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Subsets of the unit ball that are separated by more than one

by

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Abstract. The unit ball of each infinite dimensional Banach space is shown to contain an infinite subset such that the distance between each two of its points is greater than one. The question whether the number "one" may be replaced by "one plus ε" is related to the problem whether an infinite dimensional Banach space must have a subspace isomorphic to c_0 or to l_p. A related isomorphic invariant is also investigated.

0. Introduction. In this paper we shall show that the unit ball of an infinite dimensional normed linear space contains an infinite subset with the property that the distance between each two of its points is greater than one. The proof is based on a combinatorial lemma that may have independent interest. The question whether the unit ball has subsets which are $(1+\varepsilon)$-separated, where ε is a positive number depending on the normed space, is related to a question of J. Lindenstrauss [10] concerning the structure of Banach spaces: does every infinite dimensional Banach space contain a subspace which is isomorphic to c_0 or to l_p for some p, $1 \leqslant p < \infty$? We show a negative answer to the first question implies a negative answer to the second.

We shall employ the following notation. For an infinite dimensional normed space X, $U(X) = \{x \in X: \|x\| \leqslant 1\}$ denotes the unit ball and $S(X) = \{x \in X: \|x\| = 1\}$ is the unit sphere. For a number $\lambda > 0$, we say a subset A of X is *λ-separated* if for each pair of distinct points, x and y, in A, one has $\|x-y\| \geqslant \lambda$. We define $P(X) = \sup\{\lambda: U(X)$ contains an infinite λ-separated subset$\}$. In [9] it is shown that $1 \leqslant P(X) \leqslant 2$ and that $P(X)$ provides information concerning the tightest packing of balls of equal size inside the unit sphere. Two normed spaces, X and Y, are isomorphic if there is a one-to-one linear map, T, from X onto Y with T and T^{-1} bounded; in this case T is called an isomorphism and we write $X \sim Y$. If X and Y are isomorphic then the Banach–Mazur distance between X and Y is defined by $d(X, Y) = \inf\{\ln(\|T\| \cdot \|T^{-1}\|)\}$ where the inf is taken over all isomorphisms between X and Y. If $[X]$ denotes the set of all normed spaces isomorphic to X, then $([X], d)$ is a pseudo-metric space ([1], p. 242).

In part 3 of this paper we examine the isomorphic invariant $\bar{P}(X) = \{P(Y)\colon Y \sim X\}$.

We use $\boldsymbol{R}$ to denote the real numbers and $\boldsymbol{R}^\infty$ to denote the real vector space of all real sequences with finitely many non-zero terms. For an element $x = (x_1, x_2, \ldots) \in \boldsymbol{R}^\infty$, let $\|x\|_p = \left(\sum |x_i|^p\right)^{1/p}$ for $1 \leqslant p < \infty$, and $\|x\|_\infty = \max\{|x_i|\colon i = 1, 2, \ldots\}$. For each p, $1 \leqslant p < \infty$, the normed space $(\boldsymbol{R}^\infty, \|\cdot\|_p)$ is dense in l_p and $(\boldsymbol{R}^\infty, \|\cdot\|_\infty)$ is dense in c_0.

1. Subsets separated by more than unity. One may use the Riesz lemma ([17], p. 96) to show that for each infinite dimensional normed space X there is an infinite subset of $U(X)$ that is 1-separated. We shall use the following combinatorial lemma to show somewhat more; namely that $U(X)$ has an infinite subset, A, such that for each pair of distinct points, x and y, in A, we have $\|x-y\| > 1$. Credit is due to D. G. Larman for some rewarding conversations concerning this lemma and for independently supplying a proof of it.

Let $\{e_i\colon i = 1, 2, \ldots\}$ be the usual basis in $\boldsymbol{R}^\infty$, that is, $e_i = (0, \ldots, 0, 1, 0, \ldots)$ where the non-zero term appears in the ith coordinate. Let $\mathscr{U}$ be the set of sequences in $\boldsymbol{R}^\infty$ whose terms are elements of the set $\{-1, 0, 1\}$.

Lemma 1. *There is no subset A of $\mathscr{U}$ such that*

(i) *$e_i \in A$ for $i = 1, 2, \ldots$,*

(ii) *$x \in A$ implies $-x \in A$, and*

(iii) *if $\{x_1, x_2, \ldots\}$ is a sequence of elements of A then for some distinct positive integers i and j, $x_i - x_j \in A$.*

Proof. Suppose such a set A exists. Let $x = (x_1, x_2, \ldots)$ be an element of A and suppose that x_n is the last non-zero term of x. We say that an element $y = (y_1, y_2, \ldots) \in A$ is an extension of x if there is an element $z = (z_1, z_2, \ldots) \in A$ such that $y_i = z_i = x_i$ for all $i \leqslant n$ and for some $k > n$, $y_k = 1$ and $z_k = -1$. As a shorthand, we say that z verifies in the kth coordinate that y is an extension of x. A sequence $x_0, x_1, \ldots, x_n$ of elements of A is called a chain if each x_i is an extension of x_{i-1}. Choosing elements $y_1, y_2, \ldots, y_n$ in A such that for each i, y_i verifies in the k_ith coordinate that x_i extends x_{i-1}, one may check that the set $\{y_1, y_2, \ldots, y_n\}$ has the property that for each pair of integers i and j, with $1 \leqslant i < j \leqslant n$, the sequence $y_j - y_i$ has a 2 in the k_ith coordinate, and therefore is not in A. This property together with (iii) shows that there can exist no infinite chains in A. Call an element of A maximal if it has no extension. The argument above shows that each element x in A has a maximal extension (if x has no extension, we call x a maximal extension of itself.)

Now we inductively construct a sequence of maximal elements in A: define w_1 to be maximal extension of e_1, and supposing that $w_1, \ldots, w_k$ have been defined, choose an integer p such that all the non-zero terms of w_k occur before the pth coordinate and let w_{k+1} be any maximal extension of e_p. Partition the set of all two-element subsets of the natural numbers into two classes according to the rule: $\{i, j\}$ is in class one if $w_i - w_j \in A$; $\{i, j\}$ is in class two it $w_i - w_j \notin A$. Property (ii) shows that this partition is well defined. Ramsey's theorem [14] implies there is an infinite subset, M, of the natural numbers such that all the two-element subsets of M are in the same class. By (iii) this must be class one, and, by renumbering if necessary, we may assume that $M = \{1, 2, 3, \ldots\}$. Finally, consider the sequence $\{w_1 - w_2, w_3 - w_4, w_5 - w_6, \ldots\}$ of elements of A. By property (iii) there are distinct odd integers i and j such that $(w_i - w_{i+1}) - (w_j - w_{j+1})$ is an element of A. But now $w_i - w_{j+1}$ and $w_i - w_{i+1} - w_j + w_{j+1}$ are elements of A and thus $w_i - w_{i+1} - w_j + w_{j+1}$ is an extension of w_i (verified by $w_i - w_{j+1}$ in the first non-zero coordinate of w_{j+1}) contradicting the maximality of w_i. This contradiction establishes the lemma. ∎

Theorem 2. *For each infinite dimensional normed space X, there is an infinite subset $B \subset U(X)$ such that for distinct points x and y in B, $\|x-y\| > 1$.*

Proof. Let $\{b_i\} \subset X$ and $\{\beta_i\} \subset X^*$, the dual space of X, be orthonormal sequences, that is, $\|b_i\| = \|\beta_i\| = 1$ for $i = 1, 2, \ldots$ and $\beta_i(b_j) = \delta_{ij}$. The existence of such sequences is guaranteed by the results of [5]. Let L be the linear span of the set $\{b_i\colon i = 1, 2, \ldots\}$ and for each $x \in L$ define $\bar{x} = (\beta_1(x), \beta_2(x), \ldots) \in \boldsymbol{R}^\infty$. Let $V = \{\bar{x}\colon x \in L$ and $\|x\| \leqslant 1\}$ and let $A = V \cap \mathscr{U}$ where $\mathscr{U}$ is as in the discussion preceding Lemma 1. Clearly A satisfies (i) and (ii) of Lemma 1 and if each infinite sequence in $U(X)$ contains a pair of distinct points x and y with $\|x-y\| \leqslant 1$, then A also satisfies (iii). Thus, there must exist an infinite set B of points of $U(X)$, indeed, of $U(L)$, such that for distinct points x and y in B, $\|x-y\| > 1$. ∎

We do not know if the following stronger statement is true: for an infinite dimensional normed space X there is an infinite subset $\{x_1, x_2, \ldots\}$ of $U(X)$ and a sequence $\{\varepsilon_1, \varepsilon_2, \ldots\}$ of positive numbers such that for each pair of distinct natural numbers, i and j, we have $\|x_i - x_j\| > 1 + \varepsilon_i + \varepsilon_j$. Using the methods of [9], the validity of this statement would imply the existence of an infinite pairwise disjoint collection of balls inside the unit sphere of X, each with radius greater than $1/3$.

2. $(1+\varepsilon)$-separated subsets of the unit ball. This section is devoted to the question of the existence of $(1+\varepsilon)$-separated subsets of the unit ball in an arbitrary infinite dimensional normed space. Although the question is not resolved in general, it is answered for a large class of normed spaces: those containing a subspace isomorphic to $(\boldsymbol{R}^\infty, \|\cdot\|_p)$ for some p with

$1 \leqslant p \leqslant \infty$. It has been conjectured [10] that this class contains all infinite dimensional normed spaces. The main result is the following statement about Banach spaces isomorphic to c_0 or to l_p for some p with $1 \leqslant p < \infty$.

THEOREM 3. *If a Banach space X contains a subspace Y which is isomorphic to c_0, then $P(X) = 2$; if Y is isomorphic to l_p for some p with $1 \leqslant p \leqslant \infty$, then $P(X) \geqslant 2^{1/p}$.*

Proof. In case $Y \sim c_0$ or $Y \sim l_1$, this fact follows from the results of [7], 2.1, 2.2, which show that if a Banach space Z is isomorphic to c_0 (or to l_1) then for each $\delta > 0$, Z has a subspace W isomorphic to c_0 (or to l_1) with $d(c_0, W) < \delta$ (or $d(l_1, W) < \delta$). Thus, in these cases–as well as the case $Y \sim l_\infty$, since c_0 is a subspace of l_∞ – for each $\varepsilon > 0$ there is a $(2-\varepsilon)$-separated subset of $U(X)$. In case $Y \sim l_p$ for $1 < p < \infty$, we will show that for each λ with $0 < \lambda < 2^{1/p}$ there is a λ-separated subset of $U(X)$ using the following modification of R. C. James' method. Since Y is isomorphic to l_p, there are points $z_1, z_2, \ldots$ in Y and numbers m and M such that

$$m \sum |a_i|^p \leqslant \left\| \sum a_i z_i \right\|^p \leqslant M \sum |a_i|^p$$

for all sequences of numbers $\{a_i\}$. Let

$$K_n = \inf\left\{ \left\| \sum a_i z_i \right\|^p : \sum |a_i|^p = 1 \text{ and } a_i = 0 \text{ for } i < n \right\}$$

and let $K = \lim(K_n)$ as $n \to \infty$. Then $m \leqslant K \leqslant M$. Choose $\delta > 0$ so that $(1-\delta)/(1+\delta) > \lambda^p/2$, and choose p_1 so that $K_{p_1} > (1-\delta)K$. Finally, choose $\{y_n\}$ and an increasing sequence of integers $\{p_n\}$ such that

$$y_n = \sum \{a_{n,i} z_i : p_n \leqslant i \leqslant p_{n+1} - 1\}$$

where

$$\sum \{|a_{n,i}|^p : p_n \leqslant i \leqslant p_{n+1} - 1\} = 1 \quad \text{and} \quad \|y_n\| < (1+\delta)K.$$

Now for any sequence a_i we have

$$\left\| \sum a_i y_i \right\|^p \geqslant K_{p_1} \sum |a_i|^p \geqslant (1-\delta) K \sum |a_i|^p,$$

so, letting $x_i = y_i/[(1+\delta)K]$ we have $x_i \in U(X)$ for each i and

$$\left\| \sum a_i x_i \right\|^p \geqslant [(1-\delta)/(1+\delta)] \sum |a_i|^p \geqslant [(\lambda)^p/2] \sum |a_i|^p.$$

In particular, $\|x_i - x_j\| > \lambda$, for $i \neq j$. ■

We are indebted to J. R. Retherford for suggesting the method of proof employed above.

Two remarks concerning Theorem 3 are appropriate. First, the bounds for $P(X)$ found above are the best possible, since, as shown in [2] and [9], $P(c_0) = 2$ and for $1 \leqslant p < \infty$, $P(l_p) = 2^{1/p}$. Second, it should be noticed that this theorem does not guarantee the existence of $(2^{1/p})$-separated subsets in spaces isomorphic to l_p, but that for each $\varepsilon > 0$, $(2^{1/p} - \varepsilon)$-separated subsets exist. For example, a space X isomorphic to c_0 with a strictly convex norm (that such a space exists in a theorem of J. A. Clarkson [3]) has $P(X) = 2$ but no set of three points in $U(X)$ can be 2-separated.

Notice that if Y is a dense subspace of X then $P(Y) = P(X)$. Furthermore, it is easy to show that if each infinite dimensional Banach space contains a subspace isomorphic to c_0 or l_p for some p, $1 \leqslant p \leqslant \infty$, then each infinite dimensional normed space contains a subspace isomorphic to $(\boldsymbol{R}^\infty, \|\cdot\|_p)$ for some p, $1 \leqslant p \leqslant \infty$. Thus, if Lindenstrauss' conjecture is valid, the following statement shows that for each infinite dimensional normed space X, there is an $\varepsilon > 0$ and an infinite $(1+\varepsilon)$-separated subset of $U(X)$.

COROLLARY 4. *If a normed space X contains a subspace Y isomorphic to $(\boldsymbol{R}^\infty, \|\cdot\|_p)$, $1 \leqslant p < \infty$, then $P(X) \geqslant 2^{1/p}$; if Y is isomorphic to $(\boldsymbol{R}^\infty, \|\cdot\|_\infty)$ then $P(X) = 2$.*

We note in passing that recent work on Lindenstrauss' conjecture has appeared [11] in which a Banach space is exhibited that has no complemented subspace isomorphic to c_0 or l_p, $1 \leqslant p < \infty$.

While we cannot demonstrate the existence of infinite $(1+\varepsilon)$-separated subsets of the unit ball of every infinite dimensional normed space, we can show that the existence of such sets is equivalent to the negation of the combinatorial conjecture below.

CONJECTURE 5. *There exists a subset A of $\boldsymbol{R}^\infty$ such that*

(i) *A is convex,*

(ii) *$e_i \in A$ for each $i = 1, 2, \ldots$,*

(iii) *$x \in A$ implies $-x \in A$,*

(iv) *$x = (\xi_1, \xi_2, \ldots) \in A$ implies $|\xi_i| \leqslant 1$ for each $i = 1, 2, \ldots$, and*

(v) *for each sequence $\{x_1, x_2, \ldots\}$ of elements of A and $\varepsilon > 0$ there exist distinct natural numbers i and j such that $(1-\varepsilon)(x_i - x_j) \in A$.*

Notice that ε cannot be taken equal to 0, otherwise the set of sequences in A whose terms come from the set $\{-1, 0, 1\}$ satisfies the conditions of Lemma 1.

THEOREM 6. *There exists an infinite dimensional Banach space X with $P(X) = 1$ if and only if Conjecture 5 is true.*

Proof. First suppose a set A exists that satisfies Conjecture 5. Let Z denote the normed space $(\boldsymbol{R}^\infty, \|\cdot\|_1)$. Then $U(Z)$ is a subset of A and thus A is radial at 0. Clearly A is convex and symmetric. Thus, A is the unit ball of a normed space Y. Let X be the completion of Y. Since Y is dense in X, $P(Y) = P(X)$, and by property (v) of the set A, $P(Y) = 1$. On the other hand, if X is any normed space with $P(X) = 1$, we let $\{b_i\}$, $\{\beta_i\}$, L, and V be as in the proof of Theorem 2. Then $V \subset \boldsymbol{R}^\infty$ and satisfies properties (i) through (v) of Conjecture 5. ∎

We close this section with a remark on finite λ-separated subsets of the unit ball of an infinite dimensional normed space. An application of the main theorem of [6] shows that for each infinite dimensional normed space X and $\varepsilon > 0$ there are subsets of $U(X)$ of arbitrarily large finite cardinality that are $(2^{1/2}-\varepsilon)$-separated. On the other hand, a result of [15] states that for each $\varepsilon > 0$ there is an integer n such that there is no $(2^{1/2}+\varepsilon)$-separated subset of l_2 of cardinality n. Thus, in a sense, the conclusion drawn from A. Dvoretzky's theorem above is the best possible.

3. An isomorphic invariant. For each infinite dimensional normed space X we define $\overline{P}(X) = \{\lambda\colon P(Y) = \lambda$ for some space $Y \sim X\}$. Clearly, for each X, $\overline{P}(X)$ is a subset of the closed interval $[1, 2]$. If Conjecture 5 is false, then $\overline{P}(X)$ is a subset of $(1, 2]$. In this section we list some sets that are possible candidates for the isomorphic invariant $\overline{P}(X)$, and, with two exceptions, find a Banach space X corresponding to each possibility.

THEOREM 7. *For each infinite dimensional normed space X there is a number b, $1 \leqslant b \leqslant 2$, such that $\overline{P}(X)$ equals either $(b, 2]$ or $[b, 2]$.*

Proof. We shall show that if $c \in \overline{P}(X)$, then $[c, 2]$ is a subset of $P(X)$. To accomplish this, we first observe that the function $P\colon X \to P(X)$ is a continuous function from the pseudo-metric space $([X], d)$ to the real numbers. The easy details of this fact are left for the reader. Now since $([X], d)$ is connected (in fact pathwise connected [12]), it suffices to exhibit a normed space $Y \in [X]$ with $P(Y) = 2$. Let $\{b_i\} \subset X$ and $\{\beta_i\} \subset X^*$ be as in the proof of Theorem 2 and define $V =$ convex hull $(U(X) \cup \{\pm 2b_i\colon i = 1, 2, \ldots\})$. Let $\|\cdot\|$ denote the norm in X and let $\|\cdot\|'$ denote the Minkowski functional of V. For each $x \in X$ we have $\|x\| \leqslant 2\|x\|' \leqslant 2\|x\|$ and thus $Y \in [X]$ where Y denotes the vector space X with the norm $\|\cdot\|'$. It remains to show that the set $\{2b_i\colon i = 1, 2, \ldots\}$ is is 2-separated in Y. If β is a continuous linear functional on X, let $\|\beta\|$ denote the norm of β considered as an element of X^* and let $\|\beta\|'$ denote the norm of β considered as an element of Y^*. Since $\|\beta_i\| = 1$ for each $i = 1, 2, \ldots$, it follows that $\|\beta_i - \beta_j\| \leqslant 2$. Furthermore, if $x \in$ convex hull $\{\pm 2b_i\colon i = 1, 2, \ldots\}$, then $|(\beta_i - \beta_j)(x)| \leqslant 2$. Thus $\|\beta_i - \beta_j\|' \leqslant 2$. Now

$$\|2b_i - 2b_j\|' \geqslant (1/2)(\beta - \beta_j)(2b_i - 2b_j) = (1/2)(2+2) = 2$$

which completes the proof. ∎

Since $P\colon ([X], d) \to \boldsymbol{R}$ is continuous, the parameter $P(X)$ is potentially useful for determining that certain normed spaces are not nearly isometric. This technique is discussed in [13] and [18].

The results of [2] and [9] show that $2^{1/p} \in \overline{P}(l_p)$ for $1 \leqslant p < \infty$ and Theorem 3 shows that $2^{1/p} = \inf\{\lambda\colon \lambda \in \overline{P}(l_p)\}$. Theorem 3 further shows that $\overline{P}(c_0) = \overline{P}(l_1) = \{2\}$ and thus for each $b \in (1, 2]$ we have an example of a Banach space X with $\overline{P}(X) = [b, 2]$.

We shall show in Theorem 11 that for each $b \in (1, 2)$ there is a Banach space X with $\overline{P}(X) = (b, 2]$. We do not know whether there exists a normed space X with either $\overline{P}(X) = [1, 2]$ or $\overline{P}(X) = (1, 2]$. Of course, an example of either type would yield a negative answer to Lindenstrauss' conjecture. The path to Theorem 11 is segmented into three lemmas, the first of which has independent interest since it gives a method for calculating $P(X)$ for a large class of normed spaces.

If $(X_1, X_2, \ldots)$ is a sequence of normed spaces, then $X = (X_1 \oplus X_2 \oplus \oplus \ldots)_{l_p}$ denotes the normed space studied in [4], that is, X is the set of all sequences $x = (x(1), x(2), \ldots)$ such that $x(i) \in X_i$ for each $i = 1, 2, \ldots$ and

$$\|x\| = \Big(\sum\{(\|x(i)\|_{X_i})^p\colon i = 1, 2, \ldots\}\Big)^{1/p} < \infty.$$

LEMMA 8. *Let $(X_1, X_2, \ldots)$ be a sequence of normed spaces with $\dim(X_i) \geqslant 1$ for each $i = 1, 2, \ldots$ and let $X = (X_1 \oplus X_2 \oplus \ldots)_{l_p}$. Then*

$$P(X) = \max\big(2^{1/p}, \sup\{P(X_i)\colon i = 1, 2, \ldots\}\big).$$

Proof. Let $Q = \max\big(2^{1/p}, \sup\{P(X_i)\colon i = 1, 2, \ldots\}\big)$. Clearly $P(X) \geqslant Q$, since X contains subspaces isometric to l_p and to X_i for each $i = 1, 2, \ldots$ To complete the proof we shall suppose that $\{x_i\colon i = 1, 2, \ldots\}$ is a $(Q+\varepsilon)$-separated subset of $S(X)$ and obtain a contradiction via an adaptation of the 'gliding hump' method of [9], 1.5. To this end, we inductively select a subsequence $\{z_i\}$ of $\{x_i\}$ with special properties. First, since $Q^p \geqslant 2$, we may choose $\delta > 0$ so small that for any number a, $0 \leqslant a \leqslant 1$, we have

$$\{(Q+\delta)^p(a+\delta) + 2[\delta^{1/p} + (1-a+\delta)^{1/p}]^p\}^{1/p} < Q+\varepsilon. \tag{1}$$

We may assume, by passing to a subsequence of $\{x_i\}$ if necessary, that $\lim\limits_{i\to\infty}\|x_i(j)\| = a_j$ exists for each j. Next we construct a sequence of positive numbers $\{\eta_j\}$ such that for each n we have

$$\sum_{j=1}^{n}(a_j+\eta_j)^p \leqslant \Big(\sum_{j=1}^{n} a_j^p\Big) + \delta \tag{2}$$

and

$$\sum_{j=1}^{n}|a_j-\eta_j|^p \operatorname{sgn}(a_j-\eta_j) \geqslant \Big(\sum_{j=1}^{n} a_j^p\Big) - \delta. \tag{3}$$

Notice that if $x \in X$ is such that $\|x\| = 1$ and

$$\big|\|x(j)\| - a_j\big| < \eta_j \quad \text{for all } 1 \leqslant j \leqslant n,$$

then (3) implies

$$\sum_{j=n+t}^{n+t+s} \|x(j)\|^p \leqslant 1 - \sum_{j=1}^{n} \|x(j)\|^p \leqslant 1 - \sum_{j=1}^{n} a_j^p + \delta \tag{4}$$

for all positive numbers s and t. Let $M_0 = 0$, $z_1 = x_1$; choose M_1 so that $\sum_{j=M_1}^{\infty} \|z_1(j)\|^p < \delta$. Let $\{y_i\colon i = 1, 2, \ldots\}$ be a subsequence of $\{x_i\colon i = 2, 3, \ldots\}$ such that

$$\big|\|y_i(j)\| - a_j\big| < \eta_j \quad \text{for all } i \text{ and all } j \leqslant M_1.$$

Now the sequence $\{(a_1+\eta_1)^{-1} y_i(1)\colon i = 1, 2, \ldots\}$ is a subset of the unit ball of X_1 and $P(X_1) \leqslant Q$. Partitioning all two element subsets of the natural numbers into two classes according to the rule: $\{i, j\}$ belongs to class one if

$$\|y_i(1) - y_j(1)\| (a_1 + \eta_1)^{-1} \leqslant Q + \delta$$

and to class two if the opposite inequality holds, an application of Ramsey's theorem shows an infinite subset, M, of the natural numbers has all its two-element subsets in class one. By passing to another subsequence, we may assume $M = \{1, 2, \ldots\}$. Repeating this Ramsey argument in turn for the sequences

$$\{(a_j + \eta_j)^{-1} y_i(j)\colon i = 1, 2, \ldots\}, \qquad j = 2, 3, \ldots, M_1,$$

we may assume that

$$\|y_i(j) - y_k(j)\| \leqslant (Q+\delta)(a_j + \eta_j) \quad \text{for all } j \leqslant M_1.$$

Let $z_2 = y_1$ and choose $M_2 > M_1$ so that

$$\sum_{j=M_2}^{\infty} \|z_2(j)\|^p < \delta.$$

Continuing this process in the obvious fashion we construct a sequence $\{z_i\}$ in $S(X)$ and an increasing sequence of numbers $\{M_i\}$ such that

$$\sum_{j=M_k}^{\infty} \|z_k(j)\|^p < \delta \quad \text{for each } k, \tag{5}$$

$$\big|\|z_k(j)\| - a_j\big| < \eta_j \quad \text{for all } k \text{ and } j \leqslant M_{k-1}, \tag{6}$$

$$\|z_k(j) - z_m(j)\| \leqslant (Q+\delta)(a_j + \eta_j) \quad \text{for } j \leqslant M_{k-1} \text{ and } j \leqslant M_{m-1}, \tag{7}$$

and of course

$$\|z_k - z_m\| > Q + \varepsilon \quad \text{for } k \neq m. \tag{8}$$

We claim that for a fixed n and any $k < n$ one has

$$\sum_{j=M_{k-1}+1}^{M_k} \|z_n(j)\|^p > \delta, \tag{9}$$

because, if not, then letting $a = \sum_{j=1}^{M_{k-1}} a_j^p$ we have

$$\begin{aligned}
\|z_n - z_k\|^p &= \sum_{j=1}^{\infty} \|z_n(j) - z_k(j)\|^p \\
&= \sum_{j=1}^{M_{k-1}} \|z_n(j) - z_k(j)\|^p + \sum_{j=M_{k-1}+1}^{M_k} \|z_n(j) - z_k(j)\|^p + \\
&\qquad + \sum_{j=M_k+1}^{\infty} \|z_n(j) - z_k(j)\|^p \\
&\leqslant \sum_{j=1}^{M_{k-1}} [(Q+\delta)(a_j + \eta_j)]^p + \\
&\quad + \Big\{\Big(\sum_{j=M_{k-1}+1}^{M_k} \|z_n(j)\|^p\Big)^{1/p} + \Big(\sum_{j=M_{k-1}+1}^{M_k} \|z_k(j)\|^p\Big)^{1/p}\Big\}^p + \\
&\quad + \Big\{\Big(\sum_{j=M_k+1}^{\infty} \|z_n(j)\|^p\Big)^{1/p} + \Big(\sum_{j=M_k+1}^{\infty} \|z_k(j)\|^p\Big)^{1/p}\Big\}^p \\
&\hspace{6em} \text{(by (7) and Hölder's inequality)} \\
&\leqslant (Q+\delta)^p (a + \delta) + \Big\{\delta^{1/p} + \Big(\sum_{j=M_{k-1}+1}^{M_k} \|z_k(j)\|^p\Big)^{1/p}\Big\}^p + \\
&\qquad + \Big\{\Big(\sum_{j=M_k+1}^{\infty} \|z_n(j)\|^p\Big)^{1/p} + \delta^{1/p}\Big\}^p \\
&\hspace{6em} \text{(by (2), the negation of (9), and (5))} \\
&\leqslant (Q+\delta)^p (a+\delta) + 2[\delta^{1/p} + (1 - a + \delta)^{1/p}]^p \qquad \text{(by (6) and (4))} \\
&\leqslant (Q+\varepsilon)^p \qquad \text{(by (1))}
\end{aligned}$$

which contradicts (8). But the claim just established implies that $\|z_m\|^p > (m-1)\delta$, which, for m sufficiently large, contradicts the fact that each z_i has unit norm. ■

The lemma above generalizes results in [2] and [16], where only l_p spaces are considered, and also 1.7 of [9], where each space X_i was required to be finite dimensional.

Although it seems attractive to hope that if a Banach space X has $\overline{P}(X) = [2^{1/p}, 2]$ then X contains a subspace isomorphic to l_p, an application of Lemma 8 shows that if $\{p_1, p_2, \ldots\}$ is a sequence monotone decreasing to p, then $X = (l_{p_1} \oplus l_{p_2} \oplus \ldots) l_{p_1}$ has no subspace isomorphic to l_p but $\overline{P}(X) = [2^{1/p}, 2]$.

We suspect that the following form of Lemma 8 is valid for a broader class of sequence spaces than the l_p spaces, but we have no proof: if Y is a sequence space and $X = (X_1 \oplus X_2 \oplus \ldots)_Y$ then

$$P(X) = \max\big(P(Y), \sup\{P(X_i): i = 1, 2, \ldots\}\big).$$

Lemma 9. *If X is a normed space with $P(X) = s$ and $\{x_i\}$ is a sequence of points in the unit ball of X, then for each natural number n and $\varepsilon > 0$ there is a sequence of numbers $\eta = (\eta_1, \eta_2, \ldots)$ such that the terms of η come from the set $\{-1, 0, 1\}$ and exactly 2^n terms of η are non-zero and*

$$\Big\| \sum_{i=1}^{\infty} \eta_i x_i \Big\| \leqslant s^n + \varepsilon.$$

Proof. The proof is by induction on n. For $n = 0$ the lemma is obvious, and for $n = 1$, the conclusion follows immediately from the hypothesis $P(X) = s$. Suppose the lemma is valid for $n = k-1$. Fix $\varepsilon > 0$ and choose $\delta > 0$ such that $(s^{k-1}+\delta)s + \delta \leqslant s^k + \varepsilon$. Find $\eta^1 = (\eta_1^1, \eta_2^1, \ldots)$ with 2^{k-1} non-zero terms such that if $z_1 = \sum \eta_i^1 x_i$ then $\|z_1\| \leqslant s^{k-1} + \delta$. Choose K so large that all the non-zero terms of η^1 occur before the Kth coordinate and apply the lemma again to the sequence $\{x_K, x_{K+1}, \ldots\}$ to obtain a sequence $\eta^2 = (\eta_1^2, \eta_2^2, \ldots)$ with 2^{k-1} non-zero terms, all occurring after the Kth coordinate, such that if $z_2 = \sum \eta_i^2 x_i$ then $\|z_2\| \leqslant s^{k-1} + \delta$. Continuing in this fashion, we obtain a sequence $\{z_1, z_2, \ldots\}$ contained in $(s^{k-1}+\delta)U(X)$. Shrinking by a factor of $(s^{k-1}+\delta)^{-1}$ and applying the fact that $P(X) = s$, there must exist two distinct elements, z_i and z_j, such that

$$\|z_i - z_j\| \leqslant (s^{k-1}+\delta)(s) + \delta \leqslant s^k + \varepsilon.$$

The required sequence η with 2^k non-zero terms is $\eta^i - \eta^j$. ■

It may be of interest to notice that the non-zero terms of η can be made to occur in the pattern $1;\ -1;\ -1,\ 1;\ -1, 1, 1\ -1; \ldots$ where the first term is 1 and for each k the terms from 2^k+1 to 2^{k+1} are the same as those from 1 to 2^k with the signs reversed. In general, for $n > 0$ the 2^n non-zero terms of η consist of equal numbers of $+1$ and -1.

Lemma 10. *Let p be a fixed number, $1 < p < \infty$. For each natural number n and $\varepsilon > 0$ such that $2^{1/p} + \varepsilon \leqslant 2$ there is a Banach space X with $P(X) = 2^{1/p} + \varepsilon$ and $2^{1/p} \in \overline{P}(X)$ such that if Y is isomorphic to X with $P(Y) \leqslant 2^{1/p}$ then $d(X, Y) \geqslant \ln(n)$.*

Proof. Choose $q < p$ such that $2^{1/q} = 2^{1/p} + \varepsilon$ and let r be any number with $p \leqslant r < \infty$. Let U be the unit ball of the space $(\boldsymbol{R}^\infty, \|\cdot\|_r)$ and let V be the unit ball of $(\boldsymbol{R}^\infty, \|\cdot\|_q)$. For a number $m > 2$, to be chosen later, let $W = \text{convex hull}\,\big(V \cup (1/m)U\big)$ and let $|||\cdot|||$ denote the Minkowski functional of W on $\boldsymbol{R}^\infty$. We claim that, for the proper choice of m, the completion, X, of $(\boldsymbol{R}^\infty, |||\cdot|||)$ satisfies the conditions of this lemma. It is easy to show, in a manner similar to the proof of Theorem 7, that the usual basis in $(\boldsymbol{R}^\infty, |||\cdot|||)$ is a $(2^{1/q})$-separated subset of its unit ball. On the other hand, let $\{w_i\}$ be a sequence of points in W and write $w_i = \lambda_i u_i/m + (1-\lambda_i)v_i$ for each $i = 1, 2, \ldots$, where $u_i \in U$ and $v_i \in V$. By compactness of the interval $[0, 1]$ we may assume, by taking a subsequence if necessary, that for a fixed $\delta > 0$, $|\lambda_i - \lambda_j| < \delta$ for all $i, j = 1, 2, \ldots$ By Ramsey's theorem, there is an infinite subsequence of $\{u_i\}$ (which, abusing notation, we identify with $\{u_i\}$) such that

$$\|u_i - u_j\|_r \leqslant 2^{1/r} + \delta \quad \text{for all } i \neq j.$$

Picking distinct integers i and j such that $\|v_i - v_j\|_q \leqslant 2^{1/q} + \delta$ we have, since $\|\cdot\|_q \geqslant |||\cdot|||$ and $\|\cdot\|_r \geqslant |||\cdot|||/m$,

$$\begin{aligned}
|||w_i - w_j||| &\leqslant |||\lambda_i u_i - \lambda_j u_j|||/m + |||(1-\lambda_i)v_i - (1-\lambda_j)v_j||| \\
&\leqslant \lambda_i |||u_i - u_j|||/m + \delta + (1-\lambda_i)|||v_i - v_j||| + \delta \\
&\leqslant \lambda_i \|u_i - u_j\|_r + (1-\lambda_i)\|v_i - v_j\|_q + 2\delta \\
&\leqslant \lambda_i(2^{1/r} + \delta) + (1-\lambda_i)(2^{1/q} + \delta) + 2\delta \\
&\leqslant 2^{1/q} + 3\delta.
\end{aligned}$$

Noticing that $P(X)$ is the same as $P(Z)$ for any dense subspace Z of X, and letting δ tend to 0, we have $P(X) = 2^{1/q}$. Furthermore, since $V \subset U$, $(\boldsymbol{R}^\infty, |||\cdot|||)$ is isomorphic to l_r and thus $\overline{P}(X) = [2^{1/r}, 2]$ which verifies the fact that $2^{1/p} \in \overline{P}(X)$. It remains to show that if Y is isomorphic to X with $P(Y) \leqslant 2^{1/p}$ then $d(X, Y) \geqslant \ln(n)$. To do this we shall pick m in such a way that if $T: X \to Y$ is an isomorphism with $\|T\| \leqslant 1$ and $P(Y) \leqslant 2^{1/p}$, then $\|T^{-1}\| \geqslant n$. First let a be a fixed positive number and pick k so large that $2^{k/q}/(2^{k/p} + a) \geqslant n$. Then choose m such that $1/m < 2^{-k/q}$. It is not hard to verify that for this choice of m, if $\eta = (\eta_1, \eta_2, \ldots)$ is a sequence in $\boldsymbol{R}^\infty$ whose terms come from the set $\{-1, 0, 1\}$ and which has exactly 2^k non-zero terms, then $|||\eta||| = \|\eta\|_q = 2^{k/q}$. To finish the proof, let Y be isomorphic to X with $P(Y) \leqslant 2^{1/p}$ and let $T: X \to Y$ be an isomorphism with $\|T\| \leqslant 1$. Let $\{e_i: i = 1, 2, \ldots\}$ be the usual basis in $(\boldsymbol{R}^\infty, \|\cdot\|)$ so that $\{T(e_i): i = 1, 2, \ldots\}$ is a subset of $U(Y)$. By Lemma 9, for the fixed number $a > 0$, there is a sequence $\eta = (\eta_1, \eta_2, \ldots)$ whose terms come from the set $\{-1, 0, 1\}$ with 2^k non-zero terms such that

$$\Big\|\sum\{\eta_i T(e_i): i = 1, 2, \ldots\}\Big\|_Y \leqslant 2^{k/p} + a.$$

But now

$$\|T^{-1}\| \geqslant \Big\|\Big|\sum \eta_i e_i\Big\|\Big| \Big/ \Big\|T\Big(\sum \eta_i e_i\Big)\Big\|_Y \geqslant (2^{k/q})/(2^{k/p} + a) \geqslant n. \ \blacksquare$$

THEOREM 11. *For each number $b \in (1, 2)$ there is a Banach space X with $\overline{P}(X) = (b, 2]$.*

Proof. Choose p such that $2^{1/p} = b$ and let ε be a positive number such that $b + \varepsilon \leqslant 2$. For each natural number n, use Lemma 10 to construct a Banach space X_n with $P(X_n) = b + \varepsilon/n$ and $b \in \overline{P}(X_n)$ and such that if Y is isomorphic to X_n with $P(Y) \leqslant b$ then $d(X_n, Y) \geqslant \ln(n)$. Let $r \geqslant p$ and define $X = (X_1 \oplus X_2 \oplus \ldots)_{l_r}$. Let $T: X \to Y$ be an isomorphism and let $d(X, Y) = s < \infty$. Choosing n such that $\ln(n) > s$ and examining $P(T(X_n))$ we have $P(Y) > 2^{1/p} = b$. Thus $\overline{P}(X) \subset (b, 2]$. On the other hand, for an arbitrary $\delta > 0$, we may choose n such that $\varepsilon/n < \delta$ and find Banach spaces $Y_i \sim X_i$ for $1 \leqslant i \leqslant n$ with $P(Y_i) = b$. Then X is isomorphic to $Y = (Y_1 \oplus \ldots \oplus Y_n \oplus X_{n+1} \oplus X_{n+2} \oplus \ldots)_{l_r}$ under the canonical map, and by Lemma 8, $P(Y) \leqslant b + \delta$. Therefore $\overline{P}(X) = (b, 2]$. $\blacksquare$

It should be noticed that the freedom in the choice of the parameter r in the proofs of Lemma 10 and Theorem 11 allows the example constructed in Theorem 11 to be the l_p-sum of a sequence of spaces, each of which is isomorphic to l_p.

Added in proof. B. S. Tsirelson has recently constructed a Banach space with no subspace isomorphic to c_0 or any l_p, $1 \leqslant p < \infty$, thus providing us with a counterexample to the conjecture of Lindenstrauss.

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