As a final application, let us indicate how the preceding discussion can be used to describe the isometries of certain Banach spaces of Lipschitz functions. Specifically, let (X, ϱ) be a compact metric space and $\mathrm{Lip}(X, \varrho)$ denote the linear space of complex-valued continuous functions f on X for which $\|f\|_\varrho < \infty$, where

$$\|f\|_\varrho = \sup\left\{\frac{|f(x)-f(t)|}{\varrho(x, t)} : x, t \in X, x \neq t\right\}.$$

If $\mathrm{Lip}(X, \varrho)$ is equipped with the norm

$$\|f\| = \max\{\|f\|_\infty, \|f\|_\varrho\}, \quad f \in \mathrm{Lip}(X, \varrho),$$

then $(\mathrm{Lip}(X, \varrho), \|\cdot\|)$ is a Banach space. Its structure had been the subject of considerable study in recent years, and we state below one additional property of this space, which is an easy consequence of work that appears in [7], pp. 1150–1156, and in [9], Theorem 5.1, p. 1397, and of that which we have done above. The proof is omitted altogether.

THEOREM 2. *Let (X, ϱ) and (Y, δ) be compact, connected metric spaces, each of diameter at most* 1. *Then a map*

$$T: (\mathrm{Lip}(X, \varrho), \|\cdot\|) \to (\mathrm{Lip}(Y, \delta), \|\cdot\|)$$

is a linear isometry onto if and only if there is a metric space isometry h of (Y, δ) onto (X, ϱ) and a complex number a of modulus 1 *such that*

$$Tf(y) = af(h(y)) \quad (f \in \mathrm{Lip}(X, \varrho),\ y \in Y).$$

References

[1] J. W. Baker, *The separable projection property*, in *Approximation theory*, Academic Press, New York and London, 1973, pp. 247–250.

[2] N. Dunford and J. T. Schwartz, *Linear Operators*, Part I, Interscience, New York, 1958.

[3] K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice-Hall, Englewood Cliffs, N. J., 1962.

[4] W. Holsztyński, *Continuous mappings induced by isometries of spaces of continuous functions*, Studia Math. 26 (1966), pp. 133–136.

[5] A. Pełczyński, *On C(S)-subspaces of separable Banach spaces*, ibid. 31 (1968), pp. 513–522.

[6] R. R. Phelps, *Lectures on Choquet's Theorem*, Van Nostrand Mathematical Studies # 7, D. Van Nostrand, Princeton, N. J., 1966.

[7] A. K. Roy, *Extreme points and linear isometries of the Banach space of Lipschitz functions*, Canad. J. Math. 20 (1968), pp. 1150–1164.

[8] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, New York, 1966.

[9] D. R. Sherbert, *Banach algebras of Lipschitz functions*, Pacific J. Math. 13 (1963), pp. 1387–1399.

FLORIDA STATE UNIVERSITY
TALLAHASSEE, FL. 32306 U.S.A.

Received February 12, 1974 (820)

STUDIA MATHEMATICA, T. LIII. (1975)

Constructive function theory and spline systems

by

Z. CIESIELSKI (Sopot)

Abstract. Integrating $m+1$ times, $m \geq -1$, the Haar orthogonal functions we obtain a set od spline functions of order m. We complete this set of functions with the monomials $1, \ldots, t^m$. Now, applying the Schmidt orthonormalization procedure, we get a complete orthonormal set $\{f_n^{(m)}, n \geq -m\}$ of splines of order m. Differentiating and integrating k times the functions $f_n^{(m)}$, $0 \leq k \leq m+1$, we obtain new systems $\{f_n^{(m,k)}, n \geq k-m\}$ and $\{g_n^{(m,k)}\ n \geq k-m\}$ of splines óf order $m-k$ and $m+k$, respectively. All these systems are discussed as bases in various function spaces. The convergence a.e. for functions in L_1 is proved. The direct and inverse theorems of approximation theory for partial sums corresponding to the expansions with respect to $\{f_n^{(m)}, n \geq -m\}$ are established. New characterization of the Hölder classes is obtained and the linear isomorphism between these classes and suitable sequence spaces is exhibited.

1. Introduction. In this paper the investigations of spline systems started in the works [4]–[11] are continued. Some of the results were announced in [9].

We are concerned with systems of splines which are bases in various classical function spaces.

In Section 3, partial sums of the spline expansions are treated as singular integrals and the convergence almost everywhere for functions in L_1 is established. Moreover, the norm and local estimates for the sup of the corresponding partial sums are obtained.

The order of approximation by partial sums of the spline expansions is established in Theorem 4.1. In Lemma 4.3 it is shown that this order of approximation is the best possible in W_1^{m+2} unless the functions are polynomials of degree not exceeding $m+1$. The same order of approximation in the case of periodic functions was established earlier in [21] where suitable interpolating spline bases were constructed. In the case of non-periodic functions and of interpolating bases similar results were obtained recently in [12].

The norms of the biorthogonal splines are estimated from above and from below in Section 5. Local estimates for these functions are obtained in Section 6.

Theorem 7.1 establishes the relation between the coefficients of the spline series and the series themselves.

The biorthogonal functionals are treated in Section 8 as bases in L_p and C. In particular, a variational characterization of an interpolating basis is given in Theorem 8.3.

The last section contains characterization of the functions satisfying the Hölder conditions, corresponding to higher order differences, in terms of the coefficients of their spline expansions. The linear isomorphism between suitable sequence spaces and the spaces of functions satisfying Hölder conditions is established.

2. Preliminaries. In what follows the notations introduced in [10] is going to be used. For the purpose of this paper it is convenient to consider the integer parameters k and m with the domain: $m \geqslant -1$ and $0 \leqslant k \leqslant m+1$. It appears that the restriction of the considerations in [10] to $m \geqslant 0$ and $0 \leqslant k \leqslant m$ was not essential, all the definitions and results can be extended to the wider domain of the parameters k and m. For the sake of completeness we recall the necessary notations, definitions and results, extending them simultaneously to the larger domain of k and m.

For each positive integer n a partition $\pi_n = \{s_{n,i}\colon i = 0, \pm 1, \ldots\}$ is defined as follows: for $n = 1$ we put $s_{n,i} = i$ and for $n = 2^\mu + \nu$, with the integers μ and ν such that $\mu \geqslant 0$, $1 \leqslant \nu \leqslant 2^\mu$,

$$s_{n,i} = \begin{cases} \dfrac{i}{2^{\mu+1}} & \text{for} \quad i \leqslant 2\nu, \\ \dfrac{i-\nu}{2^\mu} & \text{for} \quad i > 2\nu. \end{cases} \tag{2.1}$$

Thus, for each $n > 0$ we have $s_{n,i} < s_{n,i+1}$ and $s_{n,0} = 0$, $s_{n,n} = 1$. The sequence of partitions $\pi_1, \pi_2, \ldots$ induces in $I = \langle 0, 1\rangle$ the dyadic points $t_0, t_1, \ldots$ which are ordered as follows: $t_0 = 0$, $t_1 = 1$ and for $n = 2^\mu + \nu > 1$

$$t_n = s_{n,2\nu-1} = \frac{2\nu - 1}{2^{\mu+1}}. \tag{2.2}$$

Now, let

$$I_{n,i} = \begin{cases} \langle s_{n,i-1}, s_{n,i}) & \text{for} \quad i < n, \\ \langle s_{n,i-1}, s_{n,i}\rangle & \text{for} \quad i = n, \\ (s_{n,i-1}, s_{n,i}\rangle & \text{for} \quad i > n. \end{cases} \tag{2.3}$$

For each π_n the partition of unity of splines of order $m = -1$ is defined as follows

$$N_{n,i}^{(m)}(t) = \begin{cases} 1 & \text{for} \quad t \in I_{n,i}, \\ 0 & \text{for} \quad t \notin I_{n,i}. \end{cases} \tag{2.4}$$

It is known (cf. [20]) that for each $m \geqslant 0$ the following splines of order m

$$N_{n,i}^{(m)}(t) = (s_{n,i+m+1} - s_{n,i-1})[s_{n,i-1}, \ldots, s_{n,i+m+1}; (s-t)_+^{m+1}] \tag{2.5}$$

are nonnegative and form partition of unity corresponding to π_n; the square brackets denote the divided difference of $(s-t)_+^{m+1} = (\max(0, s-t))^{m+1}$ as a function of s taken at the points $s_{n,i-1}, \ldots, s_{n,i+m+1}$. It is a consequence of (2.4) and (2.5) that supp $N_{n,i}^{(m)} = \langle s_{n,i-1}, s_{n,i+m+1}\rangle$ for $m \geqslant -1$ and $i = 0, \pm 1, \ldots$

To fixed $m \geqslant -1$ and $n \geqslant -m$ corresponds finite-dimensional space $S_n^m(I)$ of spline functions of order m defined over I. If $-m \leqslant n \leqslant 0$, then $S_n^m(I)$ is defined as the linear span over $1, t, \ldots, t^{m+n}$, and if $n > 0$ then it is spanned by the linearly independent functions $N_{n,i}^{(m)}$, $i = -m, \ldots$ $\ldots, n$. Clearly, $S_n^m(I) \subset S_{n+1}^m(I)$ and $\dim S_n^m(I) = m+n+1$. Now, let $S^m(I) = \bigcup_n S_n^m(I)$. It should be clear that $S^m(I)$ is dense in $C(I)$ for each $m \geqslant 0$, and it is dense in $L_p(I)$ for $1 \leqslant p < \infty$ and $m \geqslant -1$.

Each function $f \in S^m(I)$, $m \geqslant 0$, has absolutely continuous derivative of order m. In what follows it is always required that $D^{m+1} f \in S^{-1}(I)$ for $f \in S^m(I)$, $m \geqslant 0$, and consequently $D^{m+1} f$ is assumed to be defined everywhere in I.

Now let us consider $S_n^m(I)$, $n \geqslant -m$, as subspaces of $L_2(I)$ with the scalar product

$$(f, g) = \int_I f(t) g(t)\, dt. \tag{2.6}$$

There is orthonormal, with respect to (2.6), system of spline functions of order m, $m \geqslant -1$, $\{f_n^{(m)}, n \geqslant -m\}$, such that $f_{-m}^{(m)} = 1$, $f_n^{(m)} \in S_n^m(I) \setminus S_{n-1}^m(I)$ for $n > -m$ and $(f_n^{(m)}, f_n^{(m)}) = 1$. Moreover, let us define

$$\begin{aligned} f_n^{(m,k)} &= D^k f_n^{(m)} && \text{for} \quad n \geqslant k-m, \ 0 \leqslant k \leqslant m+1, \\ g_n^{(m,k)} &= H^k f_n^{(m)} && \text{for} \quad n \geqslant k-m, \ 0 \leqslant k \leqslant m+2, \end{aligned} \tag{2.7}$$

where D is the differentiation operator and

$$Hf(t) = \int_t^1 f(u)\, du.$$

Defining

$$Gf(t) = \int_0^t f(u)\, du,$$

we check easily that $H = G^*$, i.e.

$$(Gf, g) = (f, Hg), \qquad f, g \in L_2(I). \tag{2.8}$$

The Sobolev space over I of order $r \geqslant 0$ corresponding to the exponent p, $1 \leqslant p < \infty$, is denoted by $W_p^r(I)$. Since $DGf = f$, we infer for $f \in W_1^k(I)$ with $0 \leqslant k \leqslant m+1$, $m \geqslant -1$, that equalities (2.7) and (2.8) imply

$$(D^k f, g_j^{(m,k)}) = (f, f_j^{(m)}), \quad j \geqslant k-m. \tag{2.9}$$

Consequently, since $f_j^{(m)} \in S_n^m(I) \subset W_1^{m+1}(I)$, we obtain for $0 \leqslant k \leqslant m+1$, $m \geqslant -1$, from (2.9) and (2.7)

$$(f_i^{(m,k)}, g_j^{(m,k)}) = \delta_{i,j}, \quad i, j \geqslant k-m, \tag{2.10}$$

and it is assumed in this notation that $f_j^{(m,0)} = f_j^{(m)} = g_j^{(m,0)}$

The space of all k times continuously differentiable functions on I is denoted by $C^k(I)$, and it is used that $C(I) = C^0(I)$ and $L_p(I) = W_p^0(I)$.

The basis expansions which we shall consider are the Fourier series with respect to the orthonormal complete system of splines $\{f_n^{(m)}, n \geqslant -m\}$ and the expansions corresponding to the biorthogonal systems $\{f_j^{(m,k)}, g_i^{(m,k)}, i, j \geqslant k-m\}$ with $0 \leqslant k \leqslant m+1$, $m \geqslant -1$.

In the orthogonal case for given $m \geqslant -1$ and for a given function $f \in L_1(I)$ we have the partial sums

$$P_n^{(m)} f(t) = \int_I K_n^{(m)}(t, s) f(s)\, ds,$$

where

$$K_n^{(m)}(t, s) = \sum_{i=-m}^{n} f_i^{(m)}(t) f_i^{(m)}(s)$$

is the Dirichlet kernel. The operator $P_n^{(m)}\colon L_2(I) \to S_n^m(I)$ projects orthogonally $L_2(I)$ onto $S_n^m(I)$. It is convenient at this place to introduce notation for the projection operators, and for their kernels, corresponding to the partial sums of the biorthogonal expansions metioned above, i.e. for $0 \leqslant k \leqslant m+1$, $m \geqslant -1$ and $f \in L_1(I)$ let

$$P_n^{(m,k)} f(t) = \int_I K_n^{(m,k)}(t, s) f(s)\, ds,$$

$$K^{(m,k,l)}(t, s) = \sum_{i=k-m}^{n} f_i^{(m,k)}(t) g_i^{(m,l)}(s),$$

where $k, l \geqslant 0$, and by definition $K_n^{(m,k)} = K_n^{(m,k,k)}$. Clearly, $P_n^{(m,0)} = P_n^{(m)}$ and $K_n^{(m,0)} = K_n^{(m)}$. The operator $P_n^{(m,k)}\colon L_1(I) \to S_n^{m-k}(I)$ is a projection for each k, $0 \leqslant k \leqslant m+1$, $m \geqslant -1$. This is a consequence of (2.10).

The basic result established in [10], Theorem 2, can be extended to the wider domain of the parameters m, k, l, i.e. we have

THEOREM 2.1. *Let $m \geqslant -1$ be given. Then there are constants M_m and r_m, $0 < r_m < 1$, such that*

$$|K_n^{(m,k,l)}(t, s)| \leqslant M_m n^{k+1-l} r_m^{n|t-s|} \tag{2.11}$$

holds for $n > 0$, $m+1 \geqslant k \geqslant l \geqslant 0$ and $t, s \in I$.

The following result proved in [11] will be needed later too.

THEOREM 2.2. *To each m, $m \geqslant -1$, there are constants $M_m > 0$ and r_m, $0 < r_m < 1$, such that*

$$|f^{(m,k)}(t)| \leqslant M_m n^{k+\frac{1}{2}} r_m^{n|t-t_n|} \tag{2.12}$$

holds for all $t \in I$ and k, $0 \leqslant k \leqslant m+1$.

Another important result proved in [10], which is a consequence of Theorem 2.1, concerns the spline bases.

THEOREM 2.3. *Let $0 \leqslant k \leqslant m$. Then there is a constant $M_m > 0$ such that $\|P_n^{(m,k)}\|_\infty \leqslant M_m$, and consequently $\{f_j^{(m,k)}, j \geqslant k-m\}$ is a basis in $C(I)$ and for each $f \in C(I)$ we have*

$$f = \sum_{j=k-m}^{\infty} (f, g_j^{(m,k)}) f_j^{(m,k)}. \tag{2.13}$$

THEOREM 2.4. *Let $0 \leqslant k \leqslant m+1$, $m \geqslant -1$. Then there is $M_m > 0$ such that $\|P_n^{(m,k)}\|_p \leqslant M_m$ holds for all p, $1 \leqslant p \leqslant \infty$. Consequently, $\{f_j^{(m,k)}, j \geqslant k-m\}$ is a basis in $L_p(I)$ for each p, $1 \leqslant p < \infty$, and (2.13) holds in $L_p(I)$.*

It is worth while to realize that $\{f_j^{(m)}, j \geqslant -m\}$ for $m = -1$ is the Haar system, and for $m = 0$ the Franklin system.

Important role in the following sections is played by the Bernstein type inequality (c. f. [10]):

THEOREM 2.5. *Let $m \geqslant -1$. Then, exist constants M_m and r_m such that for each $f \in S_n^m(I)$ the inequality*

$$\|D^k f\|_p \leqslant M_m n^k \|f\|_p \tag{2.14}$$

holds for $0 \leqslant k \leqslant m+1$, $n > 0$ and $1 \leqslant p \leqslant \infty$.

It should be remembered that the constants M_m introduced above depend on m only.

3. Convergence almost everywhere and estimates for partial sums. The main result in this section is (in the case of $m = 0$; cf. [7])

THEOREM 3.1. *Let $0 \leqslant k \leqslant m+1$, $m \geqslant -1$. Then there is a constant M_m such that*

$$\int_t^1 \operatorname*{var}_{s \leqslant u \leqslant 1} K_n^{(m,k)}(t, u)\, ds \leqslant M_m \tag{3.1}$$

and

$$\int_0^t \operatorname*{var}_{0\leqslant u\leqslant s} K_n^{(m,k)}(t,u)\,ds \leqslant M_m \tag{3.2}$$

hold for $n>0$ and $t\in I$.

Proof. In the case of $m=-1$ the result follows immediately. Now, let $m\geqslant 0$ and let $t\in I_{n,i}$ for some $i=1,\ldots,n$. We are going to prove (3.1) only, the proof of (3.2) is similar.

$$\begin{aligned}\int_t^1 \operatorname*{var}_{s\leqslant u\leqslant 1} K_n^{(m,k)}(t,u)\,ds &\leqslant \sum_{h=i}^{n}\int_{I_{n,h}}\operatorname*{var}_{s\leqslant u\leqslant 1} K_n^{(m,k)}(t,u)\,ds \\ &\leqslant \sum_{h=i}^{n}|I_{n,h}|\operatorname*{var}_{s_{n,h-1}\leqslant u\leqslant 1} K_n^{(m,k)}(t,u) \\ &= \sum_{h=i}^{n}|I_{n,h}|\sum_{j=h}^{n}\operatorname*{var}_{u\in I_{n,j}} K_n^{(m,k)}(t,u).\end{aligned}\tag{3.3}$$

However, for $k>0$,

$$\begin{aligned}\operatorname*{var}_{u\in I_{n,j}} K_n^{(m,k)}(t,u) &= \int_{I_{n,j}}|D_u K_n^{(m,k)}(t,u)|\,du \\ &= \int_{I_{n,j}}|K_n^{(m,k,k-1)}(t,u)|\,du,\end{aligned}$$

whence by (2.11)

$$\operatorname*{var}_{u\in I_{n,j}} K_n^{(m,k)}(t,u) = O_m(|I_{n,j}|n^2 r_m^{n|t-s_{n,j}|}) = O_m(nq_m^{|i-j|}) \tag{3.4}$$

holds with some constants r_m and q_m satisfying the inequalities $0<r_m<q_m<1$; here and later on the index m at the capital O indicates that the bound depends on m only.

In the case of $k=0$, $K_n^{(m)}(t,s)$ is symmetric in t and s, and therefore (2.11) gives

$$\begin{aligned}\operatorname*{var}_{u\in I_{n,j}} K_n^{(m,k)}(t,u) &= \int_{I_{n,j}}|D_u K_n^{(m)}(t,u)|\,du \\ &= \int_{I_{n,j}}|D_u K_n^{(m)}(u,t)|\,du = \int_{I_{n,j}}|K_n^{(m,1,0)}(u,t)|\,du \\ &\leqslant M_m n^2\int_{I_{n,j}} r_m^{n|u-t|}\,du = O_m(nq_m^{|i-j|})\end{aligned}\tag{3.5}$$

with some r_m and q_m, $0<r_m<q_m<1$.

Combining (3.3), (3.4) and (3.5) we obtain

$$\int_t^1 \operatorname*{var}_{s\leqslant u\leqslant 1} K_n^{(m,k)}(t,u)\,ds = O_m\Big(\sum_{h=i}^{n}\sum_{j=h}^{n} q_m^{|i-j|}\Big) = O_m(1),$$

and this completes the proof.

THEOREM 3.2. *Let $m\geqslant -1$, $0\leqslant k\leqslant m+1$, $f\in L_1(I)$, and let for given $t\in I$*

$$f(t) = \lim_{h\to 0}\frac{1}{h}\int_t^{t+h} f(s)\,ds. \tag{3.6}$$

Then,

$$f(t) = \sum_{n=k-m}^{\infty}(f, g_n^{(m,k)})f_n^{(m,k)}(t). \tag{3.7}$$

In particular, (3.7) holds almost everywhere in I.

The proof is omitted since it goes exactly as the proof of Theorem 4 in [7].

THEOREM 3.3. *Let $m\geqslant -1$, $0\leqslant k\leqslant m+1$. Then there is a constant M_m such that*

$$\sup_n |P_n^{(m,k)}f(t)| \leqslant M_m \sup_{0\leqslant s\leqslant 1}\left|\frac{1}{s-t}\int_t^s f(u)\,du\right|$$

holds for every f in $L_1(I)$, for all t in I, and for $n>0$.

Proof. For given $t\in I$ we split the integral into two parts

$$P_n^{(m,k)}f(t) = \int_0^t K_n^{(m,k)}(t,s)f(s)\,ds + \int_t^1 K_n^{(m,k)}(t,s)f(s)\,ds.$$

Applying Lemma 6 of [7] (cf. [23] as well) we obtain

$$\begin{aligned}&\left|\int_t^1 K_n^{(m,k)}(t,s)f(s)\,ds\right| \\ &\quad\leqslant \sup_{t<s\leqslant 1}\left|\frac{1}{s-t}\int_t^s f(u)\,du\right|\int_t^1 [\operatorname*{var}_{s\leqslant u\leqslant 1} K_n^{(m,k)}(t,u)+K_n^{(m,k)}(t,1)]\,ds.\end{aligned}$$

Now (2.11) gives

$$\int_t^1 K_n^{(m,k)}(t,1)\,ds \leqslant M_m(1-t)nr_m^{n(1-t)} = O_m(1).$$

The combination of the last two inequalities and of (3.1) gives for some $C_m>0$

$$\left|\int_t^1 K_n^{(m,k)}(t,s)f(s)\,ds\right| \leqslant C_m\sup_{t<s\leqslant 1}\left|\frac{1}{s-t}\int_t^s f(u)\,du\right|.$$

Similar argument applies to the second integral and therefore the proof is complete.

THEOREM 3.4. *Let $p>1$, $m\geqslant -1$, $0\leqslant k\leqslant m+1$. Then there is a constant C_m such that for f in $L_p(I)$ we have*

$$\int_I \sup_{n>0} |P_n^{(m,k)}f(t)|^p\,dt \leqslant C_m^p\left(\frac{p}{p-1}\right)^p \int_I |f(t)|^p\,dt.$$

Moreover, if $|f|\log^+|f|$ is in $L_1(I)$ then there are constants C_m and M_m such that

$$\int_I \sup_{n>0} |P_n^{(m,k)}f(t)|\,dt \leqslant C_m \int_I |f(t)\log^+|f(t)||\,dt + M_m.$$

Theorem 3.4 follows from the previous theorem and from the inequalities of Hardy and Littlewood (cf. [26], pp. 244–245). Theorem 3.4 in the case of $m=-1$ was proved earlier in [24].

4. Orders of approximation. To state and to prove the main result it is necessary to introduce the best approximation and the moduli of continuity of higher orders.

For given $n>0$ and $m\geqslant -1$ the best aproximation by splines of order m, with respect to the fixed partition π_n, in $L_p(I)$ spaces is defined as follows:

$$E_{p,n}^{(m)}(f) = \inf\{\|f-g\|_p:\ g\in S_n^m(I)\}, \tag{4.1}$$

and in the case of $p=\infty$ it is understood that $E_{\infty,n}^{(m)}(f) = E_n^{(m)}(f)$, $\|\ \| = \|\ \|_\infty$.

Now let us take $m\geqslant -1$ and k such that $0\leqslant k\leqslant m+1$. According to Theorem 2.4, $\|P_n^{(m,k)}\|_p\leqslant M_m$ for some positive M_m for all p, $1\leqslant p\leqslant\infty$. Since $P_n^{(m,k)}$ is a projection of $L_p(I)$ onto $S_n^{m-k}(I)$, it follows from (4.1) by standard argument that

$$E_{p,n}^{(m-k)}(f)\leqslant \|f-P_n^{(m,k)}(f)\|_p\leqslant (1+M_m)E_{p,n}^{(m-k)}(f) \tag{4.2}$$

holds for all f in $L_p(I)$ with $1\leqslant p\leqslant\infty$ and for $n\geqslant -m+k$.

The modulus of continuity of order $r\geqslant 1$ of the function $f\in L_p(I)$, $1\leqslant p\leqslant\infty$, is defined for finite p by the formula

$$\omega_r^{(p)}(f;\delta) = \sup_{0<h\leqslant\delta}\Big(\int_0^{1-rh} |\Delta_h^r f(t)|^p\,dt\Big)^{1/p},$$

and for $p=\infty$ by the formula

$$\omega_r(f;\delta) = \omega_r^{(\infty)}(f;\delta) = \sup\{|\Delta_h^r f(t)|:\ 0\leqslant t<t+rh\leqslant 1,\ h\leqslant\delta\},$$

where it is assumed that $\delta r\leqslant 1$. The symbol Δ_h^r is used here for the forward progressive difference of order r with the increment h.

LEMMA 4.1. *Let $m\geqslant 0$ and $1\leqslant p\leqslant\infty$. Then exists M_m such that*

$$\|f-P_n^{(m)}f\|_p\leqslant M_m\frac{1}{n}\|D(f-P_n^{(m)}f)\|_p \tag{4.3}$$

holds for $n>0$ and f in $W_p^1(I)$.

Proof. Inequality (4.3) is being proved here essentially by the argument presented in [13] and for the sake of completeness the proof is given in detail.

The B-splines $M_{n,j}^{(m)}$ are simply the $N_{n,j}^{(m)}$ splines normalized as follows

$$M_{n,j}^{(m)} = \frac{m+2}{s_{n,j+1+m}-s_{n,j-1}}N_{n,j}^{(m)}.$$

It is known that (cf. [20])

$$\int_{-\infty}^{+\infty} M_{n,j}^{(m)}(t)\,dt = 1.$$

Let us consider at first the case of small n, i.e. let $0<n\leqslant m+2$. Applying (4.2) we find that there is M_m such that

$$\begin{aligned}\|f-P_n^{(m)}f\|_p &\leqslant M_m\|f-f(0)-(P_n^{(m)}f-P_n^{(m)}f(0))\|_p\\ &= M_m\|G[D(f-P_n^{(m)}f)]\|_p\leqslant M_m\|D(f-P_n^{(m)}f)\|_p,\end{aligned}$$

whence (4.3) follows with some new constant denoted by M_m as well.

Now let $n>m+2$ and let for the time being $1<p<\infty$, $q=p/(p-1)$. Moreover, let

$$g = D(f-P_n^{(m)}f),\qquad a_j = \int_{I_{n,j}} g(t)\,dt,$$

$$h = f(0)+GDP_n^{(m)}f+\sum_{j=1}^{n-m-2} a_jGM_{n,j}^{(m-1)}.$$

Clearly, $h\in S_n^m(I)$ and therefore by (4.2) there is a constant L_m such that

$$\|f-P_n^{(m)}f\|_p\leqslant L_m\|f-h\|_p = L_m\Big\|Gg-\sum_{j=1}^{n-m-2}a_jGM_{n,j}^{(m-1)}\Big\|_p,$$

the integral under the norm is being split into two parts corresponding to the intervals $\langle 0, s_{n,m+2}\rangle$ and $\langle s_{n,m+2}, 1\rangle$. The first integral is estimated as follows

$$\int_0^{s_{n,m+2}} \Big|Gg(t) - \sum_{j=1}^{n-m-2} a_j GM_{n,j}^{(m-1)}(t)\Big|^p dt$$
$$= \sum_{i=1}^{m+2} \int_{I_{n,i}} \Big|\sum_{j=1}^{i-1} a_j + \int_{s_{n,i-1}}^{t} g(u)\,du - \sum_{j=1}^{i} a_j GM_{n,j}^{(m-1)}(t)\Big|^p dt$$
$$\leqslant \sum_{i=1}^{m+2} \int_{I_{n,i}} \Big(\sum_{j=1}^{i-1} |a_j| + \int_{I_{n,i}} |g(u)|\,du\Big)^p dt$$
$$\leqslant \sum_{i=1}^{m+2} |I_{n,i}| \Big(\int_0^{s_{n,i}} |g(u)|\,du\Big)^p$$
$$\leqslant \frac{2(m+2)}{n} \Big(\int_0^{s_{n,m+2}} |g(u)|\,du\Big)^p \leqslant \Big(\frac{2(m+2)}{n}\|g\|_p\Big)^p.$$

For the second integral we have

$$\int_{s_{n,m+2}}^{1} \Big|Gg(t) - \sum_{j=1}^{n-m-2} a_j GM_{n,j}^{(m-1)}(t)\Big|^p dt$$
$$= \sum_{i=m+3}^{n} \int_{I_{n,i}} \Big|\sum_{j=1}^{i-1} a_j + \int_{s_{n,i-1}}^{t} g(u)\,du - \sum_{j=1}^{i-m-2} a_j - \sum_{j=i-m-1}^{i} a_j GM_{n,j}^{(m-1)}(t)\Big|^p dt$$
$$\leqslant \sum_{i=m+3}^{n} \int_{I_{n,i}} \Big(\sum_{j=i-m-1}^{i-1} |a_j|\big(1 - GM_{n,j}^{(m-1)}(t)\big) + \int_{I_{n,i}} |g(u)|\,du\Big)^p dt$$
$$\leqslant \frac{2}{n} \sum_{i=m+3}^{n} \Big(\int_{s_{n,i-m-2}}^{s_{n,i}} (|g(u)|\,du\Big)^p$$
$$\leqslant \frac{2}{n} \Big(\frac{2(m+2)}{n}\Big)^{p-1} \sum_{i=m+3}^{n} \int_{s_{n,i-m-2}}^{s_{n,i}} |g(u)|^p\,du \leqslant \Big(\frac{2(m+2)}{n}\|g\|_p\Big)^p.$$

Since all the estimates are uniform in p, the proof of Lemma 4.1 is complete.

LEMMA 4.2. *Let $m \geqslant -1$, $1 \leqslant p \leqslant \infty$, and let $f \in W_p^{m+2}(I)$. Then there is a constant M_m such that for $n > 0$*

$$\|f - P_n^{(m)} f\|_p \leqslant M_m \frac{1}{n^{m+2}} \|D^{m+2} f\|_p. \tag{4.4}$$

Proof. Inequality (4.4) is being proved by induction with respect to m. For $m = -1$ we have the Haar case and therefore by Theorem 7 of [7]

$$\|f - P_n^{(-1)} f\|_p \leqslant 6\omega_1^{(p)}(f; 1/n).$$

On the other hand,

$$\Big(\int_0^{1-h} (|f(t+h) - f(t)|^p\,dt\Big)^{1/p} = \Big(\int_0^{1-h} \Big|\int_t^{t+h} Df(u)\,du\Big|^p dt\Big)^{1/p}$$
$$= h\Big(\int_0^{1-h} \Big|\frac{1}{h}\int_t^{t+h} Df(u)\,du\Big|^p dt\Big)^{1/p} \leqslant h\|Df\|_p,$$

and consequently

$$\|f - P_n^{(-1)} f\|_p \leqslant \frac{6}{n}\|Df\|_p.$$

Now, let $m \geqslant 0$ and let $f \in W_p^{m+2}(I)$. The function f is absolutely continuous and therefore

$$f(t) = f(0) + \int_0^t g(u)\,du,$$

where $g = Df$ is in $W_p^{(m-1)+2}(I)$. According to the induction hipothesis we have

$$\|g - P_n^{(m-1)} g\|_p \leqslant M_{m-1} \frac{1}{n^{m+1}} \|D^{m+2} f\|_p.$$

However, $P_n^{(m,1)} g = DP_n^{(m)} f$ and therefore (4.2) leads to the following inequality

$$\|D(f - P_n^{(m)} f)\|_p \leqslant M'_m \frac{1}{n^{m+1}} \|D^{m+2} f\|_p$$

with some constant M'_m. Combining this with Lemma 4.1, we complete the proof.

LEMMA 4.3. *Let m, $m \geqslant -1$, be fixed. Moreover, let given $f \in W_1^{m+2}(I)$ be such that*

$$\|f - P_n^{(m)}\|_1 = o\Big(\frac{1}{n^{m+2}}\Big) \quad as \quad n \to \infty. \tag{4.5}$$

Then $D^{m+2} f = 0$.

Proof. For $m = -1$, $\{f_j^{(m)}, j \geqslant -m\}$ is the Haar orthonormal system. Thus, if

$$f = \sum_{j=1}^{\infty} a_j f_j^{(-1)},$$

then (4.5) is equivalent to

$$2^{\frac{1}{2}n} \sum_{j=2^n+1}^{2^{n+1}} |a_j| = o(1).$$

On the other hand, for absolutely continuous f we have (cf. [2])

$$2^{\frac{1}{2}n}\sum_{j=2^n+1}^{2^{n+1}}|a_j| = \tfrac{1}{4}\operatorname*{var}_I f + o(1) = \tfrac{1}{4}\|Df\|_1 + o(1),$$

whence if follows that $Df = 0$.

Now, let $m > -1$. Then $g = D^{m+1}f$ is in $W_1^1(I)$ and according to (4.2) we get with some $C_m > 0$

$$(4.6)\qquad \|D^{m+1}(f - P_n^{(m)}f)\|_1 = \|g - P_n^{(m,m+1)}g\|_1 \geqslant C_m\|g - P_n^{(-1)}g\|_1.$$

Moreover, if $n = 2^\mu$ then

$$\|D^{m+1}(f - P_n^{(m)}f)\|_1 \leqslant \sum_{i=\mu}^{\infty}\|D^{m+1}(P_{2^{i+1}}^{(m)} - P_{2^i}^{(m)})f\|_1,$$

whence by (2.14) and by the hypothesis

$$\|D^{m+1}(f - P_n^{(m)}f)\|_1 = O(1)\sum_{i=\mu}^{\infty}2^{i(m+1)}\|(P_{2^{i+1}}^{(m)} - P_{2^i}^{(m)})f\|_1$$

$$= o(1)\left(\sum_{i=\mu}^{\infty}\frac{1}{2^i}\right) = o\left(\frac{1}{n}\right).$$

Combining this with (4.6) we get

$$\|g - P_n^{(-1)}g\|_1 = o\left(\frac{1}{n}\right),$$

and therefore, by the first step of the proof, $Dg = D^{m+2}f = 0$.

Before the next result will be stated let us recall the definition of the Peetre interpolating K-functional (cf. [17]). We are interested in interpolating the Banach spaces $L_p(I)$ and $W_p^r(I)$ with $r \geqslant 0$ and $1 \leqslant p \leqslant \infty$. The K'-functional is defined for f in $L_p(I)$ and for $t > 0$ as follows:

$$K'\big(t; f; L_p(I), W_p^r(I)\big) = \inf\{\|f - g\|_p + t\|D^r g\|_p\colon g \in W_p^r(I)\}.$$

In this particular case the functional was estimated in [15], and it was shown that there is a constant $M_r > 0$ such that

$$(4.7)\qquad \frac{1}{M_r}\omega_r^{(p)}(f; t) \leqslant K'\big(t^r; f; L_p(I), W_p^r(I)\big) \leqslant M_r\omega_r^{(p)}(f; t)$$

holds for $0 < tr \leqslant 1$.

THEOREM 4.1. *Let m, k and p be given such that $m \geqslant -1$, $0 \leqslant k \leqslant m+1$ and $1 \leqslant p \leqslant \infty$. Then there is a constant M_m such that*

$$(4.8)\qquad \|f - P_n^{(m,k)}f\|_p \leqslant M_m\,\omega_{m-k+2}^{(p)}\left(f; \frac{1}{n}\right),\quad n \geqslant m-k+2,$$

$$(4.9)\qquad E_{p,n}^{(m)}(f) \leqslant M_m\,\omega_{m+2}^{(p)}\left(f; \frac{1}{n}\right),\quad n \geqslant m+2,$$

holds for $f \in L_p(I)$.

Proof. For given $n > 0$ and for each $g \in W_p^{m+2}(I)$ according to Theorem 2.4 and inequality (4.4) there is a constant $M_m > 0$ such that

$$\|f - P_n^{(m)}\|_p \leqslant \|f - g\|_p + \|g - P_n^{(m)}g\|_p + \|P_n^{(m)}(f-g)\|_p$$

$$\leqslant M_m\left(\|f-g\|_p + \frac{1}{n^{m+2}}\|D^{m+2}g\|_p\right),$$

whence

$$(4.10)\qquad \|f - P_n^{(m)}f\|_p \leqslant M_m K'\left(\frac{1}{n^{m+2}}; f;\ L_p(I),\ W_p^{m+2}(I)\right).$$

Now, the combination of (4.7), (4.10), and (4.2) gives Theorem 4.1.

Apparently inequality (4.9) is known to the specialists (cf. [19]) but it seems that nowhere the proof was given in detail. In the case of $m = -1$ it was already proved in [22] and [25]; for $m = 0$ and $p = \infty$ it was proved under certain restrictions in [4] and without them in [16].

5. Estimates for the L_p norms of the biorthogonal spline functions. In this section we consider the biorthogonal system $\{f_i^{(m,k)}, g_j^{(m,k)}, i, j \geqslant k-m\}$ with $m \geqslant -1$, $0 \leqslant k \leqslant m+1$ and $1 \leqslant p \leqslant \infty$.

LEMMA 5.1. *The following inequalities*

$$(5.1)\qquad \|f_n^{(m,k)}\|_p \sim n^{\frac{1}{2}+k-1/p},$$

$$(5.2)\qquad \|g_n^{(m,k)}\|_p \sim n^{\frac{1}{2}-k-1/p},\ (^1)$$

hold uniformly in n, k, and p.

Proof. It follows from (2.12) that

$$(5.3)\qquad \|f^{(m,k)}\|_p = O(n^{\frac{1}{2}+k-1/p}).$$

On the other hand, $(f_n^{(m,k)}, g_n^{(m,k)}) = 1$ and therefore, by the Hölder inequality,

$$(5.4)\qquad 1 \leqslant \|f_n^{(m,k)}\|_p\|g^{(m,k)}\|_q$$

(1) The symbol $a_n \sim b_n$ is used if and only if $a_n = O(b_n)$ and $b_n = O(a_n)$.

with $q = p/(p-1)$. This and (5.3) give

$$n^{\frac12-k-1/p} = O(\|g_n^{(m,k)}\|_p). \tag{5.5}$$

According to (5.4) it remains to estimate $\|g_n^{(m,k)}\|_p$ from above. In the case of $k = 0$ (5.3) and (5.4) imply (5.1), i.e.

$$\|f_n^{(m)}\|_p \sim n^{\frac12-1/p}. \tag{5.6}$$

Now Theorem 4.1 gives for f in $L_p(I)$

$$|(f, f_n^{(m)})|\,\|f_n^{(m)}\|_p = \|P_n^{(m)}f - P_{n-1}^{(m)}f\|_p = O\big(\omega_{m+2}^{(p)}(f; 1/n)\big),$$

whence by (5.6) we get

$$|(f, f_n^{(m)})| = O\big(n^{-\frac12+1/p}\,\omega_{m+2}^{(p)}(f; 1/n)\big). \tag{5.7}$$

In particular, for f in $W_p^k(I)$, (2.9) and (5.7) give

$$|(D^kf, g_n^{(m,k)})| = O\big(n^{-\frac12+1/p}\,\omega_{m+2}^{(p)}(f; 1/n)\big).$$

Since

$$\omega_{m+2}^{(p)}(f, 1/n) \leqslant \frac{1}{n^k}\,\omega_{m+2-k}^{(p)}(D^kf; 1/n),$$

the last inequality gives for g in $L_p(I)$

$$|(g, g_n^{(m,k)})| = O\big(n^{-\frac12-k+1/p}\,\omega_{m+2-k}^{(p)}(g; 1/n)\big). \tag{5.8}$$

Clearly, (5.8) implies

$$|(g, g_n^{(m,k)})| = O(n^{-\frac12-k+1/p}\|g\|_p),$$

whence we infer by the theorem of Riesz that

$$\|g_n^{(m,k)}\|_q = O(n^{-\frac12-k+1/p})$$

and therefore the proof is complete.

COROLLARY 5.1. *For $m \geqslant -1$ and $0 \leqslant k \leqslant m+1$ we have uniformly in n and k*

$$\|f_n^{(m,k)}\|\;\|g_n^{(m,k)}\| \sim n.$$

6. Local estimates for the biorthogonal spline functions. It is assumed in this section that $m \geqslant -1$, $0 \leqslant k \leqslant m+1$ and that t_n is defined as in (2.2). Moreover, $t_n^{(m,k)}$ and $s_n^{(m,k)}$ are any numbers in I such that $\|f_n^{(m,k)}\| = |f_n^{(m,k)}(t_n^{(m,k)})|$ and $\|g_n^{(m,k)}\| = |g_n^{(m,k)}(s_n^{(m,k)})|$.

THEOREM 6.1. *There is a constant q_m, $0 < q_m < 1$, such that the following inequalities hold uniformly in k, n, and $t \in I$*

$$|f_n^{(m,k)}(t)| = O(n^{\frac12+k}q_m^{n|t-t_n|}), \tag{6.1}$$

$$|f_n^{(m,k)}(t)| = O(n^{\frac12+k}q_m^{n|t-t_n^{(m,k)}|}), \tag{6.2}$$

$$|g_n^{(m,k)}(t)| = O(n^{\frac12-k}q_m^{n|t-t_n|}), \tag{6.3}$$

$$|g_n^{(m,k)}(t)| = O(n^{\frac12-k}q_m^{n|t-s_n^{(m,k)}|}). \tag{6.4}$$

Proof. Inequality (6.1) was proved in [11]. Now, (6.1) implies

$$\|f_n^{(m,k)}\| = O(n^{\frac12+k}q_m^{n|t_n^{(m,k)}-t_n|}),$$

whence by Lemma 5.1

$$1 = O(q_m^{n|t_n^{(m,k)}-t_n|})$$

which is equivalent to

$$n\,|t_n^{(m,k)} - t_n| = O(1). \tag{6.5}$$

Thus, $n|t-t_n| \geqslant n|t-t_n^{(m,k)}| - n|t_n - t_n^{(m,k)}| = n|t-t_n^{(m,k)}| + O(1)$ and therefore (6.1) implies (6.2). To prove (6.4) we use Theorem 2.1 to get

$$|f_n^{(m,k)}(t)\,g_n^{(m,k)}(s)| = O(nq_m^{n|t-s|}). \tag{6.6}$$

Now, Corollary 5.1 and (6.6) give

$$n \sim \|f_n^{(m,k)}\|\;\|g_n^{(m,k)}\| = O(nq_m^{n|t_n^{(m,k)}-s_n^{(m,k)}|}),$$

whence we infer

$$n\,|t_n^{(m,k)} - s_n^{(m,k)}| = O(1). \tag{6.7}$$

Inequalities (6.6) and (5.1) give

$$|g_n^{(m,k)}(s)| = O(n^{\frac12-k}q_m^{n|s-t_n^{(m,k)}|}),$$

and moreover (6.7) implies $n|s-t_n^{(m,k)}| \geqslant n|s-s_n^{(m,k)}| + O(1)$, and this proves (6.4). Finally, (6.5) and (6.7) give $n|t-s_n^{(m,k)}| \geqslant n|t-t_n| - n|t_n - t_n^{(m,k)}| - n|t_n^{(m,k)} - s_n^{(m,k)}| = n|t-t_n| + O(1)$, whence by (6.4) inequality (6.3) follows, and this completes the proof.

7. Unconditional inequalities. As in the previous sections it is assumed that $m \geqslant -1$, $0 \leqslant k \leqslant m+1$, and $1 \leqslant p \leqslant \infty$. Moreover, let

$$M_\mu^{(p)}(a) = 2^{\mu(\frac12-1/p)}\Big(\sum_{2^\mu+1}^{2^{\mu+1}} |a_n|^p\Big)^{1/p}.$$

THEOREM 7.1. *For a given real sequence* (a_n) *the following inequalities hold uniformly in* k, p, *and* $\mu \geqslant 0$:

$$\Big\|\sum_{2^\mu+1}^{2^{\mu+1}} |a_n f_n^{(m,k)}|\Big\|_p = O\big(2^{\mu k} M_\mu^{(p)}(a)\big), \tag{7.1}$$

$$2^{\mu k} M_\mu^{(p)}(a) = O\Big(\Big\|\sum_{2^\mu+1}^{2^{\mu+1}} a_n f_n^{(m,k)}\Big\|_p\Big), \tag{7.2}$$

$$\Big\|\sum_{2^\mu+1}^{2^{\mu+1}} |a_n g_n^{(m,k)}|\Big\|_p = O\big(2^{-\mu k} M_\mu^{(p)}(a)\big), \tag{7.3}$$

$$2^{-\mu k} M_\mu^{(p)}(a) = O\Big(\Big\|\sum_{2^\mu+1}^{2^{\mu+1}} a_n g_n^{(m,k)}\Big\|_p\Big). \tag{7.4}$$

Proof. The theorem in the case of $m = -1$ is trivial. For $m = 0$ it was established in [7], and the idea of the argument presented here goes back to that work.

According to Lemma 5.1 we have

$$\|f_n^{(m,k)}\|_1 = O(n^{k-\frac{1}{2}}), \qquad \|g_n^{(m,k)}\|_1 = O(n^{-k-\frac{1}{2}}), \tag{7.5}$$

and as a consequence of Theorem 6.1 we obtain

$$\begin{aligned} \Big\|\sum_{2^\mu+1}^{2^{\mu+1}} |f_n^{(m,k)}|\Big\| &= O(2^{\mu(\frac{1}{2}+k)}), \\ \Big\|\sum_{2^\mu+1}^{2^{\mu+1}} |g_n^{(m,k)}|\Big\| &= O(2^{\mu(\frac{1}{2}-k)}). \end{aligned} \tag{7.6}$$

Now, using (7.5), (7.6), (2.10) and exactly the same argument as in the proof of Theorem 6 in [7], we obtain (7.1) and (7.3).

The remaining two inequalities can be proved exactly in the same way as the inequality (32) in [7] ([2]).

8. The biorthogonal functions as spline bases in $L_p(I)$ and $C(I)$. According to (2.10) $\{g_i^{(m,k)}, f_j^{(m,k)}, i, j \geqslant k-m\}$ is a biorthogonal sequence and by Theorem 2.4 $\{f_j^{(m,k)}, j \geqslant k-m\}$ is a basis in $L_p(I)$, $1 \leqslant p < \infty$.

THEOREM 8.1. *Let* m *and* k *be given and such that* $0 \leqslant k \leqslant m+1$, $m \geqslant -1$. *Then* $\{g_j^{(m,k)}, j \geqslant k-m\}$ *is a basis in* $L_q(I)$ *with* $1 \leqslant q < \infty$, *and for* f *in* $L_q(I)$ *we have*

$$g = \sum_{j=k-m}^{\infty} (g, f_j^{(m,k)}) g_j^{(m,k)}. \tag{8.1}$$

([2]) In (32) of [7] there is a misprint, i.e. in the very right-hand side term there should be no absolute value under the norm. The same ramark concerns the relevant places in the proof of inequality (32).

Proof. The partial sum

$$Q_n^{(m,k)} g = \sum_{j=k-m}^{n} (g, f_j^{(m,k)}) g_j^{(m,k)}$$

is the adjoint to $P_n^{(m,k)}\colon L_p(I) \to L_p(I)$ and by Theorem 2.1

$$\|Q_n^{(m,k)}\|_q \leqslant M_m, \qquad 0 \leqslant k \leqslant m+1, \; m \geqslant -1, \tag{8.2}$$

holds with some $M_m > 0$ uniformly in n and q: $n > 0$, $1 \leqslant q \leqslant \infty$. Now, for finite q, $\{g_j^{(m,k)}, j \geqslant k-m\}$ is closed in $L_q(I)$. To see this, let $(f, g_j^{(m,k)}) = 0$ hold for $j \geqslant k-m$ and for given $f \in L_1(I)$. Then property (2.9) implies $(D^k G^k f, g_j^{(m,k)}) = (G^k f, f_j^{(m)}) = 0$ for $j \geqslant k-m$ and therefore

$$G^k f = \sum_{j=-m}^{k-m-1} c_j f_j^{(m,k)},$$

whence $f = 0$. Consequently, since (8.1) holds for $g = g_j^{(m,k)}$, $j \geqslant k-m$, it follows that $\{g_j^{(m,k)}, j \geqslant k-m\}$ is a basis in $L_q(I)$.

In the space $C(I)$ the situation is different. Since for $k > 0$ we have $g_j^{(m,k)}(0) = 0$ for all $j \geqslant k-m$, the set $\{g_j^{(m,k)}, j \geqslant k-m\}$ is not closed in $C(I)$. However, we have

THEOREM 8.2. *Let* $m \geqslant -1$ *and* $0 \leqslant k \leqslant m+1$. *Then the system* $\{1, g_j^{(m,k+1)}, j \geqslant k-m\}$ *is a basis in* $C(I)$ *and for* f *in* $C(I)$ *we have*

$$f(1) - f(t) = \sum_{j=k-m}^{\infty} \Big(\int_I f_j^{(m,k)}(s)\, df(s)\Big) g_j^{(m,k+1)}(t). \tag{8.3}$$

Proof. Since $\{g_j^{(m,k)}, j \geqslant k-m\}$ is closed in $L_1(I)$ (cf. the proof of Theorem 8.1), it follows that $\{1, g_j^{(m,k+1)}, j \geqslant k-m\}$ is closed in $C(I)$. Moreover, (8.3) holds for $f = 1$, $g_j^{(m,k+1)}$, $j \geqslant k-m$. Thus, it remains to show the uniform boundedness of the $C(I)$ norms of the partial sums

$$S_n^{(m,k)} f = f(1) - \sum_{j=k-m}^{n} \Big(\int_I f_j^{(m,k)}(s)\, df(s)\Big) g_j^{(m,k+1)}. \tag{8.4}$$

Let us consider two cases:

The first case: $0 \leqslant k \leqslant m$. Integration by parts in (8.4) gives

$$S_n^{(m,k)} f(t) = f(1) - f(1) P_n^{(m,k)} I_{\langle t,1\rangle}(1) + f(0) P_n^{(m,k)} I_{\langle t,1\rangle}(0) + Q_n^{(m,k+1)} f(t),$$

where $I_{\langle t,1\rangle}$ is the characteristic function of $\langle t, 1\rangle$. Thus,

$$\|S_n^{(m,k)}\| \leqslant 1 + 2\|P_n^{(m,k)}\| + \|Q_n^{(m,k+1)}\|.$$

Now, the combination of the last inequality, Theorem 2.3, and (8.2) give $\|S_n^{(m,k)}\| \leqslant 1 + 3M_m$ and this proves the case.

The second case: $k = m+1$. According to (8.4)

$$\|S_n^{(m,m+1)}\| \leqslant 1+2\,\|P_n^{(m,m+1)}\| + \sup_{s\in I}\operatorname*{var}_{t\in I} K_n^{(m,m+1,m+2)}(t,s),$$

and the last term equals

$$\sup_{s\in I}\operatorname*{var}_{t\in I} h_s(t)$$

where $h_s = P_n^{(m,m+1)} I_{\langle s,1\rangle}$.

Using Theorem 4.1 with $p=1$, we get for $n>0$ and $s\in I$

$$\|I_{\langle s,1\rangle} - h_s(t)\|_1 \leqslant M_m \frac{1}{n}. \tag{8.5}$$

According to Theorem 2.4,

$$\|h_s\| \leqslant M_m, \qquad s\in I. \tag{8.6}$$

Since $h_s \in S_n^{(-1)}(I)$, we find that it is of the form

$$h_s = \sum_{j=1}^{n} c_j(s)\, N_{n,j}^{(-1)},$$

whence by (8.6) we infer

$$|c_j(s)| \leqslant M_m, \qquad 1\leqslant j\leqslant n,\ s\in I. \tag{8.7}$$

Thus,

$$\operatorname*{var}_{t\in I} h_s(t) = \sum_{j=1}^{n-1} |c_{j+1}(s) - c_j(s)|.$$

Now, let, for given $s\in I$, i be such that $s\in I_{n,i}$. Then

$$\begin{aligned}
\|I_{\langle s,1\rangle} - h_s\|_1 &\geqslant \int_{I\setminus I_{n,i}} |I_{\langle s,1\rangle}(t) - h_s(t)|\,dt\\
&= \int_{t<s_{n,i-1}} |h_s(t)|\,dt + \int_{t>s_{n,i}} |1-h_s(t)|\,dt\\
&= \sum_{l=1}^{i-1} |c_l(s)|\,|I_{n,l}| + \sum_{l=i+1}^{n} |1-c_l(s)|\,|I_{n,l}|\\
&\geqslant \frac{1}{2n}\Big(\sum_{l=1}^{i-1} |c_l(s)| + \sum_{l=i+1}^{n} |1-c_l(s)|\Big)\\
&\geqslant \frac{1}{4n}\Big(\sum_{l=1}^{i-2} |c_{l+1}(s)-c_l(s)| + \sum_{l=i+1}^{n-1} |c_{l+1}(s)-c_l(s)|\Big)\\
&\geqslant \frac{1}{4n}\Big(\operatorname*{var}_{t\in I} h_s(t) - |c_i(s)-c_{i-1}(s)| - |c_{i+1}(s)-c_i(s)|\Big),
\end{aligned}$$

and therefore by (8.7)

$$\operatorname*{var}_{t\in I} h_s(t) \leqslant 4n\,\|I_{\langle s,1\rangle} - h_s\|_1 + 4M_m.$$

This and (8.5) imply $\operatorname{var} h_s \leqslant 8M_m$, and therefore the proof is complete.

COROLLARY 8.1. *The system $\{1, g_j^{(m,m+2)}, j>0\}$ is an interpolating basis in $C(I)$, i.e. if $S_n^{(m,m+1)}$ is defined as in* (8.4) *and $(t_n, n\geqslant 0)$ as in* (2.2), *then for $f\in C(I)$ we have*

$$S_n^{(m,m+1)} f(t_j) = f(t_j), \qquad j=0,1,\ldots,n,\ n>0. \tag{8.8}$$

Proof. We know by Theorem 8.2 that $\{1, g_j^{(m,m+2)}, j>0\}$ is a basis in $C(I)$. Therefore it remains to check (8.8). According to the definitions of $g_n^{(m,m+2)}$ and $f_n^{(m)}$ we have

$$g_n^{(m,m+2)}(t_j) = \frac{1}{(m+1)!}\int_I (s-t_j)_+^{m+1} f_n^{(m)}(s)\,ds = 0, \qquad n>j,$$

and therefore (8.3) implies (8.8).

The partial sums $S_n^{(m,m+1)}f$ are related to the splines defined in [1] as a solution of certain variational problem (cf. Theorem 5.4.1 in [1]). In particular, if we denote by $V_n^m(f)$, for given $f\in C(I)$, the set of all $g\in W_2^{m+2}(I)$ such that $g(t_i) = f(t_i)$ for $i=0,\ldots,n$, and $D^k g(0) = D^k g(1) = 0$ for $k=1,\ldots,m+1$, then we have

THEOREM 8.3. *Let $n>0$, $m\geqslant -1$ and $f\in C(I)$ be given. Then there is unique $g_0\in V_n^m(f)$ such that*

$$\|D^{m+2} g_0\|_2 = \inf\{\|D^{m+2} g\|_2 \colon g\in V_n^m(f)\},$$

and $g_0 = S_n^{(m,m+1)} f$.

Proof. If follows that for $g\in V_n^m(f)$

$$(D^{m+2} g, f_n^{(m)}) = (-1)^{m+1}\int_I f_n^{(m,m+1)}(t)\,df(t),$$

and $S_n^{(m,m+1)} f = S_n^{(m,m+1)} g$. Consequently,

$$\|D^{m+2} g\|_2^2 = \|D^{m+2} S_n^{(m,m+1)} f\|_2^2 + \sum_{j=n+1}^{\infty} (D^{m+2} g, f_j^{(m)})^2.$$

Thus, g_0 has to satisfy the conditions $(D^{m+2} g_0, f_j^{(m)}) = 0$, $j>n$, i.e. $D^{m+2} g_0 = D^{m+2} S_n^{(m,m+1)} f$ or else $g_0 = S_n^{(m,m+1)} f$.

9. Isomorphisms between function and sequence spaces. Let us consider for given f in $L_1(I)$ the Fourier series

$$f = \sum_{j=-m}^{\infty} a_j f_j^{(m)}, \qquad a_j = (f, f_j^{(m)}). \tag{9.1}$$

The aim of this section is to characterize in terms of the coefficients a_j the functions for which $\omega_{m+2}^{(p)}(f;h)=O(h^a)$ holds with some a, $0<a<m+1+1/p$, $p<\infty$. In the case of $p=\infty$ the characterization concerns the functions f in $C(I)$ for which $\omega_{m+2}(f;h)=O(h^a)$ with some a, $0<a<m+1$.

To establish the final result a series of lemmas is needed. The first one is known and therefore the proof is omitted.

LEMMA 9.1. *Let the integers $i\geqslant 0$ and $j\geqslant 0$ be given and let $f\in W_p^i(I)$, $1\leqslant p\leqslant\infty$. Then*

$$\Big(\int_0^{1-(i+j)h}|\Delta_h^{i+j}f(s)|^p\,ds\Big)^{1/p}\leqslant h^i\Big(\int_0^{1-jh}|\Delta_h^jD^if(s)|^p\,ds\Big)^{1/p}$$

holds for $0<(i+j)h\leqslant 1$.

LEMMA 9.2. *Let $m\geqslant -1$, $n>0$ and let $f\in S_n^m(I)$. Then there is a constant M_m such that for each k, $0\leqslant k\leqslant m+1$,*

$$\int_0^{1-kh}|\Delta_h^kf(s)|^p\,ds\Big)^{1/p}\leqslant M_m(nh)^k\|f\|_p$$

holds for $0<kh\leqslant 1$ and $1\leqslant p\leqslant\infty$.

Proof. The combination of (2.14) and Lemma 9.1 with $j=0$ and $i=k$ gives the required inequality.

LEMMA 9.3. *Let $m\geqslant -1$, $n>0$ and let $f\in S_n^m(I)$. Then there is a constant M_m such that*

$$\Big(\int_0^{1-(m+2)h}|\Delta_h^{m+2}f(s)|^p\,ds\Big)^{1/p}\leqslant M_m(nh)^{m+1+1/p}\|f\|_p$$

holds for $0<(m+2)h\leqslant 1$ and $1\leqslant p\leqslant\infty$.

Proof. Application of Lemma 9.1 with $i=m+1$ and $j=1$ gives

$$\Big(\int_0^{1-(m+2)h}|\Delta_h^{m+2}f(s)|^p\,ds\Big)^{1/p}\leqslant h^{m+1}\Big(\int_0^{1-h}|\Delta_hD^{m+1}f(s)|^p\,ds\Big)^{1/p}.$$

Since $D^{m+1}f$ is in $S_n^{-1}(I)$, we infer from (27) of [7] that

$$\Big(\int_0^{1-h}|\Delta_hD^{m+1}f(s)|^p\,ds\Big)^{1/p}\leqslant 4(nh)^{1/p}\|D^{m+1}f\|_p.$$

Combining the last two inequalities with (2.14), we obtain the inequality of Lemma 9.3.

THEOREM 9.1. *Let m, p, and f be given such that $m\geqslant -1$, $1\leqslant p\leqslant\infty$, $f\in L_p(I)$ if $p<\infty$, and $f\in C(I)$ if $p=\infty$. Then exists a constant M_m such that*

$$\omega_k^{(p)}(f;1/n)\leqslant M_m\frac{1}{n^k}\Big((\|f\|_p+\sum_{i=m+2}^{n}i^{k-1}E_{p,i}^{(m)}(f)\Big),\tag{9.2}$$

$$\omega_{m+2}^{(p)}(f;1/n)\leqslant M_m\frac{1}{n^{m+1+1/p}}\Big(\|f\|_p+\sum_{i=m+2}^{n}i^{m+1/p}E_{p,i}^{(m)}(f)\Big)\tag{9.3}$$

hold for $n\geqslant m+2$ and $1\leqslant k\leqslant m+1$.

This theorem can be proved with the help of Lemmas 9.2 and 9.3 in a similar way as Theorem 10 of [7] and therefore the details of the proof will be omitted.

Inequality (9.3) in the case of $m=-1$ was proved in [14] and in the case of $m=0$ in [7].

THEOREM 9.2. *Let $m\geqslant 0$ and $0<a<m+1$. Moreover, let $f\in L_p(I)$ if $1\leqslant p<\infty$, and $f\in C(I)$ if $p=\infty$, and let a_j be given as in (9.1). Then the following conditions are equivalent:*

(i) $E_{p,n}^{(m)}(f)=O(n^{-a})$,

(ii) $\|f-P_n^{(m)}f\|_p=O(n^{-a})$,

(iii) $\omega_{m+1}^{(p)}(f;\delta)=O(\delta^a)$,

(iv) $M_\mu^{(p)}(a)=O(2^{-a\mu})$,

where $M_\mu^{(p)}(a)$ is defined as in Section 7.

Proof. According to (4.2), (i) is equivalent to (ii), and by (9.2), (i) implies (iii). Now, applying (iii) to (4.8) and combining the result with (7.2), we get (iv). Finally, (iv), according to (7.1), implies (ii).

THEOREM 9.3. *Let $f\in L_p(I)$ if p is finite, $1\leqslant p<\infty$, and let $f\in C(I)$ if $p=\infty$. Moreover, let $0<a<m+1+1/p$ and $m\geqslant -1$. Then the following conditions are equivalent:*

(i) $E_{p,n}^{(m)}(f)=O(n^{-a})$,

(ii) $\|f-P_n^{(m)}f\|_p=O(n^{-a})$,

(iii) $\omega_{m+2}^{(p)}(f;\delta)=O(\delta^a)$,

(iv) $M_\mu^{(p)}(a)=O(2^{-a\mu})$,

where $M_\mu^{(p)}(a)$ is defined as in Section 7.

The proof is very much like the previous one and therefore it is omitted. Theorem 9.3 for $m=-1$ was proved in [14].

Remarks. Theorems 9.2 and 9.3 remain true after replacing capital O by small o. Moreover, in both cases of O and o in the theorems conditions (i), (ii), and (iv) are equivalent if only $a > 0$. Condition (iii) implies each of the remainnig conditions under the same restriction, i.e. $a > 0$.

COROLLARY 9.1. *Let $m \geqslant 0$ be given, and let f be in $L_p(I)$ if $1 \leqslant p < \infty$ and in $C(I)$ if $p = \infty$. Then from Theorems 9.2 and 9.3 follows the known result: If $0 < a < m+1$, then condition $\omega_{m+2}^{(p)}(f;\delta) = O(\delta^a)$ is equivalent to $\omega_{m+1}^{(p)}(f;\delta) = O(\delta^a)$.*

The same is true with O replaced by o.

COROLLARY 9.2. *Let $m > 0$. Then $\omega_{m+1}(f;\delta) = O(\delta^m)$ if and only if $E_n^{(m)}(f) = O(n^{-m})$. The same holds with O replaced by o.*

In particular, for $m = 1$ we obtain a characterization of the Zygmund class in the non-periodic case

$$\omega_2(f;\delta) = O(\delta) \quad \text{iff} \quad E_n^{(1)}(f) = O(1/n). \tag{9.4}$$

In [7], p. 316, an example of $f \in C(I)$ was constructed such that

$$\omega_2(f; 1/2^{n+1}) \geqslant n/2^n \quad \text{and} \quad E_n^{(0)}(f) = O(1/n). \tag{9.5}$$

COROLLARY 9.3. *Let $m \geqslant 0$, $1 \leqslant p < \infty$, and let f be in $L_p(I)$. Then $\omega_{m+2}^{(p)}(f;\delta) = O(\delta^{m+1})$ if and only if $E_n^{(m)}(f) = O(n^{-(m+1)})$. The same holds true after replacing O by o. In particular, if $m = 0$ then for finite p we have*

$$\omega_2^{(p)}(f;\delta) = O(\delta) \quad \text{iff} \quad E_{p,n}^{(0)}(f) = O(1/n). \tag{9.6}$$

It is very interesting to compare Corollaries 9.2 and 9.3 and characterizations (9.4) and (9.6). It follows from (9.5) that the order of splines in (9.4) cannot be lowered down.

To state our isomorphism result we need the following lemma.

LEMMA 9.4. *Let $m \geqslant -1$ and let $0 < a \leqslant m+1+1/p$. Then there is a constant M_m such that*

$$M_m^{-1} n^a \|f_n^{(m)}\|_p \leqslant \sup_{0<h(m+2)\leqslant 1} \frac{1}{h^a}\, \omega_{m+2}^{(p)}(f_n^{(m)}; h) \leqslant M_m n^a \|f_n^{(m)}\|_p. \tag{9.7}$$

Moreover, if $m \geqslant 0$ and $0 < a \leqslant m+1$, then for $n > m+2$

$$M_m^{-1} n^a \|f_n^{(m)}\|_p \leqslant \sup_{0<h(m+1)\leqslant 1} \frac{1}{h^a}\, \omega_{m+1}^{(p)}(f_n^{(m)}; h) \leqslant M_m n^a \|f_n^{(m)}\|_p. \tag{9.8}$$

Proof. The left-hand side of (9.7) is being proved as follows. According to Theorem 4.1,

$$\|f_n^{(m)}\|_p = \|f_n^{(m)} - P_{n-1}^{(m)} f_n^{(m)}\|_p \leqslant M'_m \omega_{m+2}^{(p)}\big(f_n^{(m)}; 1/(n-1)\big),$$

whence we infer

$$M_m^{-1} n^a \|f_n^{(m)}\|_p \leqslant n^a \omega_{m+2}^{(p)}(f_n^{(m)}; 1/n),$$

$M_m = M'_m 2^{m+2}$. To prove the right-hand side of (9.7) we use Lemma 9.3 which for $h < 1/n$ gives

$$\frac{1}{h^a}\, \omega_{m+2}^{(p)}(f_n^{(m)}; h) \leqslant M'_m n^a \|f_n^{(m)}\|_p,$$

and for $1/n \leqslant h \leqslant 1/(m+2)$ we use the inequality

$$\frac{1}{h^a}\, \omega_{m+2}^{(p)}(f_n^{(m)}; h) \leqslant 2^{m+2} n^a \|f_n^{(m)}\|_p.$$

The left-hand side of (9.8) is a consequence of (9.7) and of the inequality $\omega_{m+2}^{(p)}(f;h) \leqslant 2\omega_{m+1}^{(p)}(f;h)$. The right-hand side of (9.8) for $0 < h < 1/n$ we obtain using Lemma 9.2

$$\frac{1}{h^a}\, \omega_{m+1}^{(p)}(f_n^{(m)}; h) \leqslant M'_m n^a \|f_n^{(m)}\|_p,$$

and for $1/n \leqslant h \leqslant 1/(m+1)$

$$\frac{1}{h^a}\, \omega_{m+1}^{(p)}(f_n^{(m)}; h) \leqslant 2^{m+1} n^a \|f_n^{(m)}\|_p,$$

and therefore the proof is complete.

In what follows we are going to employ some more notation. For given $k \geqslant 0$ and a, $0 < a \leqslant k$, let

$$H_p^{k,a} = \{f \in L_p(I):\ \|f\|_p^{k,a} < \infty\},$$

where

$$\|f\|_p^{k,a} = \|f\|_p + \sup_{0<kh\leqslant 1} \frac{1}{h^a}\, \omega_k^{(p)}(f;h).$$

The space $\langle H_p^{k,a}, \|\ \|_p^{k,a}\rangle$ is a complete Banach space. Moreover, let

$$H_{p,0}^{k,a} = \{f \in H_p^{k,a}:\ \omega_k^{(p)}(f;h) = o(h^a)\}.$$

The space $\langle H_{p,0}^{k,a}, \|\ \|_p^{k,a}\rangle$ is a closed subspace of $H_p^{k,a}$.

In the case of $p = \infty$ the definitions are modified in the obvious way, namely the $L_\infty(I)$ space is replaced by $C(I)$. For simplicity we introduce $H^{k,a} = H_\infty^{k,a}$, $H_0^{k,a} = H_{\infty,0}^{k,a}$ and $\|\ \|^{k,a} = \|\ \|_\infty^{k,a}$. We denote by $f_{1,n}^{(m)}$ and $f_{2,n}^{(m)}$ functions proportional to $f_n^{(m)}$ and normalized in the norms $\|\ \|_p^{m+1,a}$ and $\|\ \|_p^{m+2,a}$, respectively.

The space of all bounded real sequences $b = (b_{-m}, \ldots, b_0, \ldots)$ is denoted by l_∞, and $\|b\| = \sup\{|b_n|:\ n \geqslant -m\}$; $c_0 = \{b \in l_\infty:\ b_n = o(1)\}$, and the norm in c_0 is the same as in l_∞. Moreover, let $l_\infty^p = \{b \in l_\infty \colon \|b\|^p < \infty\}$, where

$$\|b\|^{(p)} = \sum_{j=-m}^{2^\mu} |b_j| + \sup_{\nu \geqslant \mu} \Big(\sum_{j=2^\nu+1}^{2^\nu+1} (|b_j|^p)\Big)^{1/p},$$

where $2^\mu < m+2 \leqslant 2^{\mu+1}$. Moreover,

$$c_0^p = \Big\{b \in l_\infty^p \colon \Big(\sum_{j=2^\nu+1}^{2^\nu+1} |b_j|^p\Big)^{1/p} = o(1)\Big\}.$$

The space $\langle l_\infty^p, \|\ \|^{(p)}\rangle$ is a complete Banach space and c_0^p with the same norm its closed subspace. Of course, $l_\infty^\infty = l_\infty$ and $c_0^\infty = c_0$.

THEOREM 9.4. *Let $m \geqslant 0$, $0 < a < m+1$ and $1 \leqslant p \leqslant \infty$. Then $\langle H_p^{m+1,a}, \|\ \|_p^{m+1,a}\rangle$ is linearly isomorphic to $\langle l_\infty^p, \|\ \|^{(p)}\rangle$, and $\langle H_{p,0}^{m+1,a}, \|\ \|_p^{m+1,a}\rangle$ is linearly isophormic to $\langle c_0^{(p)}, \|\ \|^{(p)}\rangle$. In both cases the isomorphism is given by the formulae*

$$f = \sum_{n=-m}^{\infty} b_n f_{1,n}^{(m)}, \qquad b_n = \|f_n^{(m)}\|_p^{m+1,a}(f, f_n^{(m)}). \tag{9.9}$$

Proof. According to (9.8) and (5.1) we get

$$\|f_n^{(m)}\|_p^{m+1,a} \sim n^{\frac{1}{2}+a-1/p},$$

whence by Theorem 9.2

$$\Big(\sum_{2^\mu+1}^{2^\mu+1} |b_j|^p\Big)^{1/p} = O(1) \quad (= o(1)),$$

if and only if $f \in H_p^{m+1,a}$ $(f \in H_{p,0}^{m+1,a})$.

THEOREM 9.5. *Let $m \geqslant -1$, $0 < a < m+1+1/p$ and $1 \leqslant p \leqslant \infty$. Then $\langle H_p^{m+2,a}, \|\ \|_p^{m+2,a}\rangle$ is linearly isomorphic to $\langle l_\infty^p, \|\ \|^{(p)}\rangle$, and $\langle H_{p,0}^{m+2,a}, \|\ \|^{m+2,a}\rangle$ is linearly isomorphic to $\langle c_0^p, \|\ \|^{(p)}\rangle$. In both cases the isomorphism is given by the formulae*

$$f = \sum_{n=-m}^{\infty} b_n f_{2,n}^{(m)}, \qquad b_n = \|f_n^{(m)}\|_p^{m+2,a}(f, f_n^{(m)}). \tag{9.10}$$

The proof is similar to the previous one.

COROLLARY 9.4. *The Zygmund class, i.e. the space $H_p^{2,1}$, $1 \leqslant p \leqslant \infty$, is linearly isomorphic to l_∞^p. Moreover, $H_{p,0}^{2,1}$, $1 \leqslant p \leqslant \infty$, is linearly isomorphic to c_0^p.*

Most of the results of this section were established in the case of $m = 0$ in [5] and [7].

It is worth to mention at this place that in the language of interpolation spaces, according to (4.7), the spaces $H_p^{k,a}$, $0 < a < k$, could be introduced as the intermediate spaces for $L_p(I)$ and $W_p^k(I)$ (cf. [3] and [17]). The results of this section and the general theory of interpolating spaces [18] show that the theorems on isomorphisms can be extended to Besov spaces and perhaps to several variables too.

References

[1] J. H. Ahlberg, E. N. Nilson, J. L. Walsh, *Theory of splines and their applications*, New York 1967.

[2] S. V. Botschkarev, *On the coefficients of the Fourier–Haar series* (in Russian), Mat. Sbor. 80 (1969), pp. 97–116.

[3] P. L. Butzer, H. Berens, *Semi-groups of operators and approximation*, Berlin 1967.

[4] Z. Ciesielski, *On Haar functions and on Schauder basis of the space $C_{\langle 0,1\rangle}$*, Bull. Acad. Polon. Sci. 4 (1959), pp. 227–232.

[5] — *On the isomorphisms of the spaces H_a and m*, ibidem 4 (1960), pp. 217–222.

[6] — *Properties of the orthonormal Franklin system*, Studia Math. 23 (1963), pp. 141–157.

[7] — *Properties of the orthonormal Franklin system. II*, ibidem 27 (1966), pp. 289–323.

[8] — *A construction of basis in $C^{(1)}(I^2)$*, ibidem 33 (1969), pp. 243–247.

[9] — *Spline bases in function spaces*, Proc. of the Conference on Approximation Theory, Poznań 1972 (to appear in 1975).

[10] J. Ciesielski and J. Domsta, *Construction of an orthonormal basis in $C^m(I^d)$ and $W_p^m(I^d)$*, Studia Math. 41 (1972), pp. 211–224.

[11] — — *Estimates for the spline orthonormal functions and for their derivatives*, ibidem 44 (1972), pp. 315–320.

[12] J. Domsta, *Approximation by interpolating spline bases*, in preparation for Studia Mathematica.

[13] G. Freud and V. Popov, *Some questions that are related to the approximation by spline functions and polynomials* (in Russian), Studia Sci. Math. Hungar. 5 (1970), pp. 161–171.

[14] B. I. Golubov, *Best approximation of functions in L_p metric by Haar and Walsh polynomials* (in Russian), Mat. Sbor. 87 (1972), pp. 254–274.

[15] H. Johnen, *Inequalities connected with the moduli of smootheness*, Matematiceski Vestnik 19 (1973), pp. 289–303.

[16] V. A. Matveev, *The Schauder series* (in Russian), Mat. Zametki 2 (1967), pp. 267–278.

[17] J. Peetre, *On the theory of interpolation spaces*, Revista Union Mathematica Argentina 23 (1967), pp. 49–66.

[18] — *On the connection between the theory of interpolation spaces and approximation theory*, Proc. of the Conf. on Constructive Theory of Functions, Budapest 1969.

[19] K. Scherer, *A comparison approach to direct theorems for polynomial spline approximation*, Poznań 1972 (to appear).

[20] I. J. Schoenberg, *On spline functions. Inequalities*, pp. 255–291, New York 1967.

[21] Yu. N. Subbotin, *Approximation by splines and smooth bases in $C(0, 2\pi)$* (in Russian), Mat. Zametki 12 (1972), pp. 43–51.

[22] B. Sz.-Nagy, *Approximation properties of orthogonal expansions*, Acta Sci. Math. 15 (1953/54), pp. 31–37.
[23] R. Taberski, *Singular integrals depending on two parameters*, Prace Matematyczne 7 (1962), pp. 173–179.
[24] P. L. Ulianov, *The Haar series with monotone coefficients* (in Russian), Izv. Akad. Nauk SSSR 28 (1964), pp. 925–950.
[25] — *The Haar series* (in Russian), Mat. Sbor. 63 (1964), pp. 356–391.
[26] A. Zygmund, *Trigonometric series*, Vol. I, London/New York 1959.

INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES

Received March 14, 1974 (807)