = m + |\alpha| = k$. Therefore, in view of the boundedness of R_j on L^p, $1 < p < \infty$, and the continuity of I^m, we obtain that $D^\alpha f = I^m(I^{-m}D^\alpha f)$ belongs to L^p.

Conclusion (a) now follows by a standard argument. Part (b) follows from (a) by an argument analogous to the proof of Theorem 2.2.13 of [4]. (See also Theorem 2.5 below.) ∎

The next theorem, and the remarks following it, extend slightly a result in [1].

THEOREM 1.4. *Let $1 < p < \infty$.*

(a) *If $p \leqslant q \leqslant \left(\dfrac{1}{p} - \dfrac{t-r}{n}\right)^{-1} < \infty$, then $\mathscr{L}^p_t \subset \mathscr{L}^q_r$ and the inclusion is continuous.*

(b) *If* $0 < s-n/p < 1$, *each* $f \in \mathscr{L}_s^p$ *is continuous and, for all* x,

$$|f(x)| \leqslant C_{p,s}\|f\|_{p,s}, \qquad |f(x+h)-(f(x)| \leqslant C_{p,s}\|f\|_{p,s}|h|^{s-n/p}.$$

Proof. (a) We must show that $I^s\colon L^p \to L^q$ is bounded, where $0 < s = t-r < n/p < n$. There are two cases.

Case 1. If $q^{-1} = p^{-1}-s/n$, the conclusion stems from the Hardy–Littlewood–Sobolev Theorem (e.g. [6], p. 119) since, by (1.0),

$$|I^s f| = |g_s * f| \leqslant c_s(|x|^{s-n} * |f|).$$

Case 2. If $q^{-1} > p^{-1}-s/n$, then by a theorem of W. H. Young (e.g. [4], p. 8) it suffices to check that $g_s \in L^a$ where $p^{-1}-q^{-1} = 1-a^{-1}$. Since $a^{-1} = 1-p^{-1}+q^{-1} > 1-s/n$, so that $-n < a(s-n)$, the conclusion follows directly from (1.0).

(b) For any $f \in \mathscr{L}_s^p$, $f = I^s u = g_s * u$ where $u \in L^p$ and $n/p < s < n/p+1 < n+1$. By (1.0), $g_s \in L^q$, $q = p/(p-1)$, since $n+(s-n)q \geqslant s-n(q-1)/q = s-n/p > 0$ and $\log|x|$ is locally in L^q for all $q < \infty$. Hence, f is continuous and, for all x,

$$|f(x)| \leqslant \|g_s\|_q\|u\|_p = C_{p,s}\|f\|_{p,s}.$$

Let $\Delta_h g_s(x) = g_s(x+h)-g_s(x)$. If $s \neq n$, (1.0) and the mean-value theorem give the estimates

$$|\Delta_h g_s(x)| \leqslant \begin{cases} c_s(|x+h|^{s-n}+|x|^{s-n}), & |x| \leqslant 2|h|,\\ c_s|x|^{s-n-1}|h|, & |x| > 2|h|.\end{cases}$$

Since $n+(s-n-1)q \leqslant sp-n-p < 0$, if follows that $\Delta_h g_s \in L^q$ and a simple computation yields $\|\Delta_h g_s\|_q \leqslant C'_{p,s}|h|^{s-n/p}$. Thus,

$$|\Delta_h f(x)| = |\Delta_h g_s * u| \leqslant C'_{p,s}|h|^{s-n/p}\|f\|_{p,s}.$$

For $s = n$, we have likewise

$$|\Delta_h g_n(x)| \leqslant \begin{cases} c'_n\big|\log|x+h|-\log|x|\big|, & |x| \leqslant 2|h|,\\ c'_n|x|^{-1}|h|, & |x| > 2|h|.\end{cases}$$

Since $n/q = n-n/p$, it suffices, by the preceding argument, to verify the estimate

$$I_q(h) \equiv \int\limits_{|x|\leqslant 2|h|} |\Delta_h \log|x||^q\, dx \leqslant C_q|h|^n$$

which in fact holds for any $q \geqslant 1$.

Now, $\log\big|1-|h|\,|x|^{-1}\big| \leqslant \log(|x+h|/|x|) \leqslant \log(1+|h|\,|x|^{-1})$ implies that

$$\big|\log(|x+h|/|x|)\big| \leqslant \log(1+|h|\,|x|^{-1})+\big|\log\big|1-|h|\,|x|^{-1}\big|\big|.$$

Hence, letting $r = |x|$ and then $y = |h|r^{-1}$, we can easily see that

$$I_q(h) \leqslant 2^{q-1}C|h|^n \int\limits_{\frac{1}{2}}^{\infty} \{[\log(1+y)]^q+\big|\log|1-y|\big|^q\}y^{-n-1}dy = C_q|h|^n. \ \blacksquare$$

Remarks. 1.5 (a). In part (a) of the preceding proof, Case 1 is false when $p = 1$, whereas Case 2 remains true (that is, $\mathscr{L}_t^1 \subset \mathscr{L}_r^q$ continuously if $q^{-1} > 1-(t-r)/n$) since Young's theorem also holds for $p = 1$.

1.5 (b). Theorem 1.4 (b) also holds for $p = 1$ or ∞. In fact, for $p = 1$, $n < s < n+1$ implies that g_s is bounded and $\|\Delta_h g_s\|_\infty \leqslant C_s|h|^{s-n}$. For $p = \infty$, $0 < s < 1$ implies that $g_s \in L^1$ and that $\|\Delta_h g_s\|_1 \leqslant C_s|h|^s$.

§ 2. The space $\mathscr{H}_s^p$, $p = 1, \infty$. In view of the results in [6] (Chapter VII, § 3) we denote by $H^1 = H^1(E^n)$ the class of all $f \in L^1$ such that $R_j(f) \in L^1$, $j = 1, \ldots, n$, equipped with the norm $\|f\|_{1,0} = \|f\|_1+\sum_{j=1}^{n}\|R_j f\|_1$. We let $\mathscr{S}_0^1 = \{f \in \mathscr{S}\colon \hat{f} \in \mathscr{D}(E_0^n)\}$. Then (see [6], pp. 230–233) H^1 is a Banach space and $\mathscr{S}_0^1$ is dense in H^1. Moreover, if $m \in C^n(E_0^n)$ satisfies the estimates $|\zeta|^{|\alpha|}|D^\alpha m(\zeta)| \leqslant C$ for $0 \leqslant |\alpha| \leqslant n$, the operator T_m, given by $(T_m f)^\wedge = m(\zeta)\hat{f}$, is bounded on H^1. In fact ([3], § 3), any $m(\zeta)$ satisfying the assumptions of Mihlin's Fourier-multiplier theorem, and any $m(\zeta) = [\text{p.v.}\, K(x)]^\wedge$ where $K(x) = \Omega(x)|x|^{-n}$ is a Calderón–Zygmund kernel, give rise to Fourier-multipliers T_m which are bounded on H^1. In particular, $R_j\colon H^1 \to H^1$ continuously, $j = 1, \ldots, n$.

We denote by $H^\infty = H^\infty(E^n)$ the class of all functions of bounded mean oscillation, where we identify functions which differ by an additive constant. Then, H^∞ is a Banach space with norm

$$\|f\|_* = \sup\{|Q|^{-1}\int\limits_Q |f(x)-f_Q|\,dx\}$$

where f_Q = mean value of f on Q and the sup is taken over all cubes $Q \subset E^n$. By a theorem of Fefferman ([2]; [3], § 2) we know that H^∞ is the dual of H^1, with dual pairing

$$(2.0) \qquad \langle f, g\rangle = \int f(x)g(x)\,dx, \qquad f \in \mathscr{S}_0^1 \text{ and } g \in H^\infty,$$

which extends, by continuity, to a continuous bilinear form on $H^1 \times H^\infty$. Moreover, $g \in H^\infty$ if and only if $g = g_0+\sum_{j=1}^{n} R_j(g_j)$ where $g_0, g_1, \ldots, g_n$ are in L^∞.

It is well known (e.g. [3], § 1) that $R_j\colon L^\infty \to H^\infty$ continuously. Hence, for any $g \in H^\infty$, $g = g_0+\sum_1^n R_j(g_j)$ as above, we have

$$\|g\|_* \leqslant \|g_0\|_*+\sum_1^n\|R_j g_j\|_* \leqslant 2\|g_0\|_\infty+\sum_1^n c_j\|g_j\|_\infty$$

and a familiar argument shows that $\|g\|_*$ is equivalent to the norm

$$\|g\|_{\infty,0} = \inf\{\|g_0\|_\infty + \sum_{j=1}^{\infty} \|g_j\|_\infty\}$$

which, from now on, will be taken as our norm on H^∞, where the inf is taken over all possible representations of g in terms of $g_0, g_1, \ldots, g_n$ in L^∞.

The following lemma is implicit in [3].

Lemma 2.0. *If $g \in L^1$, then $Tf = g*f$ is bounded on H^1, and extends to a bounded operator on H^∞, with norm $\leqslant \|g\|_1$.*

Proof. Taking Fourier transforms, we see that $R_jT = TR_j$ on $\mathscr{S}_0^1$. Hence, for any $f \in \mathscr{S}_0^1$,

$$\|Tf\|_{1,0} = \|g*f\|_1 + \sum_1^n \|R_j(Tf)\|_1$$
$$\leqslant \|g\|_1\Big\{\|f\|_1 + \sum_1^n \|R_jf\|_1\Big\} = \|g\|_1 \|f\|_{1,0}.$$

Since $\mathscr{S}_0^1$ is dense in H^1, T is bounded on H^1 with norm $\leqslant \|g\|_1$. Clearly, the same holds for the operator $\check{T}f = \check{g}*f$, where $\check{g}(x) = g(-x)$. Since H^∞ is the dual of H^1 with pairing given by (2.0), the dual of T is an extension of T to a bounded operator on H^∞ with norm $\leqslant \|g\|_1$. ∎

With $p = 1$ or ∞, and any $k \in \boldsymbol{N}$, we consider the Sobolev-type spaces $H_k^p = \{f \in H^p\colon (D^\alpha f) \in H^p$, for all $|\alpha| \leqslant k\}$ with norms

$$\|f\|_{p,k} = \|f\|_{p,0} + \sum_{|\alpha|\leqslant k} \|D^\alpha f\|_{p,0}.$$

The completeness of H^1 and H^∞ imply, in the usual way, that H_k^1 and H_k^∞ are Banach spaces. Furthermore, a slight adaptation of the proof of the density of $\mathscr{S}_0^1$ in H^1 ([6], p. 231) shows that $\mathscr{S}_0^1$ is dense in H_k^1 for all $k \in \boldsymbol{N}$.

As shown in [3], many singular convolution operators (such as the Calderón–Zygmund operators and hence the transforms R_j) extend to bounded operators on H^∞. This fact and Lemma 2.0 yield an easy consequence.

Lemma 2.1. *Let $Tf = K*f$, where $K \in L^1$ or else $K(x)$ is a Calderón–Zygmund kernel. Then for $p = 1$ or ∞ and all $k \in \boldsymbol{N}$, $T\colon H_k^p \to H_k^p$ continuously.*

Proof. Viewing T and D^α as Fourier multipliers on $\mathscr{S}'$, we have that $D^\alpha T = TD^\alpha$ on $\mathscr{S}'$. For any $|\alpha| \leqslant k$ and $f \in H_k^p$, $D^\alpha f \in H^p$ and hence $T(D^\alpha f) \in H^p$ also. Therefore, $D^\alpha(Tf) = T(D^\alpha f) \in H^p$ and

$$\|Tf\|_{p,k} = \|Tf\|_{p,0} + \sum_{|\alpha|\leqslant k} \|D^\alpha(Tf)\|_{p,0}$$
$$\leqslant \|T\|\{\|f\|_{p,0} + \sum_{|\alpha|\leqslant k} \|D^\alpha f\|_{p,0}\} = \|T\|\,\|f\|_{p,k},$$

where $\|T\|$ denotes the norm of T on H^p, $p = 1$ or ∞. ∎

We examine now the action of the fractional integral I^1 on the spaces H_k^p.

Lemma 2.2. *If $p = 1$ or ∞ and $k \geqslant 0$, the operators $I^1\colon H_k^p \to H_{k+1}^p$ and $I^{-1}\colon H_{k+1}^p \to H_k^p$ are continuous.*

Proof. Since $I^1 f = g_1 * f$ with $g_1 \in L^1$, Lemmas 2.0 and 2.1 show that I^1 is bounded on H_k^p, for all $k \geqslant 0$. Let $n \geqslant 2$ (the case $n = 1$ is identical). By formula (1.1), for all $f \in H_k^p$, $D_j(I^1 f) = R_j f - R_j(K_1 f) + K_2 f$ where $K_1 f$ and $K_2 f$ are convolutions with kernels in $\mathscr{S}$. Thus, by Lemma 2.1, $D_j I^1$ is bounded on H_k^p for any $j = 1, \ldots, n$ and hence $I^1\colon H_k^p \to H_{k+1}^p$ continuously.

Similarly, by (1.1′), for all $f \in H_{k+1}^p$, $I^{-1} f = \Lambda f - \Lambda(K_3 f) + K_4 f$ where $\Lambda = \sum_1^n R_j D_j$, $K_3 f$ and $K_4 f$ are convolutions with kernels in $\mathscr{S}$. Since $R_j D_j = D_j R_j$ on $\mathscr{S}'$ and $R_j(D_j f) \in H_k^p$ for all $f \in H_{k+1}^p$, we see by Lemma 2.1 that $\|\Lambda f\|_{p,k} \leqslant C\|f\|_{p,k+1}$ and the conclusion follows readily. ∎

Viewing H^p, $p = 1$ or ∞, as linear subspaces of $\mathscr{S}'$, we may define, for any real s, the spaces $\mathscr{H}_s^p = \mathscr{H}_s^p(E^n)$ as the images $\mathscr{H}_s^p = I^s(H^p)$ equipped with the norms $\|f\|_{p,s} = \|I^{-s} f\|_{p,0}$ which make them Banach spaces isometrically isomorphic to H^p. From Lemma 2.2, the equality (with equivalence of the norms)

$$\mathscr{H}_k^p = H_k^p, \quad \text{for} \quad p = 1 \text{ or } \infty \text{ and all } k \in \boldsymbol{N}, \tag{2.1}$$

follows at once.

Theorem 2.3. *Let $p = 1$ or ∞, and let r, s, t be real. Then:*

(a) *$I^s\colon \mathscr{H}_r^p \to \mathscr{H}_{r+s}^p$ is an isometric isomorphism.*

(b) *If $s < t$, $\mathscr{H}_t^p \subset \mathscr{H}_s^p$ and the inclusion is continuous.*

(c) *$\mathscr{S}_0^1$ and $\mathscr{H}_\infty^1 = \bigcap_s \mathscr{H}_s^1$ are dense in $\mathscr{H}_s^1$.*

(d) *$\mathscr{H}_{-s}^\infty$ is the dual of $\mathscr{H}_s^1$ with dual pairing*

$$\langle f, g\rangle = \int fg\,dx, \quad \text{for all } f \in \mathscr{H}_\infty^1 \text{ and } g \in \mathscr{H}_\infty^\infty \tag{2.2}$$

which can be extended to a continuous bilinear form on $\mathscr{H}_s^1 \times \mathscr{H}_{-s}^\infty$.

Proof. (a) and (b) follow as in Theorem 1.1, using Lemma 2.0. Part (c) is proved in the same way using the fact that, for any real s, $I^s\colon \mathscr{S}_0^1 \to \mathscr{S}_0^1$ is one-one and onto. (Notice that if $f \in \mathscr{S}_0^1$, then $d(\zeta)^{-s}\hat{f}(\zeta)$ is again in $\mathscr{D}(E_0^n)$ for any real s.) For part (d), we have as before that, for all $f \in \mathscr{H}_\infty^1$ and $g \in \mathscr{H}_\infty^\infty$, $\langle f, g\rangle = \langle I^{-s} f, I^s g\rangle$, $s > 0$. The rest of the proof follows by the same argument as for Theorem 1.1, except that instead of Hölder's inequality and the duality of L^q with L^p we now use Theorem 2 of [3] on the duality of H^∞ with H^1. ∎

Corresponding to (1.4) we now have that, *for all real s*,

(2.3) $$D_j\colon \mathscr{H}^p_s \to \mathscr{H}^p_{s-1} \text{ is continuous, if } p = 1 \text{ or } p = \infty.$$

The proof is similar to that of (1.4), using however Lemma 2.0 and the boundedness of R_j on H^1 and on H^∞.

We omitted to define H^p_k and $\mathscr{H}^p_s$ for $1 < p < \infty$ only because the corresponding definitions would yield again the spaces L^p_k and $\mathscr{L}^p_s$, respectively, with new norms equivalent to the old ones. In view of this, we now state only a partial analogue of Theorem 1.2.

THEOREM 2.4. *Let T be a Fourier-multiplier operator which is bounded on H^1 (or on H^∞) with norm $\|T\|$. Then, for all $s > 0$,*

(a) *if $p = 1$ (or $p = \infty$), $T\colon \mathscr{H}^p_s \to \mathscr{H}^p_s$ is bounded with norm $\leqslant \|T\|$.*

(b) *T extends, by continuity, to a bounded operator on $\mathscr{H}^1_{-s}$ with norm $\|T\|$.*

Proof. As Fourier multipliers, T and I^s commute on $\mathscr{S}'$, for all real s. Hence, for any $f = I^s u$ in $\mathscr{H}^p_s$, $\|Tf\|_{p,s} = \|I^{-s}(Tf)\|_{p,0} = \|Tu\|_{p,0} \leqslant \|T\|\,\|u\|_{p,0} = \|T\|\,\|f\|_{p,s}$ and (a) is proved. Now, by part (c) of Theorem 2.3, $\mathscr{S}^1_0$ is dense in $\mathscr{H}^1_{-s}$. Thus, for any $f \in \mathscr{S}^1_0$, $\|Tf\|_{1,-s} = \|I^s(Tf)\|_{1,0} \leqslant \|T\|\,\|I^s f\|_{1,0} = \|T\|\,\|f\|_{1,-s}$ and the conclusion follows. ∎

For any $k \in \boldsymbol{N}$, let us define H^∞_{-k} to be the dual of H^1_k. Then, formula (2.1) and Theorem 2.3 (d) show that, with equivalence of norms,

(2.4) $$H^\infty_{-k} = \mathscr{H}^\infty_{-k}, \qquad k \in \boldsymbol{N}.$$

Another representation of H^∞_{-k} is given by the following analogue of Theorem 1.3.

THEOREM 2.5. *For any $k \in \boldsymbol{N}$, H^∞_{-k} coincides with the space of all $g \in \mathscr{S}'$ of the form $g = g_0 + \sum_{|a|=k} D^a g_a$, where g_0 and g_a are in H^∞, and the norm*

$$\|g_0\|_{\infty,0} + \sum_{|a|=k} \|g_a\|_{\infty,0}$$

is equivalent to $\|g\|_{\infty,-k}$.

Proof. As in the proof of Theorem 1.3 (a), we may verify that $H^1_k = \{f \in H^1\colon D^a f \in H^1 \text{ for all } |a| = k\}$ with (equivalent) norm $\|f\|_{1,k} = \|f\|_{1,0} + \sum_{|a|=k} \|D^a f\|_{1,0}$. Moreover, by definition each $g \in H^\infty_{-k}$ is a bounded linear functional $g(f) = \langle f, g\rangle$ on H^1_k.

Consider the product space $H^1 \times \prod_{|a|=k} H^1$ with norm $\|h\| = \|h_0\|_{1,0} + \sum_{|a|=k} \|h_a\|_{1,0}$ where $h = (h_0, h_a)_{|a|=k}$ and $h_0, h_a \in H^1$. Then the mapping $f \to (f, D^a f)_{|a|=k}$ gives an isometric embedding of H^1_k into our product space. So, by the Hahn–Banach theorem, we can extend each $g \in H^\infty_{-k}$ to a bounded linear functional $\tilde{g}$ on $H^1 \times \prod_{|a|=k} H^1$, that is, to an element

$g = (g_0, \tilde{g}_a)_{|a|=k}$ of the product space $H^\infty \times \prod_{|a|=k} H^\infty$. Hence, for all $f \in H^1_k$,

$$\begin{aligned}\langle f, g\rangle = \langle f, \tilde{g}\rangle &= \langle f, g_0\rangle + \sum_{|a|=k} \langle D^a f, \tilde{g}_a\rangle \\ &= \langle f, g_0\rangle + \sum_{|a|=k} \langle f, (-1)^k D^a \tilde{g}_a\rangle \\ &= \langle f, g_0 + \sum_{|a|=k} D^a (-1)^k \tilde{g}_a\rangle\end{aligned}$$

so that $g = g_0 + \sum_{|a|=k} D^a g_a$ where g_0 and $g_a = (-1)^k \tilde{g}_a$ are in H^∞.

Conversely, by (2.3) and Theorem 2.3 (b), any g of this form belongs to $\mathscr{H}^\infty_{-k}$. Moreover, by (2.4), $\mathscr{H}^\infty_{-k} = H^\infty_{-k}$. Finally, using (2.3), we see that

$$\|g\|_{\infty,-k} \leqslant \|g_0\|_{\infty,-k} + \sum_{|a|=k} \|D^a g_a\|_{\infty,-k} \leqslant C_k\{\|g_0\|_{\infty,0} + \sum_{|a|=k} \|g_a\|_{\infty,0}\}$$

and the equivalence of the norms follows. ∎

Our next objective is to show that H^1, and hence the spaces $\mathscr{H}^1_s$, are also the duals of certain Banach spaces. In particular, we shall see that H^1 (and hence H^∞) is not reflexive.

Let $C_0(E^n)$ denote the space of all continuous functions on E^n which vanish at ∞, equipped with the sup norm, and let $\mathscr{B}(E^n)$ denote the Banach space of all finite Borel measures ν on E^n, with norm $\|\nu\|_1 = \int |d\nu(x)|$. We consider now the class $G^\infty = G^\infty(E^n)$ of all locally integrable functions g which can be represented in the form $g = g_0 + \sum_{j=1}^{n} R_j g_j$ for some $g_0, g_1, \ldots, g_n$ in $C_0(E^n)$. If we equip G^∞ with the (H^∞) norm

$$\|g\|_{\infty,0} = \inf\Big\{\|g_0\|_\infty + \sum_1^n \|g_j\|_\infty\Big\}$$

where the inf is taken over all possible representations of g, then G^∞ is a Banach space, in fact a proper subspace of H^∞. (The function $\log|x|$ is in H^∞ but not in G^∞.)

Let $\mathscr{B}^1(E^n)$ denote the class of all $\mu \in \mathscr{B}(E^n)$ for which there are $\mu_j \in \mathscr{B}(E^n)$ satisfying $\hat{\mu}_j(\zeta) = \zeta_j|\zeta|^{-1}\hat{\mu}(\zeta)$, $j = 1, \ldots, n$. As in [6], p. 221, we then say that $\mu_j = R_j\mu$ are the Riesz transforms of μ. Since $\mathscr{B}(E^n)$ is the dual of $C_0(E^n)$, it follows that, equipped with the norm $\|\mu\|_{1,0} = \|\mu\|_1 + \sum_1^n \|R_j\mu\|_1$, $\mathscr{B}^1(E^n)$ is the dual of G^∞. On the other hand, by the n-dimensional F. and M. Riesz Theorem (see [7], Theorem E), such $\mu \in \mathscr{B}^1(E^n)$ (and μ_j) are absolutely continuous with respect to the Lebesgue measure and so they correspond to L^1 functions. Hence, with dual pairing

(2.0′) $$\langle g, f\rangle = \int g f\,dx, \qquad g \in G^\infty \text{ and } f \in H^1,$$

it follows that H^1 is the dual of G^∞. Since the dual of H^1 is H^∞, we have in particular that H^1 is not reflexive.

Remark 2.6. Just as the elements of H^∞ can be characterized as functions of bounded mean oscillation (B.M.O.), it is easy to see (approximating $g_0, g_1, \ldots, g_n$ by smooth functions and using the boundedness of R_j: $L^\infty \to H^\infty$) that each $g \in G^\infty$ has continuous mean oscillation (C.M.O.). Specifically, for each $g \in G^\infty$ there exists a positive set function $C(Q)$ such that

$$|Q|^{-1} \int_Q |g(x) - g_Q|\, dx \leqslant C(Q)$$

where $C(Q)$ is bounded above and $C(Q) \to 0$, as $|Q| \to 0$ or $+\infty$, and as the center of Q tends to ∞. Moreover, it is also known (but more difficult to verify) that all functions in C.M.O. are in fact elements of G^∞. (This characterization of G^∞ seems to be due to Herz, Strichartz and Sarason.)

LEMMA 2.7. *For any real $s > 0$, I^s: $G^\infty \to G^\infty$ is continuous.*

Proof. Since I^s commutes with R_j and is a convolution with the integrable kernel g_s, it suffices to show that, for any g in $C_0(E^n)$, the continuous function $I^s g$ vanishes at infinity. Fix any $\varepsilon > 0$ and let $c_s = \|g_s\|_1$. Since $g \in C_0(E^n)$, there is an $N = N(\varepsilon) \in \boldsymbol{N}$ such that $\sup\{|g(x)|: |x| > N\} < \varepsilon (c_s)^{-1}$. Let now $g = g_0 + g_1$ where $g_0 = g$ if $|x| \leqslant N$, $g_0 = 0$ otherwise. Since g_0 is a bounded function with bounded support and g_s is rapidly decreasing away from the origin, it is easy to see that $I^s g_0 = g_s * g_0$ vanishes at infinity. Hence, the same is true for $I^s g$, since $\|I^s g_1\|_\infty \leqslant c_s \|g_1\|_\infty < \varepsilon$. ■

For any real s, we define the spaces $\mathscr{G}_s^\infty = \mathscr{G}_s^\infty(E^n)$ to be the images of G^∞ under I^s equipped with the same norm as $\mathscr{H}_s^\infty$, so that the $\mathscr{G}_s^\infty$ are Banach spaces isometrically isomorphic to G^∞.

THEOREM 2.8. *For any real s, $\mathscr{H}_{-s}^1$ is the dual of $\mathscr{G}_s^\infty$ with respect to the duality induced by* (2.0′).

Proof. It suffices to consider the case $s > 0$. By Lemma 2.7, if $t > s$ then I^{t-s} is a continuous map of G^∞ into itself. Hence $\mathscr{G}_t^\infty \subset \mathscr{G}_s^\infty$ continuously. Moreover, if $\mathscr{G}_\infty^\infty = \bigcap_t \mathscr{G}_t^\infty$, then I^s maps $\mathscr{G}_\infty^\infty$ into itself (for any real s) and, for all $g \in \mathscr{G}_\infty^\infty$ and $f \in \mathscr{H}_\infty^1$, $\langle g, f\rangle = \langle I^s g, I^{-s} f\rangle$. The rest of the proof now follows by the same argument used in Theorem 1.1 (d), using the fact that H^1 is the dual of G^∞. ■

We shall prove next an embedding theorem for the spaces $\mathscr{H}_s^1$ which makes use of the following result.

LEMMA 2.9. *For any $f \in H^1$ and $g \in H^\infty$, $f * g$ is a bounded, uniformly continuous function satisfying*

$$\|f * g\|_\infty \leqslant \|f\|_{1,0} \|g\|_{\infty,0}. \tag{2.5}$$

Proof. Let g be represented by $g_0 + \sum_1^n R_j g_j$, where g_0 and the g_j are in L^∞, so that

$$f * g = f * g_0 + \sum_1^n f * R_j g_j.$$

Let us note that

$$f * R_j g_j = -(R_j f) * g_j.$$

In fact, by translation invariance, it suffices to verify this formula at $x = 0$. Moreover, since $\mathscr{S}_0^1$ is dense in H^1 and R_j is continuous on H^1, we may assume that $f \in \mathscr{S}_0^1$. But then, by the duality (2.0) and the fact that the R_j are (principal value) convolutions with odd kernels, the formula is readily verified.

Thus, $f * g = f * g_0 - \sum_1^n (R_j f) * g_j$. Since f and the $R_j f$ are in L^1, it follows that $f * g$ is a bounded, uniformly continuous function such that

$$\|f * g\|_\infty \leqslant \|f\|_1 \|g_0\|_\infty + \sum_1^n \|R_j f\|_1 \|g_j\|_\infty \leqslant \|f\|_{1,0} \left(\|g_0\|_\infty + \sum_1^n \|g_j\|_\infty\right).$$

Consequently, taking the inf over all possible representations of g, we obtain (2.5). ■

THEOREM 2.10. (a) *If $1 \leqslant q \leqslant \left(1 - \dfrac{t-r}{n}\right)^{-1} < \infty$, then $\mathscr{H}_t^1 \subset \mathscr{L}_r^q$ and the inclusion is continuous.*

(b) *If $0 \leqslant s - n \leqslant 1$, each f in $\mathscr{H}_s^1$ is uniformly continuous and, for all x, $|f(x)| \leqslant C_s \|f\|_{1,s}$, $|f(x+h) - f(x)| \leqslant C_s \|f\|_{1,s} |h|^{s-n}$.*

Proof. (a) Letting $s = t - r$, we must show that I^s is bounded from H^1 into L^q. If $s = 0$, then $q = 1$ and the continuous inclusion $\mathscr{H}_t^1 \subset \mathscr{L}_t^1$ follows directly from the definition of H^1. Let now $0 < s < n$. If $q^{-1} = 1 - s/n$, we apply Theorem H of [7] and argue as in the proof of Theorem 1.4 (a). If $q^{-1} > 1 - s/n$, the conclusion follows from Remark 1.5 (a) since $\mathscr{H}_t^1 \subset \mathscr{L}_t^1$ continuously.

(b) By virtue of Remark 1.5 (b) for $p = 1$, it suffices to consider the extreme cases $s = n$ and $s = n+1$. If $s = n$ and $f \in \mathscr{H}_n^1$, we have that $f = g_n * u$ where $u \in H^1$ and, by formula (1.0), $g_n \in H^\infty$. Hence, Lemma 2.9 gives desired conclusion, since $\|u\|_{1,0} = \|f\|_{1,n}$.

If $s = n+1$, we note that g_{n+1} is bounded. Moreover, $g_{n+1} = I^1 g_n$ so that, by formula (1.1), $D_j g_{n+1} = R_j g_n - S_j g_n$ where $S_j = R_j K_1 - K_2$ is again a bounded operator on H^∞. Therefore, with $f = g_{n+1} * u$, the mean-value theorem and Lemma 2.9 imply that

$$|f(x+h) - f(x)| \leqslant C_n |h| \|u\|_{1,0} = C_n |h| \|f\|_{1,n+1}. \ ■$$

In order to obtain H^1 analogues of formula (2.4) and Theorem 2.5, we consider once more the space G^∞. The key fact is given by the following theorem. (The proof below is due to C. Fefferman.)

THEOREM 2.11. *The Riesz transforms R_j are bounded operators on G^∞.*

Proof. Recall that G^∞ is equipped with the H^∞ norm and that R_j is bounded on H^∞. Since R_j maps $L^2_\infty = \bigcap_s L^2_s$ into itself, it suffices to show that this class is dense in G^∞.

Clearly, any function in $C_0(E^n)$ is the uniform limit of continuous functions with compact support. Consequently, given any $g \in G^\infty$, $g = g_0 + \sum_1^n R_j g_j$, and any $\varepsilon > 0$, there exists a function $h = h_0 + \sum_1^n R_j h_j$, where h_0 and h_j are continuous and have compact support, such that

$$\|g-h\|_{\infty,0} \leqslant \|g_0-h_0\|_\infty + \sum_1^n \|g_j-h_j\|_\infty < \varepsilon .$$

Now, if $\{\varphi_t\}$, $t>0$, is an approximate identity with $\varphi \in \mathscr{D}$, we have that $\varphi_t * h = (\varphi_t * h_0) + \sum_1^n R_j(\varphi_t * h_j)$. Since for $j = 0, 1, \ldots, n$ we have that $\|h_j - (\varphi_t * h_j)\|_\infty \to 0$ as $t \to 0$, and $(\varphi_t * h_j) \in \mathscr{D}$, we may assume by the preceding argument that the functions h_0 and h_j are in $\mathscr{D}$. In particular, the functions h_0 and $R_j h_j$ belong to L^2_∞; hence, so does h. ■

For any $k \in N$, we let G^∞_k denote the kth Sobolev space of G^∞ equipped with the usual norm, and we define H^1_{-k} to be the dual of G^∞_k with respect to the duality induced by formula (2.0′).

LEMMA 2.12. *For any $k \in \mathbf{N}$, $G^\infty_k = \mathscr{G}^\infty_k$ with equivalence of the norms.*

Proof. By Lemma 2.7, I^1 is bounded on G^∞. Hence, as in the proof of Lemma 2.1, I^1 is bounded on G^∞_k for any k. By (1.1), $D_j I^1 = R_j - R_j K_1 + K_2$ where K_1 and K_2 are convolution operators with kernels in $\mathscr{S}$. Thus, with the same proof of Lemma 2.7, we see that K_1 and K_2 are bounded on G^∞. Since, by Theorem 2.11, R_j is bounded on G^∞, it follows that $D_j I^1$ is bounded on G^∞ and hence on G^∞_k. Therefore, $I^1\colon G^\infty_k \to G^\infty_{k+1}$ continuously.

Similarly, by (1.1′), for all $f \in G^\infty_{k+1}$, $I^{-1}f = \Lambda f - \Lambda K_3 f + K_4 f$ where $K_3 f$ and $K_4 f$ are convolutions with kernels in $\mathscr{S}$, and $\Lambda = \sum_1^n R_j D_j$. Now, the boundedness of $D_j\colon G^\infty_{k+1} \to G^\infty_k$ and of R_j on G^∞_k show that $\Lambda\colon G^\infty_{k+1} \to G^\infty_k$ continuously, and from this the boundedness of $I^{-1}\colon G^\infty_{k+1} \to G^\infty_k$ follows readily. Consequently, $I^k\colon G^\infty \to G^\infty_k$ is a continuous isomorphism and the conclusion is an immediate consequence of the definition of $\mathscr{G}^\infty_k = I^k(G^\infty)$. ■

THEOREM 2.13. *For any $k \in \mathbf{N}$, $H^1_{-k} = \mathscr{H}^1_{-k}$ with equivalence of the norms. Moreover, H^1_{-k} coincides with the space of all $f \in \mathscr{S}'$ of the form*

$f = f_0 + \sum_{|\alpha|=k} D^\alpha f_\alpha$, where f_0 and f_α are in H^1, and the norm $\|f_0\|_{1,0} + \sum_{|\alpha|=k} \|f_\alpha\|_{1,0}$ is equivalent to $\|f\|_{1,-k}$.

Proof. The first conclusion follows at once from Lemma 2.12 and Theorem 2.8. In view of the proofs of Lemma 2.12 and of Theorem 1.3 (a), we verify without difficulty that $G^\infty_k = \{f \in G^\infty\colon D^\alpha f \in G^\infty \text{ for all } |\alpha| = k\}$ with equivalent norm

$$\|f\|_{\infty,k} = \|f\|_{\infty,0} + \sum_{|\alpha|=k} \|D^\alpha f\|_{\infty,0}.$$

Using the fact that H^1 is the dual of G^∞, the rest of the proof is identica to that of Theorem 2.5, with G^∞ in place of H^1 and H^∞ replaced by H^1. ■

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