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Weighted norm inequalities for the Hardy-Littlewood maximal function for one parameter rectangles

by

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Abstract. The paper studies conditions on a non-negative function w so that the transformation which sends a function to its Hardy–Littlewood maximal function is a bounded operator from $L^p(wdx)$ to $L^p(wdx)$. The Hardy–Littlewood maximal function of a function f, with respect to a family of geometric shapes $\mathscr{R}$, is defined as

$$f^*(x) = \operatorname*{Sup}_{\{R\in\mathscr{R}:\,x\in R\}} \frac{1}{|R|}\int_R |f|(y)\,dy.$$

The family of shapes considered are one-parameter rectangles in Euclidean n-space, with generalizations to collections of shapes similar to such rectangles. If $1 < p < \infty$, a necessary and sufficient condition on w is that

$$\operatorname*{Sup}_{R\in\mathscr{R}} \left(\frac{1}{|R|}\int_R w(x)\,dx\right)\left(\frac{1}{|R|}\int_R w(x)^{-1/(p-1)}\,dx\right)^{p-1} < \infty,$$

which is the A_p condition. An analogous result is proved for the case $p = 1$.

1. Introduction. In [5], Jessen, Marcinkiewicz, and Zygmund prove L^p norm inequalities for the Hardy–Littlewood maximal function defined over collections of n-dimensional rectangles. When the rectangles are n-dimensional squares, Muckenhoupt [6] has obtained necessary and sufficient conditions for weighted norm inequalities. In this paper, we prove weighted norm inequalities for the maximal function taken over a class of one-parameter rectangles. Similar results are then obtained for certain classes of metric balls.

Before stating the main results, we first list several definitions.

Let $\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t)$ be n continuous, monotone non-decreasing functions of one parameter, $t > 0$, which increase to ∞ and decrease to 0 with t. Define

$$\begin{aligned} R(0,t) &= \{x = (x_1, \ldots, x_n)\in E^n\colon\ |x_i| \leqslant \varphi_i(t)/2,\ i = 1, 2, \ldots, n\}, \\ \mathscr{R} &= \{R(x,t)\colon\ R(x,t) = x + R(0,t),\ x\in E^n,\ t > 0\}. \end{aligned} \tag{1.1}$$

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Such a collection of rectangles, $\mathscr{R}$, is a one parameter class of rectangles in E^n. Similarly, define $k\cdot R(x,t) = x + k\cdot R(0,t)$ where

$$k\cdot R(0,t) = \{x\in E^n\colon\ |x_i| \leqslant k\varphi_i(t)/2,\ i = 1, 2, \ldots, n\}.$$

Where no confusion should arise, a rectangle of the form $R(x,t)$ will be represented by $R(t)$ or R. Examples of collections satisfying (1.1) are the collection of all n-dimensional squares and one where $\varphi_i(t) = t^{a_i}$, for a_i a positive constant, $1 \leqslant i \leqslant n$. Such one parameter collections were introduced in [5].

Next, we define the Hardy–Littlewood maximal function, $f^*(x)$, with respect to a class of rectangles satisfying (1.1). If $\mathscr{R}$ is such a collection then

$$f^*(x) = \sup \frac{1}{|R|}\int_R |f(y)|\,dy \tag{1.2}$$

where the supremum is taken over all $R\in\mathscr{R}$ which contain x.

The A_p condition with respect to a collection of rectangles, $\mathscr{R}$, satisfying (1.1) is defined for all non-negative functions as follows: For $1 < p < \infty$, $w\in A_p$ if

$$\left(\frac{1}{|R|}\int_R w(x)\,dx\right)\left(\frac{1}{|R|}\int_R w(x)^{-1/(p-1)}\,dx\right)^{p-1} \leqslant c \tag{1.3}$$

for all $R\in\mathscr{R}$, where c is independent of R. A function, w, satisfies the A_1 condition if

$$w^*(x) \leqslant cw(x) \tag{1.4}$$

for almost every $x\in E^n$, where c is independent of x. These conditions were first considered in [6] and [7] for cubes, and in [4] for cubes with $p = 1$. It is interesting to note that an A_p function may not take on the values 0 or ∞ on a set of positive measures unless it is the whole space, due to conditions (1.3) and (1.4).

The major result of this paper is the following theorem.

(1.5) THEOREM. *Suppose* $1 < p < \infty$ *and* $\mathscr{R}$ *satisfies* (1.1). *Then, there exists a constant, c, independent of f, such that*

$$\int_{E^n} [f^*(x)]^p w(x)\,dx \leqslant c\int_{E^n} |f(x)|^p w(x)\,dx$$

if and only if $w\in A_p$.

As shown in Section 3, a weak type result is proved for the case $p = 1$. Theorem (1.5) has been proved for arbitrary rectangles and a weight function which is identically one in [5] when $1 < p < \infty$, and for cubes and a weight function satisfying A_p for cubes in [6]. The proof of (1.5) is contained in Section 5.

In Section 2, a covering lemma essentially due to Jessen, Marcinkiewicz, and Zygmund is stated. In addition, an interesting result pertaining measures generated by A_p functions is demonstrated. This covering lemma is used to prove a weak type result in Section 3.

Several theorems relating A_p classes are considered in Section 4. The most interesting result is the following theorem.

(1.6) THEOREM. *Let* $1 < p < \infty$ *and* $w\in A_p$. *Then* $w\in A_{p-\varepsilon}$, *for some* $\varepsilon > 0$.

This was proved for cubes by Muckenhoupt [6], and then simplified by Coifman and Fefferman [1]. Though the proof is similar to Coifman and Fefferman's, a new result was needed for its completion.

The final section is devoted to generalizing the previous results to certain classes of metric balls. Such collections are considered by de Guzmán in [2]. A few lemmas are proved which imply (1.5) is true for such collections. The method of proof demonstrates that the results of this paper can be extended to any class of shapes which is geometrically similar to a collection of rectangles satisfying (1.1).

Standard notations will be used in this paper, so that $|E|$ will represent the Lebesgue measure of a set E, p' will satisfy the equation $\dfrac{1}{p}+\dfrac{1}{p'} = 1$, and $0\cdot\infty$ will equal 0. c will represent a constant, though not necessarily one such constant.

Before continuing, I would like to extend my warmest thanks to Dr. Richard Wheeden. This paper never would have been completed without his effort and guidance, for which I am deeply indebted.

2. The covering lemma. The following theorem is a generalization of a result of Jessen, Marcinkiewicz, and Zygmund, in [5], for Lebesgue measure. See also [8], vol. 2, p. 309.

(2.1) THEOREM. *Let* $\mathscr{R}$ *satisfy* (1.1). *Suppose* μ *is a measure such that*

$$\mu(k\cdot R) \leqslant c\cdot\mu(R) \tag{2.2}$$

for all $R\in\mathscr{R}$, *where c is dependent only on k. Suppose E is a measurable set such that* $0 < \mu(E) < \infty$. *For each* $y\in E$, *associate some* $R_y\in\mathscr{R}$ *which contains y. Then, there are a finite number of points,* $y_1, \ldots, y_N\in E$, *and a positive constant,* γ, *such that* $R_{y_1}, \ldots, R_{y_N}$ *are disjoint and*

$$\sum_{i=1}^{N} \mu(R_{y_i}) > \gamma\mu(E).$$

The proof is the same as for Lebesgue measure since μ satisfies (2.2).

Let $w\in A_p$, for some p, $1 \leqslant p < \infty$. Define

$$m_w(E) = \int_E w(x)\,dx \tag{2.3}$$

for any Lebesgue measurable set, E. Then, m_w is a measure which, as pointed out to this author by Dr. B. Muckenhoupt, satisfies (2.2). The proof is as follows.

If w is 0 or ∞ almost everywhere, the result is obvious. Thus, suppose $0 < w(x) < \infty$ almost everywhere and that $1 < p < \infty$. Let $R = R(x, t)$ be a fixed element of $\mathscr{R}$. Through x, draw n hyperplanes parallel to the coordinate hyperplanes and consider one of the 2^n quadrants, R_i, of R so formed. Clearly,

$$1 \leqslant \left(\frac{1}{|R_i|}\int_{R_i} w(x)\,dx\right)\left(\frac{1}{|R_i|}\int_{R_i} w(x)^{-1/(p-1)}\,dx\right)^{p-1}$$

by Hölder's inequality, so that

$$\left(\frac{1}{|R_i|}\int_{R_i} w(x)^{-1/(p-1)}\,dx\right)^{-(p-1)} \leqslant \frac{1}{|R_i|}\int_{R_i} w(x)\,dx. \tag{2.4}$$

By the A_p condition and (2.4), since $|R| = 2^n|R_i|$,

$$\begin{aligned}\frac{1}{|R|}\int_R w(x)\,dx &\leqslant c\left(\frac{1}{|R|}\int_R w(x)^{-1/(p-1)}\,dx\right)^{-(p-1)}\\ &\leqslant c\left(\frac{1}{2^n|R_i|}\int_{R_i} w(x)^{-1/(p-1)}\,dx\right)^{-(p-1)}\\ &\leqslant c\,2^{n(p-1)}\frac{1}{|R_i|}\int_{R_i} w(x)\,dx.\end{aligned} \tag{2.5}$$

Hence,

$$\int_R w(x)\,dx \leqslant c\,2^{np}\int_{R_i} w(x)\,dx. \tag{2.6}$$

Clearly, R_i contains one corner, q_i, of R. Let $R_i' = R(q_i, t)$. In the same manner as above

$$\int_{R_i'} w(x)\,dx \leqslant c\,2^{np}\int_{R_i} w(x)\,dx. \tag{2.7}$$

Notice that $2R = \bigcup_{i=1}^{2^n} R_i'$. Therefore,

$$\begin{aligned} m_w(2R) &= \int_{2R} w(x)\,dx = \int_{\bigcup_{i=1}^{2^n} R_i'} w(x)\,dx\\ &= \sum_{i=1}^{2^n}\int_{R_i'} w(x)\,dx \leqslant c\,2^{np}\sum_{i=1}^{2^n}\int_{R_i} w(x)\,dx = c\,2^{np}m_w(R).\end{aligned} \tag{2.8}$$

If $p = 1$, fix an $R = R(x, t)$ in $\mathscr{R}$ and define R_i and R_i' as before. For each i, by the A_1 condition,

$$\begin{aligned}\frac{1}{|R_i'|}\int_{R_i'} w(x)\,dx &\leqslant c\cdot\operatorname*{ess\,inf}_{x\in R_i'} w(x)\\ &\leqslant c\cdot\operatorname*{ess\,inf}_{x\in R_i} w(x) \leqslant c\,\frac{1}{|R_i|}\int_{R_i} w(x)\,dx.\end{aligned} \tag{2.9}$$

Since $|R_i'| = 2^n|R_i|$, from (2.9) we obtain

$$\int_{R_i'} w(x)\,dx \leqslant c\,2^n\int_{R_i} w(x)\,dx. \tag{2.10}$$

Hence, as in (2.8),

$$m_w(2R) \leqslant c\,2^n m_w(R). \tag{2.11}$$

The next result is Lebesgue's Differentiation Theorem with respect to a class of rectangles and a weight function which is identically one.

(2.12) Theorem. *Let $f \in L^1(dx)$ and $\mathscr{R}$ be a collection of rectangles satisfying* (1.1). *Then, for almost every x,*

$$\lim_{\substack{|R|\to 0\\ \{R:\ x\in R\}}}\frac{1}{|R|}\int_R f(y)\,dy = f(x).$$

The theorem follows in a routine manner from the case $d\mu = dx$ of Theorem (2.1). It is also a special case of Theorem 2.4 of [2].

3. A weak type result. We now prove a weak type result for the Hardy–Littlewood maximal function and A_p functions, defined with respect to a suitable class of rectangles. The proof is similar to the case proved by Muckenhoupt, in [6], where the rectangles are cubes, using a different covering lemma.

(3.1) Theorem. *Let $\mathscr{R}$ satisfy* (1.1) *and $1 \leqslant p < \infty$. There is a constant, c, independent of f, such that for all $a > 0$*

$$\int_{\{x:\ f^*(x)>a\}} w(x)\,dx \leqslant ca^{-p}\int_{E^n}|f(x)|^p w(x)\,dx$$

if and only if $w \in A_p$.

Suppose $w \in A_p$ and let $E = \{x: f^*(x) > a\}$. We may assume without loss of generality that f is non-negative and that E is bounded. For each $x \in E$, there is an R_x such that $\frac{1}{|R_x|}\int_{R_x} f(y)\,dy > a$. By (2.1), since the R_x's cover E, there is a finite disjoint sequence, $R_1, \ldots, R_N$, and a positive constant, γ, such that

$$\frac{1}{|R_i|}\int_{R_i} f(x)\,dx > a, \quad \text{for} \quad 1 \leqslant i \leqslant N,$$

and

$$m_w(E) < \gamma \sum_{i=1}^{N} m_w(R_i).$$

Thus,

$$(3.2)\qquad m_w(E) \leqslant \frac{\gamma}{a^p} \sum_{i=1}^{N} m_w(R_i)\, a^p \leqslant \frac{\gamma}{a^p} \sum_{i=1}^{N} m_w(R_i) \left(\frac{1}{|R_i|} \int_{R_i} f(x)\,dx\right)^p.$$

If $p > 1$, an application of Hölder's inequality to the integral of f in (3.2) yields

$$m_w(E) \leqslant \frac{\gamma}{a^p} \sum_{i=1}^{N} \left[\frac{1}{|R_i|^p} \left(\int_{R_i} w(x)\,dx\right) \left(\int_{R_i} w(x)^{-1/(p-1)}\,dx\right)^{p-1}\right] \int_{R_i} f(x)^p w(x)\,dx.$$

Since $w \in A_p$ and the R_i's are disjoint, we obtain

$$m_w(E) \leqslant c' a^{-p} \int_{E^n} f(x)^p w(x)\,dx.$$

If $p = 1$, (3.2) becomes

$$m_w(E) \leqslant \frac{\gamma}{a} \sum_{i=1}^{N} \int_{R_i} f(x) \left(\frac{1}{|R_i|} \int_{R_i} w(t)\,dt\right) dx.$$

The result is immediate since $\frac{1}{|R_i|} \int_{R_i} w(t)\,dt \leqslant w^*(x)$ for almost every x in R_i and $w \in A_1$.

For the necessity, recall that, for $1 < p < \infty$, $w \in A_p$ if and only if

$$(3.3)\qquad \left(\int_R w(x)\,dx\right)\left(\int_R w(x)^{-1/(p-1)}\,dx\right)^{p-1} \leqslant c\,|R|^p,$$

for all $R \in \mathscr{R}$. Let R be a fixed element of $\mathscr{R}$ and $A = \int_R w(x)^{-1/(p-1)}\,dx$. If $A = 0$, (3.3) is true for any c. If $0 < A < \infty$, let $f(x) = w(x)^{-1/(p-1)} \cdot \chi_R(x)$. Then $f^*(x) \geqslant A/|R|$ for all $x \in R$, and we have

$$(3.4)\qquad \int_R w(x)\,dx \leqslant c\,|R|^p A^{-p} \int_R w(x)^{-p/(p-1)} w(x)\,dx.$$

Since $A < \infty$, $w(x) > 0$ almost everywhere on R and the integral on the right side of (3.4) is A. Multiplying both sides of (3.4) by A^{p-1} proves (3.3). If $A = \infty$, $w(x)^{-1/p}$ is not in $L^{p'}$ on R, so there is a function, $g(x)$, which is in L^p on R and 0 outside of R, such that $\int_R g(x) w(x)^{-1/p}\,dx = \infty$. Let $f(x) = g(x) w(x)^{-1/p}$. Since $f(x)^p w(x) \leqslant g(x)^p$, $f(x)^p w(x)$ is integrable on R while $f^*(x) \equiv \infty$. Therefore, we have $\int_R w(x)\,dx = 0$, so (3.3) is true for any c.

If $p = 1$, the proof is exactly the same as in [6]. It is interesting to note that the necessity proof does not depend on the geometric shape being considered. The only place the shape is important is in selecting a covering lemma to prove the sufficiency part of (3.1).

4. Relations among A_p classes. In this section we prove some theorems relating different A_p classes. Though many of the proofs are routine extensions of analogous facts in [1], they are included for completeness. The interesting result of this section is Lemma (4.3). Its proof was necessitated due to the fact that collections of rectangles satisfying (1.1) may not be rich enough to use something like the Calderón–Zygmund Covering Lemma.

(4.1) THEOREM. *Let $1 \leqslant p < \infty$ and $w \in A_p$. Then, $w \in A_r$ for all $r > p$.*

For $p > 1$, the proof is an immediate consequence of Hölder's inequality. If $p = 1$ and $r > p$, then

$$\left(\frac{1}{|R|} \int_R w(x)^{-1/(r-1)}\,dx\right)^{r-1} \leqslant \frac{1}{\operatorname*{ess\,inf}_{x \in R} w(x)}$$

and

$$\frac{1}{|R|} \int_R w(x)\,dx \leqslant c \operatorname*{ess\,inf}_{x \in R} w(x).$$

The proof is now obvious.

The following four lemmas are used for the proof of Theorem (1.6).

(4.2) LEMMA. *Let $1 \leqslant p < \infty$ and $w \in A_p$. Define*

$$M_R(w) = \frac{1}{|R|} \int_R w(x)\,dx.$$

Then, there exist a and β, $0 < a < 1$ and $0 < \beta < 1$, independent of R, such that

$$|\{x \in R\colon w(x) > \beta M_R(w)\}| \geqslant a\,|R|.$$

Let $E = \{x \in R\colon w(x) \leqslant \beta M_R(w)\}$. Since E is a subset of R, if $p > 1$

$$c \geqslant \left(\frac{1}{|R|} \int_R w(x)\,dx\right) \left(\frac{1}{|R|} \int_E \left(\frac{1}{w(x)}\right)^{1/(p-1)} dx\right)^{p-1}$$

$$\geqslant M_R(w) \left(\frac{1}{|R|} \int_E \left(\frac{1}{\beta M_R(w)}\right)^{1/(p-1)} dx\right)^{p-1} = \frac{1}{\beta} \left(\frac{|E|}{|R|}\right)^{p-1}.$$

Hence, $|E| \leqslant (c\beta)^{1/(p-1)} |R|$. Let $E' = \{x \in R\colon w(x) > \beta M_R(w)\}$. Then, since E' is the complement of E in R,

$$|E'| \geqslant [1 - (c\beta)^{1/(p-1)}]\,|R|.$$

The result follows by choosing $\beta < \min(1/c, 1)$. If $w \in A_1$, then $w \in A_r$ for all $r > 1$ by (4.1). The proof is immediate.

(4.3) LEMMA. *Let $\mathscr{R}$ satisfy* (1.1) *and $R_1 = R(p_1, t_1)$ and $R_0 = R(p_0, t_0)$ be elements of $\mathscr{R}$ such that R_0 is contained in the interior of R_1. Then, there exist $R(p_t, t)$, $t_0 \leqslant t \leqslant t_1$, such that p_t is a continuous function of t, $p_{t_0} = p_0$, $p_{t_1} = p_1$, and*

$$R_0 \subset R(p_t, t) \subset R_1. \tag{4.4}$$

For simplicity, assume $p_1 = (0, 0, \ldots, 0)$ in R^n. If it is possible to continuously expand and shift R_0 to a rectangle $R(p_t, t)$, satisfying (4.4), for which p_t is on one of the coordinate axes, then the proof is practically completed. For, suppose there is a t', $t_0 \leqslant t' \leqslant t_1$, such that $R(p_{t'}, t')$ satisfies (4.4) and $p_{t'}$ lies on the x_1-axis; i.e. $p_{t'} = (x_{t'}, 0, \ldots, 0)$. Let t^* be such that

$$\frac{\varphi_1(t^*)}{2} + |x_{t'} - 0| = \frac{\varphi_1(t_1)}{2}.$$

Continuously expand $R(p_{t'}, t')$ to $R(p_{t'}, t^*)$ by increasing t' to t^*. Notice that, since $t^* < t_1$ and $p_{t'}$ is on the x_1-axis,

$$\left|0 + \frac{\varphi_i(t^*)}{2}\right| \leqslant \frac{\varphi_i(t_1)}{2}, \quad \text{for} \quad 2 \leqslant i \leqslant n.$$

Thus, $R(p_{t'}, t^*) \subset R_1$. Keeping its center on the x_1-axis, continuously slide $R(p_{t'}, t^*)$ towards $(0, 0, \ldots, 0)$ as much as possible so that each translate contains R_0. Let $R(p_{t^*}, t^*)$ be the translate whose center is closest to the origin. Repeat this process for $R(p_{t^*}, t^*)$. This will eventually make $p_t = 0$ since R is always shifted by the same amount. At this point, continuously expand t to t_1 so that $R(p_t, t)$ will coincide with R_1. Similarly, the result follows if $p_{t'}$ lies on any other axis.

Suppose p_0 does not lie on any axis. For simplicity, let p_0 be in the first quadrant of R_1; i.e., $p_0 = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)})$ where $0 \leqslant x_i^{(0)}$ for each i. For $1 \leqslant i \leqslant n$, since R_1 has boundaries $|x_i| = \varphi_i(t_1)/2$, let $c_i = \varphi_i(t_1)/2 - x_i^{(0)}$ denote the distance from p_0 to the border of R_1 crossing the x_i-axis in the first quadrant. Recall that R_0 is contained in the interior of R_1 so that $c_i > \varphi_i(t_0)/2$, for $1 \leqslant i \leqslant n$. By the continuity of the φ_i's, there exist t_{c_i}'s such that

$$\frac{\varphi_i(t_{c_i})}{2} = c_i, \quad 1 \leqslant i \leqslant n.$$

Suppose, for the sake of argument, that $t_{c_1} = \min\{t_{c_1}, \ldots, t_{c_n}\}$. Continuously increase t to t_{c_1}, so that $R(p_0, t_{c_1})$ will extend to the hyperplane $x_1 = \varphi_1(t_1)/2$. Clearly

$$x_i^{(0)} + \frac{\varphi_i(t_{c_1})}{2} \leqslant x_i^{(0)} + \frac{\varphi_i(t_{c_i})}{2} = x_i^{(0)} + c_i = \frac{\varphi_i(t_1)}{2}, \quad \text{for} \quad 1 \leqslant i \leqslant n.$$

Therefore, since p_0 is in the first quadrant of R_1, $R(p_0, t_{c_1}) \subset R_1$.

The method of translating $R(p_0, t_{c_1})$ towards $(0, 0, \ldots, 0)$ is as follows. Continuously slide $p_0 = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)})$ to $(x_1', x_2^{(0)}, \ldots, x_n^{(0)})$, where

$$x_1' = x_1^{(0)} - \left[\frac{\varphi_1(t_{c_1})}{2} - \frac{\varphi_1(t_0)}{2}\right].$$

Setting

$$x_i' = x_i^{(0)} - \left[\frac{\varphi_i(t_{c_1})}{2} - \frac{\varphi_i(t_0)}{2}\right], \quad \text{for} \quad 2 \leqslant i \leqslant n,$$

the shifting is completed by continuously sliding $(x_1', \ldots, x_{i-1}', x_i^{(0)}, x_{i+1}^{(0)}, \ldots \ldots, x_n^{(0)})$ to $(x_1', \ldots, x_{i-1}', x_i', x_{i+1}^{(0)}, \ldots, x_n^{(0)})$, that is, first slide along the x_1-axis, then the x_2-axis, etc. Let $p' = (x_1', x_2', \ldots, x_n')$. Observe that for each $p = (x_1, x_2, \ldots, x_n) \in R_0$,

$$\begin{aligned} |x_i - x_i'| &\leqslant |x_i - x_i^{(0)}| + |x_i^{(0)} - x_i'| \\ &\leqslant \frac{\varphi_i(t_0)}{2} + \left[\frac{\varphi_i(t_{c_1})}{2} - \frac{\varphi_i(t_0)}{2}\right] = \frac{\varphi_i(t_{c_1})}{2}. \end{aligned}$$

Hence $R_0 \subset R(p', t_{c_1})$. Since $t_{c_1} < t_1$, $0 \leqslant x_i' \leqslant x_i^{(0)}$, and

$$x_i^{(0)} + \frac{\varphi_i(t_{c_1})}{2} \leqslant \frac{\varphi_i(t_0)}{2}, \quad \text{for} \quad 1 \leqslant i \leqslant n,$$

we have that $R(p', t_{c_1}) \subset R_1$. Now, either the center of the translate of $R(p_0, t_{c_1})$ crosses an axis, in which case we are done, or at least two coordinates of the center of the translate remain greater than 0. If necessary, repeat the process using $R(p', t_{c_1})$. Since the center will be moved by one of n fixed positive constants, there will only be a finite number of such shifts possible before the center crosses an axis.

If p_0 is in any other quadrant of R_1, the argument follows by symmetry.

(4.5) LEMMA. *Let $1 \leqslant p < \infty$, $w \in A_p$, and*

$$\frac{1}{|R|} \int_R w(x)\,dx \leqslant \lambda.$$

Then, there exist positive constants c and β, independent of R and λ, such that

$$\int_{\{x \in R:\ w(x) > \lambda\}} w(x)\,dx \leqslant c\lambda |\{x \in R:\ w(x) > \beta\lambda\}|.$$

Let $E = \{x \in R\colon w(x) > \lambda\}$. By (2.12), for almost every $x \in E$ which lies in the interior of R there is an $R'_x \subset R$ such that $\frac{1}{|R'_x|}\int_{R'_x} w(y)\,dy > \lambda$. Thus, by the absolute continuity of the integral and Lemma (4.3), since $R'_x \subset R$, there is an $R_x \subset R$ such that $\frac{1}{|R_x|}\int_{R_x} w(y)\,dy = \lambda$. Since the R_x's cover E, by (2.1) there is a disjoint sequence, $R_1, \ldots, R_N$, and a positive constant, γ, such that

$$\frac{1}{|R_i|}\int_{R_i} w(x)\,dx = \lambda, \text{ for } 1 \leqslant i \leqslant n \quad \text{and} \quad m_w(E) < \gamma \sum_{i=1}^{N} m_w(R_i).$$

By (4.2), since $m_w(R_i) = M_{R_i}(w)|R_i| = \lambda|R_i|$,

$$m_w(E) \leqslant \frac{\gamma}{a}\lambda \sum_{i=1}^{N} |\{x \in R_i\colon w(x) > \beta\lambda\}| \leqslant \frac{\gamma}{a}\lambda|\{x \in R\colon w(x) > \beta\lambda\}|.$$

Therefore,

$$\int_{\{x\in R\colon w(x)>\lambda\}} w(x)\,dx \leqslant c'\lambda|\{x \in R\colon w(x) > \beta\lambda\}|.$$

(4.6) LEMMA. *Let $1 \leqslant p < \infty$. Given $w \in A_p$, there are positive contants c and δ, independent of R, such that*

$$\left(\frac{1}{|R|}\int_R w(x)^{1+\delta}\,dx\right)^{1/(1+\delta)} \leqslant c\left(\frac{1}{|R|}\int_R w(x)\,dx\right).$$

Before proving (4.6), a proposition is stated and proved.

(4.7) PROPOSITION. *Suppose $1 \leqslant p < \infty$ and $w \in A_p$. Define w_N by*

$$w_N(x) = \begin{cases} w(x) & \text{if} \quad w(x) < N, \\ N & \text{if} \quad w(x) \geqslant N. \end{cases}$$

Then $w_N \in A_p$, with an A_p constant dependent only on p and the A_p constant for w.

Suppose $1 < p < \infty$. Since $w_N(x) \leqslant \min(w(x), N)$,

$$\text{(4.8)} \quad \begin{aligned} &1)\ \frac{1}{|R|}\int_R w_N(x)\,dx \leqslant \frac{1}{|R|}\int_R w(x)\,dx, \\ &2)\ \frac{1}{|R|}\int_R w_N(x)\,dx \leqslant N. \end{aligned}$$

In addition,

$$\text{(4.9)} \quad \frac{1}{|R|}\int_R w_N(x)^{-1/(p-1)}\,dx = \frac{1}{|R|}\int_{\{x\in R\colon w(x)<N\}} w(x)^{-1/(p-1)}\,dx + \frac{1}{|R|}\int_{\{x\in R\colon w(x)\geqslant N\}} N^{-1(p-1)}\,dx \leqslant \frac{1}{|R|}\int_R w(x)^{-1/(p-1)}\,dx + N^{-1/(p-1)}.$$

Hence, by (4.9), we obtain

$$\text{(4.10)} \quad \begin{aligned} &\left(\frac{1}{|R|}\int_R w_N(x)\,dx\right)\left(\frac{1}{|R|}\int_R w_N(x)^{-1/(p-1)}\,dx\right)^{p-1} \\ &\leqslant \left(\frac{1}{|R|}\int_R w_N(x)\,dx\right)\left[\frac{1}{|R|}\int_R w(x)^{-1/(p-1)}\,dx + N^{-1/(p-1)}\right]^{p-1} \\ &\leqslant 2^{p-1}\left(\frac{1}{|R|}\int_R w_N(x)\,dx\right)\left[\left(\frac{1}{|R|}\int_R w(x)^{-1/(p-1)}\,dx\right)^{p-1} + N^{-1}\right]. \end{aligned}$$

The application of the two conditions of (4.8) to (4.10) yields

$$\text{(4.11)} \quad \begin{aligned} &\left(\frac{1}{|R|}\int_R w_N(x)\,dx\right)\left(\frac{1}{|R|}\int_R w_N(x)^{-1/(p-1)}\,dx\right)^{p-1} \\ &\leqslant 2^{p-1}\left[\left(\frac{1}{|R|}\int_R w(x)\,dx\right)\left(\frac{1}{|R|}\int_R w(x)^{-1/(p-1)}\,dx\right)^{p-1} + N\cdot N^{-1}\right] \\ &\leqslant 2^{p-1}[C+1]. \end{aligned}$$

Thus $w_N \in A_p$.

If $p = 1$, we need only consider two cases. If $w_N(x) < N$, then $w_N(x) = w(x)$ so that $w_N(x)^* \leqslant w(x)^* \leqslant c\cdot w(x) = c\cdot w_N(x)$. Condition 2 of (4.8) shows that the desired result is obtained if $w_N(x) = N$.

We may now prove (4.6). Assume that w is a bounded A_p function. By (4.5), there are constants c and β such that

$$\text{(4.12)} \quad \int_{M_R(w)}^{\infty} \lambda^{\delta-1}\left(\int_{\{x\in R\colon w(x)>\lambda\}} w(x)\,dx\right)d\lambda \leqslant c\int_{M_R(w)}^{\infty} \lambda^{\delta}|\{x \in R\colon w(x) > \beta\lambda\}|\,d\lambda \leqslant \frac{c'}{1+\delta}\int_R w(x)^{1+\delta}\,dx.$$

In addition,

$$(4.13)\quad \int_{M_R(w)}^{\infty} \lambda^{\delta-1}\Big(\int_{\{x\in R:\, w(x)>\lambda\}} w(x)\,dx\Big)\,d\lambda$$
$$\geqslant \int_0^{\infty} \lambda^{\delta-1}\Big(\int_{\{x\in R:\, w(x)>\lambda\}} w(x)\,dx\Big)\,dx - \int_0^{M_R(w)} \lambda^{\delta-1}|R|M_R(w)\,dx$$
$$\geqslant \frac{1}{\delta}\int_R w(x)^{1+\delta}\,dx - \frac{1}{\delta}|R|M_R(w)^{1+\delta}.$$

The combination of (4.12) and (4.13) yields

$$(4.14)\quad \Big(\frac{1}{\delta}-\frac{c'}{1+\delta}\Big)\int_R w(x)^{1+\delta}\,dx \leqslant \frac{1}{\delta}M_R(w)^{1+\delta}|R|.$$

The desired result for such a w follows by making δ so small that

$$\frac{1}{\delta}-\frac{c'}{1+\delta}>0.$$

If w is an arbitrary A_p function, define a sequence of functions $\{w_N\}$ by

$$w_N(x) = \begin{cases} w(x) & \text{if } w(x)<N,\\ N & \text{if } w(x)\geqslant N.\end{cases}$$

Then, $\{w_N\}$ is a monotone, non-decreasing sequence of bounded functions having w as its limit. The result follows by the Monotone Convergence Theorem, (4.7), and (4.14).

We can now prove (1.6). Let $w\in A_p$ for some p, $1<p<\infty$. Then, $w^{-1/(p-1)}\in A_{p'}$. By (4.6), there are positive constants c and δ such that

$$\Big(\frac{1}{|R|}\int_R \big(w(x)^{-1/(p-1)}\big)^{1+\delta}\,dx\Big)^{1/(1+\delta)} \leqslant \frac{c}{|R|}\int_R w(x)^{-1/(p-1)}\,dx.$$

Since $w\in A_p$, we obtain

$$\Big(\frac{1}{|R|}\int_R w(x)\,dx\Big)\Big(\frac{1}{|R|}\int_R w(x)^{(1+\delta)/(p-1)}\,dx\Big)^{(p-1)/(1+\delta)} \leqslant c.$$

Setting $r=\dfrac{p-1}{1+\delta}+1$, $w\in A_r$ for some $r<p$.

5. The major theorem. Theorem (1.5) can now be deduced from (3.1), (4.1), and (1.6) by using the Marcinkiewicz interpolation theorem, as shown by Muckenhoupt in [6]. Let $1<p<\infty$ and $w\in A_p$. By (1.6), there is an $\varepsilon>0$ such that $w\in A_{p-\varepsilon}$. By (3.1) and (4.1),

$$m_w\{x:\, f^*(x)>\alpha\} \leqslant c\alpha^{-q}\int_{E^n}|f(x)|^q w(x)\,dx$$

for $q=p-\varepsilon$ or $p+\varepsilon$. The result follows from the Marcinkiewicz interpolation theorem, [8], p. 111.

6. Generalizations. Let ϱ: $E^n\times E^n\to[0,\infty)$ be a translation invariant metric in E^n. Define

$$B(0,r) = \{x\in E^n:\ \varrho(0,x)\leqslant r\}$$

and suppose that the balls $B(0,r)$, $r>0$, are convex, compact bodies symmetric with respect to the coordinate hyperplanes, expand continuously to E^n as r tends to ∞, and contract continuously to $(0,\dots,0)$ as r tends to 0. Let $\mathscr{B}$ be the collection of all balls of the form $B(x,r)=x+B(0,r)$, for $r>0$ and $x\in E^n$, and suppose ϱ is so defined that for all $B(x,r)\in\mathscr{B}$, there is a b such that

$$(6.1)\quad |B(x,2r)|\leqslant b|B(x,r)|.$$

Then, it is possible to generalize all the results for one parameter rectangles to such a class of metric balls. As for rectangles, $B(r)$ or B will be used to represent $B(x,r)$ whenever it would cause no confusion.

Let $B(0,r)$ be a fixed metric ball. The non-zero coordinates of the points of intersection of $B(0,r)$ with the coordinate axes $x_1,\dots,x_n$ are

$$\{\pm a_1,\dots,\pm a_n\} = \{\pm a_1(r),\dots,\pm a_n(r)\},$$

respectively. Define a rectangle

$$R(0,r) = \{x\in E^n:\ |x_i|\leqslant a_i,\ i=1,2,\dots,n\}.$$

Such an R is a one parameter rectangle which, by symmetry and convexity, contains $B(0,r)$. Furthermore, $R(0,r)$ is contained in $B(0,nr)$. To see this, notice that each x in $R(0,r)$ is of the form $(\beta_1a_1,\beta_2a_2,\dots,\beta_na_n)$, where $-1\leqslant\beta_i\leqslant 1$ for each i. Therefore,

$$\varrho(x) = \varrho(\beta_1a_1,\dots,\beta_na_n)$$
$$\leqslant \varrho(\beta_1a_1,0,\dots,0)+\dots+\varrho(0,\dots,0,\beta_na_n)\leqslant nr$$

since $\varrho(0,\dots,0,\beta_ia_i,0,\dots,0)\leqslant r$, for $1\leqslant i\leqslant n$.

Let $R(x,r)=x+R(0,r)$, for $r>0$ and $x\in E^n$. Then, given such a collection of balls, $\mathscr{B}$, a collection of rectangles, $\mathscr{R}$, satisfying (1.1), is naturally generated such that for each $B(x,r)\in\mathscr{B}$ there is an $R(x,r)\in\mathscr{R}$ with the properties that

$$(6.2)\quad \begin{aligned} &1)\ B(x,r)\subset R(x,r)\subset B(x,nr),\\ &2)\ |R(x,r)|\leqslant|B(x,nr)|\leqslant\beta|B(x,r)|,\end{aligned}$$

where β is dependent only on the constant, b, of (6.1) and n. This second condition is a consequence of condition 1 and (6.1).

The Hardy–Littlewood maximal function with respect to $\mathscr{B}$ is defined as

$$f_{\mathscr{B}}^*(x) = \sup \frac{1}{|B|} \int_B |f(y)|\,dy \tag{6.3}$$

when the supremum is taken over all $B \in \mathscr{B}$ which contain x. As a consequence of (6.2), notice that

$$f_{\mathscr{B}}^*(x) = \sup \frac{1}{|B|} \int_B |f(y)|\,dy \leqslant \sup \frac{\beta}{|R|} \int_R |f(y)|\,dy = \beta \cdot f^*(x). \tag{6.4}$$

The A_p condition with respect to $\mathscr{B}$ is analogous to the condition for rectangles. If $1 < p < \infty$, $w \in A_p(\mathscr{B})$ if

$$\left(\frac{1}{|B|} \int_B w(x)\,dx\right)\left(\frac{1}{|B|} \int_B w(x)^{-1/(p-1)}\,dx\right)^{p-1} \leqslant c \tag{6.5}$$

for all $B \in \mathscr{B}$, where c is independent of B. A function, w, will be in $A_1(\mathscr{B})$ if

$$w_{\mathscr{B}}^*(x) \leqslant c w(x) \tag{6.6}$$

for almost every $x \in E^n$, where c is independent of x.

(6.7) LEMMA. *Let $1 \leqslant p < \infty$. $w \in A_p(\mathscr{B})$ if and only if $w \in A_p(\mathscr{R})$.*

Suppose $w \in A_p(\mathscr{R})$. If $p = 1$, the proof follows from (6.4). Therefore, let $1 < p < \infty$. For each $B \in \mathscr{B}$,

$$\left(\frac{1}{|B|} \int_B w(x)\,dx\right)\left(\frac{1}{|B|} \int_B w(x)^{-1/(p-1)}\,dx\right)^{p-1}$$
$$\leqslant \left(\frac{\beta}{|R|} \int_R w(x)\,dx\right)\left(\frac{\beta}{|R|} \int_R w(x)^{-1/(p-1)}\,dx\right)^{p-1} \leqslant \beta^p \cdot c.$$

Hence, $w \in A_p(\mathscr{B})$ for $1 \leqslant p < \infty$.

Now, suppose $w \in A_p(\mathscr{B})$. If $p = 1$, by (6.1) and (6.2)

$$w^*(x) = \sup \frac{1}{|R(r)|} \int_{R(r)} w(x)\,dx \leqslant \sup \frac{\beta}{|B(nr)|} \int_{B(nr)} w(x)\,dx = \beta \cdot w_{\mathscr{B}}^*(x).$$

Thus, $w \in A_1(\mathscr{R})$.

If $1 < p < \infty$, using (6.1), we obtain

$$\frac{1}{|B(nr)|} \int_{B(nr)} w(x)\,dx \leqslant c\left(\frac{1}{|B(nr)|} \int_{B(nr)} w(x)^{-1/(p-1)}\,dx\right)^{-(p-1)}$$

$$\leqslant c\left(\frac{1}{|B(nr)|} \int_{B(r)} w(x)^{-1/(p-1)}\,dx\right)^{-(p-1)}$$
$$\leqslant c\left(\frac{1}{\beta |B(r)|} \int_{B(r)} w(x)^{-1/(p-1)}\,dx\right)^{-(p-1)}$$
$$\leqslant c\beta^{p-1} \frac{1}{|B(r)|} \int_{B(r)} w(x)\,dx.$$

Therefore, by (6.1) and (6.2),

$$\frac{1}{|R(r)|} \int_{R(r)} w(x)\,dx \leqslant c' \frac{1}{|B(r)|} \int_{B(r)} w(x)\,dx.$$

Similarly, since $w^{-1/(p-1)} \in A_{p'}(\mathscr{B})$, where $1 < p' < \infty$,

$$\frac{1}{|R(r)|} \int_{R(r)} w(x)^{-1/(p-1)}\,dx \leqslant c'' \frac{1}{|B(r)|} \int_{B(r)} w(x)^{-1/(p-1)}\,dx.$$

Thus, $w \in A_p(\mathscr{R})$, for $1 \leqslant p < \infty$.

The results previously obtained for rectangles can be extended to balls because of (6.4) and (6.6). If $w \in A_p(\mathscr{B})$, then $w \in A_p(\mathscr{R})$ so that m_w, with respect to $\mathscr{R}$, satisfies (2.2). Therefore, for $1 < p < \infty$ and $w \in A_p$, by (1.5),

$$\int_{E^n} [f_{\mathscr{B}}^*(x)]^p w(x)\,dx \leqslant c \int_{E^n} [f^*(x)]^p w(x)\,dx \leqslant c' \int_{E^n} |f(x)|^p w(x)\,dx. \tag{6.8}$$

If $w \in A_1$, using (3.1) and (6.6),

$$m_w\{x\colon f_{\mathscr{B}}^*(x) > a\} \leqslant m_w\{x\colon cf^*(x) > a\} \leqslant c' a^{-1} \int_{E^n} |f(x)| w(x)\,dx. \tag{6.9}$$

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Self-decomposable probability measures on Banach spaces

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Abstract. Self-decomposable probability measures (laws) on a real, separable Banach space E are defined and identified as the limit laws of certain normed sums of independent, uniformly infinitesimal, E-valued random variables. It is shown that self-decomposable measures are infinitely divisible, and a characterization of such measures in terms of their Lévy–Khinchine representations is given on the spaces for which such a representation is known to exist. Finally, a representation theorem due to K. Urbanik for certain measures associated with self-decomposable probability measures on finite-dimensional spaces is generalized to separable Banach spaces.

In § 1 we introduce the notion of a self-decomposable probability measure and obtain a necessary and sufficient condition for a self-decomposable law to be stable in terms of its "component". In § 2 we first show the class of self-decomposable measures on a real, separable Banach space can be identified with the class L ([2], p. 145) on the space. It is then shown that a self-decomposable measure and its "components" are infinitely divisible. This result is of interest since it is not known whether the limit laws of uniformly infinitesimal triangular arrays of random variables with values in a separable Banach space are always infinitely divisible (see [9]). § 3 is devoted to characterizing self-decomposable probability measures on certain Orlicz sequence spaces in terms of their Lévy–Khinchine representations as given in [7]. The paper ends with the extension to the present context of the work of K. Urbanik ([13], [14]) on the representation of self-decomposable probability measures in § 4.

1. Notation and preliminaries. We shall denote by E a real separable Banach space and by R and R^+ the space of real numbers and strictly positive real numbers, respectively, with the usual topology. E^* will

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