Contents of volume LIII, number 2

STUDIA MATHEMATICA

Managing Editors: **Z. Ciesielski, W. Orlicz** (*Editor-in-Chief*), **A. Pełczyński, W. Żelazko**

The journal prints original papers in English, French, German and Russian, mainly on functional analysis, abstract methods of mathematical analysis and on the theory of probabilities. Usually 3 issues constitute a volume.

The papers submitted should be typed on one side only and they should be accompanied by abstracts, normally not exceeding 200 words. The authors are requested to send two copies, one of them being the typed, not Xerox copy. Authors are advised to retain a copy of the paper submitted for publication.

Manuscripts and the correspondence concerning editorial work should be addressed to

STUDIA MATHEMATICA
ul. Śniadeckich 8
00-950 Warszawa, Poland

Correspondence concerning exchange should be addressed to

INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
ul. Śniadeckich 8
00–950 Warszawa, Poland

The journal is available at your bookseller or at

"ARS POLONA-RUCH"
Krakowskie Przedmieście 7
00–068 Warszawa, Poland

PRINTED IN POLAND

WROCŁAWSKA DRUKARNIA NAUKOWA

Isomorphism of spaces of bounded continuous functions

by

ALAIN ETCHEBERRY (Berkeley, Calif.)*

Abstract. Milutin proved in 1952 that all Banach spaces $C(K)$, K compact metrizable and uncountable, are isomorphic to $C(\Delta)$ (Δ the Cantor set). Later, Pełczyński and Ditor extended and simplified the methods. In this paper Milutin's theorem is extended to the setting of bounded continuous functions on certain metrizable topological spaces. In particular, when X is separable, the following is proved: a) If X is polish, uncountable and not locally compact at any point, then the Banach spaces $BC(X)$ and $BC(N^\infty)$ are isomorphic; b) If X is polish, locally compact and non-compact, and such that every non-empty open subset of X is uncountable, then the Banach spaces $BC(X)$ and $BC(\Delta\times N)$ are isomorphic. Here N is the discrete space of natural numbers and $N^\infty = N\times N\times \ldots$ is the polish space of irrational numbers. The strict topology plays an important role in the proof. There are extensions to the non-separable case, as well as to other topologies. The case of the spaces $BC(X)/C_0(X)$, as well as several applications and open problems, are also considered.

The subject of this work is an extension of the isomorphism theorem of Milutin to the setting of bounded continuous functions on metrizable spaces. The methods, except for the use of the strict topology, are extensions or variants of those used by Milutin ([13] and [14]), Pełczyński [18] and Ditor [5] (see also Bade [2] and Semadeni [20]).

If X is a Hausdorff topological space and E is a Banach space, we denote by $C(X; E)$ the vector space of all continuous E-valued functions on X, and by $BC(X; E)$ the subspace of all bounded functions in $C(X; E)$. We denote by $\|f\|$ the uniform (supremum) norm of a function f in $BC(X; E)$. If S is a subset of X, we denote by $\|f\|_S$ the norm of the restriction of f to S. When E is either the reals $\boldsymbol{R}$ or the complexes $\boldsymbol{C}$, we write generically $C(X)$ and $BC(X)$. All our results will be valid in both the real and complex case.

Remember that a *polish space* is a topological space homeomorphic to a separable complete metric space. The symbol Δ will denote the Cantor set $\{0, 1\}^N$ with the product topology; the symbol N denotes the discrete space of all natural numbers, and N^∞ the cartesian product of countably many copies of N, with the product topology. The space N^∞ is known

* Author partially supported by NSF Grant GP-29012.

to be homeomorphic to the subspace of irrational numbers in $\boldsymbol{R}$ (see Section 2 for a discussion of this point). The spaces Δ, $\Delta\times N$ and N^∞ are polish spaces; the first one is compact and the second one locally compact.

We state and prove first the separable version of the main result of this paper, and leave the non-separable case for Section 5. We state the theorem in a form slightly more general than the given in the abstract. To save space we leave out the (known) compact case, although all cases could be treated in parallel.

THEOREM A. *Let X be a polish space.*

(a) *Let X contain an uncountable closed subset which is not locally compact at any point. Then the Banach spaces $BC(X)$ and $BC(N^\infty)$ are isomorphic.*

(b) *Let X be locally compact and contain a closed non-compact subset, every non-empty (relatively) open subset of which has at least two points (i.e., is uncountable). Then the Banach spaces $BC(X)$ and $BC(\Delta\times N)$ are isomorphic.*

Statement (b) implies that the Banach spaces $BC(\Delta\times N)$, $BC(\boldsymbol{R}^n)$ $(n=1,2,\ldots)$, $BC(O)$ (O a non-empty open subset of $\boldsymbol{R}^n$) are isomorphic. Statement (a) implies that the Banach spaces $BC(N^\infty)$, $BC(l^2)$, $BC(\boldsymbol{R}^\infty)$ ($\boldsymbol{R}^\infty$ the countable product of lines), $BC(O)$ (O a non-empty open subset of a separable, infinite dimensional Banach space or Fréchet space) are isomorphic.

It will be shown in Section 5 that in the locally compact case the isomorphisms can be chosen so that they respect functions vanishing at infinity. This implies an isomorphism result for the corresponding quotient spaces $BC(X)/C_0(X)$, i.e., for the spaces $C(\beta X-X)$ (βX being the Stone–Čech compactification of X).

The proof of the theorem uses the ideas in Ditor's approach to the Milutin theorem. However, in the case at hand it is slightly more efficient to use, for the extended "Milutin lemma", a construction like that in Kuratowski [10], vol. I, p. 437, rather than the more elegant (but equivalent) inverse limits used by Ditor.

The proof is based on Theorems B and C below and on a version of the decomposition technique of Borsuk and Pełczyński.

THEOREM B. *With X as in Theorem A. The Banach space $BC(X)$ contains a closed subspace isometric (i.e., isometrically isomorphic) to the space $BC(Y)$, Y being N^∞ in case* (a) *and $\Delta\times N$ case* (b). *This subspace is the range of a contractive projection of norm one from $BC(X)$.*

This will be proved in Section 2, and is an immediate application of the Borsuk–Dugundji linear extension theorem, in the form proved by Michael [11]; as soon as we prove that under the stated conditions X contains a closed subset homeomorphic to N^∞ (case (a)) or $\Delta\times N$ (case (b)).

For the statement of Theorem C we need some terminology. Let X and Y be Hausdorff topological spaces and $\pi\colon X\to Y$ a continuous surjection. We define the linear injection of norm one $\pi^0\colon BC(Y)\to BC(X)$ by $\pi^0(g)=g\pi$, for every g in $BC(Y)$. A linear map $L\colon BC(X)\to BC(Y)$ is called an *averaging operator* for π (or π^0), if $L\pi^0=\mathrm{Id}_{BC(Y)}$. It follows immediately that L is surjective and that $L(1_X)=1_Y$. In general, for a subset S of a set T, we denote by 1_S the characteristic function of the set S. The map L is called *regular* if, in addition, $\|L\|\leqslant 1$ (so $\|L\|=1$, and L is a positive operator).

THEOREM C (Extended Milutin lemma). *Let X be a polish space. Then there is a zero-dimensional polish space X_∞, in fact a closed subspace of N^∞, with the following properties: There is a continuous surjection $\Pi\colon X_\infty\to X$ admitting a norm continuous averaging operator. Moreover, in case* (a), *X_∞ can be taken to be N^∞ and, in case* (b), *to be $\Delta\times N$.*

The importance of this theorem lies in the fact that it implies that $BC(N^\infty)$ (case (a)) or $BC(\Delta\times N)$ (case (b)) is isometric to a complemented subspace of $BC(X)$. Indeed, $L\Pi^0$ is a projection onto the subspace $\Pi^0BC(Y)$ (with $Y=N^\infty$ or $\Delta\times N$) of $BC(X)$, as

$$(L\Pi^0)(L\Pi^0)=L(\Pi^0L)\Pi^0=L\Pi^0.$$

Theorem C is the heart of the whole argument and is based on an extension of the Pełczyński "localization lemma" [17]. This theorem will be proved in Section 4. In the proof we will need, in case (a), a Stone–Weierstrass-type theorem for the strict topology. This topology will be discussed in Section 3.

In Section 5 we treat several extensions of the main theorem: (i) The non-separable analogues; (ii) Isomorphism theorems for the strict, compact-open and uniform topologies (the last two on $C(X)$); (iii) The case of quotient spaces.

In Section 6 we give some applications of the main theorem: (i) To the contractibility of the corresponding linear groups; (ii) To the existence of fixed bounds on the norms of the isomorphisms; (iii) To an extension of a theorem of Pełczyński on complemented subspaces of $C(K)$.

In Section 7 we state some open problems. In particular, it is left open whether the Banach spaces $BC(\boldsymbol{R})$ and $BC(l^2)$ (say) are isomorphic (l^2 being a separable Hilbert space).

The present work is essentially the author's Ph.D. thesis, written at the University of California (Berkeley) under the supervision of Professor William G. Bade. The author wishes to state his deepest appreciation to Professor Bade for his advice.

1. The proof of Theorem A from Theorems B and C. If E and F are Banach spaces, we put on the product $E\times F$ the product topology and the norm defined by $\|(x,y)\| = \max\{\|x\|, \|y\|\}$. The symbol "$\sim$" will stand for isomorphism and "$\simeq$" for isometry (i.e., isometric isomorphism).

The following relations are easy to verify.

(1) *If* $E\sim F$, *then* $BC(X;E)\sim BC(X;F)$.

(2) $$BC(X;E\times F)\simeq BC(X;E)\times BC(X;F).$$

For the sake of definitness let us deal with case (a) in the theorem. In case (b) one simply replaces N^∞ by $N\times\Delta$.

The compact case, which will not be repeated here, is very similar, with N^* (the one point compactification of N) replacing N in the arguments (see references).

We have

(3) $$BC(N;E)\simeq E\times BC(N;E).$$

Observing that $N\times N^\infty$ is homeomorphic to N^∞, we also have

(4) $$BC(N^\infty)\simeq BC(N\times N^\infty)\simeq BC\big(N;BC(N^\infty)\big),$$

and thus

(5) $$BC(N^\infty)\simeq BC(N^\infty)\times BC\big(N;BC(N^\infty)\big).$$

By Theorem C, the space $BC(X)$ is isomorphic to a complemented subspace of $BC(N^\infty)$. Thus we have

(6) $$BC(N^\infty)\sim BC(X)\times U,$$

for some closed subspace U of $BC(N^\infty)$.

On the other hand, by Theorem B, the space $BC(N^\infty)$ is isometric to a complemented subspace of $BC(X)$, that is,

(7) $$BC(X)\sim V\times BC(N^\infty).$$

The following decomposition argument then shows that $BC(X)\sim BC(N^\infty)$, and achieves the proof of the theorem.

$$\begin{aligned}
BC(X) &\sim V\times BC(N^\infty) && \text{(by 7)}\\
&\simeq V\times BC(N^\infty)\times BC\big(N;BC(N^\infty)\big) && \text{(by 5)}\\
&\sim BC(X)\times BC\big(N;BC(N^\infty)\big) && \text{(by 7)}\\
&\sim BC(X)\times BC\big(N;BC(X)\times U\big) && \text{(by 6)}\\
&\simeq BC(X)\times BC\big(N;BC(X)\big)\times BC(N;U) && \text{(by 2)}\\
&\simeq BC\big(N;BC(X)\big)\times BC(N;U) && \text{(by 3)}\\
&\simeq BC\big(N;BC(X)\times U\big) && \text{(by 2)}\\
&\sim BC\big(N;BC(N^\infty)\big) && \text{(by 6)}\\
&\simeq BC(N^\infty). && \text{(by 4).}
\end{aligned}$$

2. Linear extensions and the proof of Theorem B. We quote the statement of the linear extension theorem of Borsuk and Dugundji, in the form proved by Michael [11].

2.1. THEOREM. *Let X be a metrizable space and Y a closed subset of X. Then there is a linear map $T\colon C(Y)\to C(X)$ such that:*

(a) *$T(f)$, restricted to Y, is f,*

(b) $T(1_Y) = 1_X$.

(c) *the range of $T(f)$ is contained in the closed convex hull of the range of f.*

The operator T is continuous if both spaces have the compact-open topology or the topology of uniform convergence.

The map T, restricted to $BC(Y)$, is a linear map of norm one.

The *compact-open* topology on $C(X)$ is the locally convex topology induced by the seminorms $p_K(f) = \|f\|_K$, K a compact subset of X.

The space $TBC(Y)$ is a subspace of $BC(X)$ isometric to $BC(Y)$. The space $TBC(Y)$ is, in fact, complemented in $BC(X)$. The map $P = TR$, $R\colon BC(X)\to BC(Y)$, R the restriction map, is a projection of norm one:

$$P^2 = (TR)(TR) = T(RT)R = TR = P$$

since, clearly, $RT = \mathrm{Id}_{BC(Y)}$.

To obtain Theorem B we have only to prove then that, under the hypotheses of the theorem, X contains a closed subset homeomorphic to N^∞ or $\Delta\times N$.

2.2. PROPOSITION. *Let X be an uncountable polish space which is not locally compact at any point. Then X contains a closed subset homeomorphic to N^∞.*

Proof. Fix on X a complete metric. Since X is not compact, there is a countable discrete subset $\{x_1, x_2, \ldots\}$ of X. We can then find mutually disjoint closed balls B_n $(n = 1, 2, \ldots)$, with the following properties: B_n has center x_n; each B_n has diameter at most one; the diameters of the balls B_n tend to zero as n tends to infinity. With these properties, the set $\bigcup\limits_1^\infty B_n$ is closed.

By hypothesis none of the B_n is compact so, for each n, we can find a countable discrete subset of X which we can assume (by shrinking B_n a bit) to be contained in the interior of B_n. We then construct mutually disjoint closed balls B_{nk} $(k = 1, 2, \ldots)$ with the following properties: B_{nk} has center x_{nk}; B_{nk} has diameter at most $\frac{1}{2}$; for fixed n, the diameter of B_{nk} tends to zero as k tends to infinity.

Proceeding by induction we get at the nth stage, for each fixed choice of indices $i(1), \ldots, i(n-1)$, the following: (a) a discrete countable set $\{x_{i(1)\ldots i(n-1),1},\ x_{i(1)\ldots i(n-1),2}, \ldots\}$ in X, contained in the interior of the closed balls $B_{i(1)\dots i(n-1)}$; (b) mutually disjoint closed balls $B_{i(1)\dots i(n-1),k}$ $(k = 1, 2, \dots)$ with the following properties: $B_{i(1)\dots i(n-1),k}$ has center $x_{i(1)\dots i(n-1),k}$; each of them is contained in the interior of $B_{i(1)\dots i(n-1)}$; each of these balls has diameter at most $1/n$; the diameter of $B_{i(1)\dots i(n-1),k}$ tends to zero as k tends to infinity.

We now define, for all n,

$$B(n) = \bigcup\{B_{i(1)\dots i(n)};\ i(1), \dots, i(n) \in N\}.$$

This is, by construction, a closed set in X. We assert that the closed set

$$S = \bigcap_1^\infty B(n)$$

is homeomorphic to N^∞.

Consider the map $f\colon N^\infty \to S$ defined by

$$f\big(i(1), i(2), \dots\big) = \bigcap\{B_{i(1)\dots i(n)}\colon n = 1, 2, \dots\}.$$

The above intersection reduces, by construction, to a single point, so f is well defined. It is also clearly a bijection. Observe next that $f(N_{i(1)\dots(n)}) \subset B_{i(1)\dots i(n)}$ and that the diameter of $B_{i(1)\dots i(n)}$ is at most $1/n$; this shows that f is continuous. Finally, f is also an open map (onto S), because

$$f(N_{i(1)\dots i(n)}) = \operatorname{Int} B_{i(1)\dots i(n)} \cap S,$$

a relatively open set. This completes the proof of the proposition.

Remarks. We have used in the proof the notation $N_{i(1)\dots i(n)}$. These sets, defined as $\{(n_1, n_2, \dots) \in N\colon n_1 = i(1), \dots, n_n = i(n)\}$, as n and the $i(k)$ range over N, form the usual base of open and closed sets for the zero-dimensional polish space N^∞. These sets have diameter $1/(n+1)$ for the complete compatible metric defined as follows: For $m = (m_1, m_2, \dots)$ and $n = (n_1, n_2, \dots)$ in N^∞, $d(m, m) = 0$ and $d(m, n) = 1/n$ if n is the first index i such that $n_i = m_i$.

In the locally compact case we have to use the following result.

2.3. Proposition. *Let X be a polish locally compact and non-compact space. If every non-empty open set in X has at least two points, then X contains a closed subset homeomorphic to $\Delta \times N$.*

Proof. Since X is not compact, it contains a countable discrete subset $\{x_1, x_2, \dots\}$. By discreteness we can find mutually disjoint compact balls V_n centered at these points, such that their diameters tend to zero when n tends to infinity. In every one of these balls V_n we can find a closed subset Δ_n homeomorphic to the Cantor set. This is possible because each V_n is uncountable. By construction the set $\bigcup_1^\infty V_n$ is closed, and so is $\bigcup_1^\infty \Delta_n$. This last set is homeomorphic to $\Delta \times N$, ending the proof.

Remark. A subset of a topological space is called *discrete* if every point of the space has a neighborhood intersecting the subset in at most one point. Such a subset is necessarily closed.

There is a classical characterization of N^∞, similar to that of the Cantor set. This has been extended in the following way by A. H. Stone in [22], p. 6. Let us write W^∞ for a countable product of copies of a discrete space W of cardinality w, with the product topology. This is a metrizable topologically complete zero-dimensional space of weight w. A complete metric and a base of open and closed sets can be defined similarly to N^∞. (The weight of a topological space is the least cardinality of an open base.) Stone proved that a metrizable, zero-dimensional absolute G_δ of weight w, is homeomorphic to W^∞ if every non-empty open subset contains a discrete subset of cardinality w. One can replace the condition of being an absolute G_δ by the equivalent one of having a complete metric (i.e., topological completeness). The space N^∞ is then the particular case of countable weight with non-empty open subsets containing discrete sequences. Observe also that it would be easy to give similar characterization of $\Delta \times N$ or $\Delta \times W$.

3. The strict topology. In this section we review without proofs some results on the strict topology on $BC(X)$ that will be needed in the sequel. This topology, studied initially by Buck and others in the locally compact case, has recently been extended to completely regular or more general topological spaces by several authors; it is closely related to the compact-open topology (the topology of uniform convergence on compact sets). We mention only: Cooper [4]; Fremlin, Garling and Haydon [7]; Hoffmann–Jørgensen [9]; Sentilles [21]; and Summers [23], and refer to them for proofs an further papers on the subject. In particular, Sentilles [21] has the best overview of the field. The topology we define below is actually called the *substrict topology* by Sentilles. However, since we are only interested in the case of metrizable X, where this topology coincides with the topology Sentilles calls *strict*, we will keep the name.

Let X be a completely regular Hausdorff space. We define the *strict topology* β on $BC(X)$ in three equivalent ways (see [21] and references there). We will write $BC(X)_\beta$ for the space $BC(X)$ with this locally convex Hausdorff topology.

(a) A base of neighborhoods at zero for β is given by the following family of absolutely convex absorbent subsets U of $BC(X)$. For every positive integer n there is a compact-open neighborhhod V_n of zero such that $U \cap nB \supset V_n \cap nB$, B being the closed unit ball of $BC(X)$. This is the finest locally convex (in fact, the finest linear) topology on $BC(X)$ which coincides with the compact-open topology on the norm bounded subsets of $BC(X)$.

(b) The mixed topology (in the sense of Wiweger, see [21] for references) $\gamma(\mathrm{CO}, \|\cdot\|)$ (CO the *compact-open topology*). A base of neighborhoods at zero is given as follows. For every sequence (U_n^*) of CO-neighborhoods of zero, let

$$U^\gamma = U(U_1^*, \ldots, U_n^*) = \bigcup_{n=1}^{\infty} (U_1^* \cap B + \ldots + U_n^* \cap nB).$$

(c) A base of neighborhoods at zero for β is the following. For any sequence (K_n) of compact subsets of X and any sequence $0 < a_n \to \infty$ of reals, let

$$W\big((K_n), (a_n)\big) = \bigcup_{n=1}^{\infty} \{f\colon \|f\|_{K_n} \leqslant a_n\}.$$

Equivalently, consider the seminorms

$$p(f) = \sup\{b_n \|f\|_{K_n} : n \geqslant 1\},$$

where $0 < b_n \to 0$ (e.g., $b_n = 1/a_n$).

For later purposes we need to extend the definition of the strict topology to the vector valued case. Let E be a vector space provided with a Banach space norm $\|\cdot\|$ and a second (weaker) locally convex Hausdorff topology τ. Let (a') be the topology defined on $BC(X; E)$ exactly as in (a), but replacing the CO topology by the τ-CO topology, that is, by the topology of τ-uniform convergence on compact subsets of X. In other words, in this case V_n is a set of the form

$$\{f\colon f(x) \in W, \text{ for every } x \in K\},$$

with K compact in X and W a τ-neighborhood of zero. Similarly, we define (b'), making U_n^* an absolutely convex τ-neighborhood of zero. The proof of the equality of these two topologies is exactly as before (see [21] for details and references). We do not need, for our purposes, to have an analogue of (c). We designate the space $BC(X; E)$ with the strict topology by $BC(X; E)_\beta$.

It can be shown that the strict topology on $BC(X)$ has the same bounded sets as the norm topology ([9] or [21]); that the strict topology coincides, in the locally compact case, with the topology originally defined by Buck (e.g., [4]); and that the space dual to $BC(X)_\beta$ is a Banach space and can be identified with the space $M(X)$ of all regular real or complex valued measures defined on the Borel sets of X (total variation norm) (see [7] or [9]). We will specifically need the following facts about this topology.

3.1. THEOREM. *A linear map T from $BC(X)_\beta$ into a locally convex space F is strict continuous iff T, restricted to each nB (n a positive integer, B the unit ball of $BC(X)$), is CO-continuous. Exactly the same thing is true for $BC(X; E)$, with CO replaced by τ-CO.*

See [4], the proof in the vector valued case is exactly the same.

3.2. THEOREM. *Let A be a subalgebra of $BC(X; \boldsymbol{R})$ which separates the points of X and such that, for every x in X, there is a function f in A with $f(x) \neq 0$. Then A is strict dense in $BC(X; \boldsymbol{R})$.*

This is a theorem of Stone–Weierstrass type. The simplest proof (using only the definition (c) of β) is that in [9]. If A is a self-adjoint subalgebra of $BC(X; \boldsymbol{C})$ with the above properties then, of course, A is also strict dense in $BC(X; \boldsymbol{C})$. As in the classical norm case we have as a consequence the following result.

3.3. COROLLARY. *If X is separable and metrizable then $BC(X)$ is separable in both the* CO *and strict topologies.*

See [23] for a more general result and related material. The space $BC(X)$ need not, of course, be separable in the norm topology.

3.4. THEOREM. *The locally convex space $BC(X)_\beta$ is complete iff X is a k_f-space. In particular, this is the case if X is metrizable.*

Recall that a topological space X is called a *k_f-space* if every bounded function, continuous on each compact subset of X, must be continuous on X. Among the k_f-spaces are the first countable topological spaces (e.g., Dugundji [6], p. 248), in particular the metrizable ones. For the proof see [7], [9], or [21].

4. Averaging operators and Milutin's lemma. In this section we extend Milutin's lemma to the setting of polish spaces and bounded continuous functions on them. We begin with an extension of Pełczyński's localization lemma [17].

4.1. PROPOSITION (Localization lemma). *Let X and Y be Hausdorff topological spaces, with Y completely regular, and (Y_α) a locally finite open covering of Y. Let $\pi\colon X \to Y$ be a continuous surjection, X_α a closed subset of X such that $\pi(X_\alpha) = Y_\alpha^-$, and π_α the restriction of π to X_α. Assume that for each α we have a regular averaging operator $L_\alpha\colon BC(X_\alpha) \to BC(Y_\alpha^-)$ for π_α. Then there is a regular averaging operator $L\colon BC(X) \to BC(Y)$ for π. Moreover, if the L_α are also assumed to be strictly continuous, L is then strictly continuous.*

Proof. Let (λ_α) be a partition of unity subordinated to the locally finite open covering (Y_α). Since λ_α is zero off Y_α, defining the function $\lambda_\alpha L_\alpha(fX_\alpha)$ to be zero off Y_α makes it continuous on Y.

Define, for f in $BC(X)$,

$$L(f) = \sum_\alpha \lambda_\alpha L_\alpha(f|X_\alpha).$$

Then $L(f)$ is in $BC(Y)$ and $L\pi^0$ is the identity on $BC(Y)$. This last statement follows from the following computation:

$$(L\pi^0)(g) = \sum \lambda_\alpha L_\alpha\big(\pi^0(g)\,|\,X_\alpha\big) = \sum \lambda_\alpha L_\alpha\big(\pi^0(g\,|\,Y_\alpha^-)\big) = \Big(\sum \lambda_\alpha (g\,|\,Y_\alpha^-)\Big) = g.$$

The following computation shows that L is regular if all the L_α are,

$$\begin{aligned}\|L(f)\| &= \sup_{y\in Y}\Big|\sum \lambda_\alpha(y)L_\alpha(f|X_\alpha)(y)\Big|\\ &\leqslant \sup_{y\in Y}\sum \lambda_\alpha(y)\,|L_\alpha(f|X_\alpha)(y)|\\ &\leqslant \sup_{y\in Y}\Big(\sum \lambda_\alpha(y)\Big)\,\|f\| = \|f\|.\end{aligned}$$

From the regularity of L we prove its strict continuity by showing (3.1) that, for every net (f_β) in $BC(X)$ with $\|f_\beta\|\leqslant 1$ such that f_β converges to zero uniformly on compact sets of X, we have that $L(f_\beta)$ converges to zero uniformly on the compact sets of Y. Indeed, if K is a compact set in Y, only finitely many of the Y_α, say $Y_1, \ldots, Y_{n(K)}$, can intersect K. Then, for every y in K, we have

$$(Lf_\beta)(y) = \sum_{i=1}^{n(K)} \lambda_i(y)L_i(f_\beta|X_i)(y),$$

and the uniform convergence of this net on K follows from that of each $L_i(f_\beta|X_i)$. This ends the proof of the proposition.

Remarks. Observe that, if $\pi\colon X\to Y$ is a continuous surjection, the map π^0 has norm one, in particular it is norm continuous and its range is norm closed. It is also strictly continuous: By (3.1) it suffices to check that, if (f_α) is a net in $BC(Y)$ with $\|f_\alpha\|\leqslant 1$ and f_α converging to zero uniformly on compact sets in Y, then $\pi^0 f_\alpha$ converges to zero uniformly on compact sets in X (since also $\|\pi^0 f_\alpha\|\leqslant 1$). Indeed, if K is a compact set in X, $\pi(K)$ is compact in Y and, since f_α converges to zero uniformly on $\pi(K)$, we have that $\pi^0 f_\alpha = f_\alpha\pi$ converges to zero uniformly on K. Also the image of π^0 is strictly closed because strict convergence of f_α implies pointwise convergence of f_α, and consequently, if each f_α is constant on each fiber $\pi^{-1}(y)$, so is the limit function. If L is a strictly continuous averaging operator for π, then $\pi^0 BC(Y)$ is a complemented subspace of $BC(X)_\beta$. The map $P = \pi^0 L$ being a strictly continuous projection from $BC(X)$ onto $\pi^0 BC(Y)$.

We are now in a position to prove Theorem C, and thus complete the proof of Theorem A.

Proof of Theorem C. The proof of this theorem will be divided into several parts.

(a) *Construction of the space* X_∞. Fix a complete metric on X. Then there is a family $\{A'_\alpha\colon \alpha\in N\}$ of (possibly empty) open subsets of X of diameter at most one, forming a locally finite covering of X. Let A_α be the closure of A'_α in X. Define a new topological space X_1 to be the disjoint union (topological sum) of the A_α.

Cover each A_α in the same way by a family $\{A'_{\alpha\beta}\colon \beta\in N\}$ of open subsets of A_α of diameter at most $\frac{1}{2}$, forming a locally finite covering of A_α. We think of the $A'_{\alpha\beta}$ as being open sets in X_1; a suitable complete metric is obtained on X_1 by defining $d(x, y)$ to be the distance induced from X if x and y are in the same A_α, and equal to one otherwise. Let $A_{\alpha\beta}$ be the closure of $A'_{\alpha\beta}$ (in A_α or X_1).

Proceeding by induction, we define at the nth stage closed subsets $A_{\alpha(1)\ldots\alpha(n)}$ of $A_{\alpha(1)\ldots\alpha(n-1)}$, of diameter at most $1/n$, these sets being the closures of the sets of a locally finite open covering of $A_{\alpha(1)\ldots\alpha(n-1)}$. Then define X_n to be the disjoint union of the $A_{\alpha(1)\ldots\alpha(n)}$, with n fixed and $\alpha(1), \ldots, \alpha(n)$ in N.

Observe now that for every sequence of elements of N, that is, for every element $\alpha = (\alpha(1), \alpha(2), \ldots)$ of N^∞, we have that

$$A_{\alpha(1)}\supset A_{\alpha(1)\alpha(2)}\supset A_{\alpha(1)\alpha(2)\alpha(3)}\supset \ldots,$$

and that the diameters of these tend to zero. By completeness of X, whenever all the sets in one such chain are non-empty, the corresponding element α of N^∞ defines a unique point of X which we denote by $\pi(\alpha)$, namely, $\pi(\alpha)$ is the point $\bigcap_{n=1}^{\infty} A_{\alpha(1)\ldots\alpha(n)}$. Similarly, for fixed $\beta(1), \ldots, \beta(n)$ in N and for any $\alpha = (\beta(1), \ldots, \beta(n), \alpha(n+1), \ldots)$ in N^∞ such that all the corresponding sets are non-empty, we obtain a unique element of X_n (in fact, of $A_{\beta(1)\ldots\beta(n)}$) which we denote by $\pi_n(\alpha)$.

Let X_∞ be the subset of N^∞ for which all the sets $A_{\alpha(1)\ldots\alpha(n)}$ are non-empty. The above construction then gives functions $\pi\colon X_\infty\to X$ and $\pi_n\colon X_\infty\to X_n$; these functions are clearly surjective. We also have

$$(*)\qquad (X_\infty\cap N_{\beta(1)\ldots\beta(n)})\subset A_{\beta(1)\ldots\beta(n)},$$

for every n (in fact we have equality here). Indeed, if $\alpha = (\alpha(1), \ldots)$ is in X_∞ then $\pi(\alpha)$ is in $A_{\alpha(1)\ldots\alpha(n)}$ for every n; if also α is in $N_{\beta(1)\ldots\beta(n)}$ then $\alpha(1) = \beta(1), \ldots, \beta(n) = \beta(n)$ and $\pi(\alpha)$ is in $A_{\beta(1)\ldots\beta(n)}$.

(b) *The set X_∞ is closed in N^∞, in particular zero-dimensional.* If α is in $N^\infty - X_\infty$, then $A_{\alpha(1)\ldots\alpha(n)}$ is empty for some n. Then by $(*)$, $\pi(X_\infty\cap N_{\alpha(1)\ldots\alpha(n)})$ is also empty, hence $X_\infty\cap N_{\alpha(1)\ldots\alpha(n)}$ must be empty. Since each one of the sets $N_{\alpha(1)\ldots\alpha(n)}$ is open and contains α, every point of $N^\infty - X_\infty$ belongs to some open set disjoint from X_∞, thus X_∞ is closed in N^∞. It is then automatically zero-dimensional.

(c) *The maps $\pi\colon X_\infty\to X$ and $\pi_n\colon X_\infty\to X_n$ are continuous.* Let α be in X_∞ and fix a positive ε. There is an integer n such that the diameter of $A_{\alpha(1)\ldots\alpha(n)}$ is less than ε. Then, by $(*)$, the diameters of $\pi(X_\infty\cap N_{\alpha(1)\ldots\alpha(n)})$ must also be less than ε. Since $N_{\alpha(1)\ldots\alpha(n)}$ is a neighborhood of α, it follows that π is continuous at α. The case of π_n is similar.

(d) *If K' is a compact set in X, so is $\pi^{-1}(K')$ in X_∞. Similarly for π_n.* In particular, the fibers $\pi^{-1}(x)$ of π are compact. Let K' be a compact set in X and $K = \pi^{-1}(K')$. There is a natural continuous surjection π_0^1: $X_1 \to X$ defined through the covering. There are similar maps π_{n-1}^n: $X_n \to X_{n-1}$ and, by composition, π_m^n: $X_n \to X_m$ $(n \geqslant m)$.

Now $(\pi_0^1)^{-1}(K')$ can intersect only finitely many of the A_α (because K' is compact and the A_α form a locally finite covering), say for α in I_1, a finite subset of N. Similarly,

$$(\pi_0^2)^{-1}(K') = (\pi_1^2)^{-1}[(\pi_0^1)^{-1}(K')]$$

is a compact subset of X_2. Thus, this set can intersect only finitely many of the sets $A_{\alpha\beta}$ for α in I_1, say only for those β in I_2, I_2 a finite subset of N. Proceeding by induction, we obtain for each n a finite set I_n of elements of N, with properties similar to those described for I_1 and I_2. The set $K = \pi^{-1}(K')$ is a closed subset of X_∞, and the fact that it is compact follows from the fact that K is a subset of the compact set $\prod_{n=1}^{\infty} I_n$ in N^∞.

(e) *The map π^0 from $BC(X)$ into $BC(X_\infty)$ is a strict isomorphism onto its range. Similarly for $(\pi_n)^0$.* We already know that π^0 is strict continuous and has strict closed range. It remains to be proved that $(\pi^0)^{-1}$ is strict continuous. Since $(\pi^0)^{-1}$ is an isometry (from the range of π^0), it suffices to prove that if a net $\pi^0 f_a$ converges to zero uniformly on compact sets, the same thing is true for the net f_a in $BC(X)$. This is now clear because of the previously proved fact that, if K' is a compact set in X, then $\pi^{-1}(K')$ is a compact set in X_∞.

(f) *The map π: $X_\infty \to X$ admits an averaging operator. This operator is both strict and norm continuous.* The localization lemma, applied to the functions π_{n-1}^n; $X_n \to X_{n-1}$ and the sets $A_{\alpha(1)\ldots\alpha(n)}$ gives regular averaging operators L_{n-1}^n: $BC(X_n) \to BC(X_{n-1})$. The hypotheses of the lemma are trivially verified. Then, by composition, one obtains regular averaging operators

$$L_n\colon BC(X_n) \to BC(X), \quad \text{and} \quad L_n^m\colon BC(X_m) \to BC(X_n) \quad (m \geqslant n).$$

Define a subset M of $BC(X_\infty)$ by

$$M = \bigcup_1^\infty (\pi_n)^0 BC(X_n).$$

We construct a linear map L: $M \to BC(X)$ as follows: if f is in M then f is in some $(\pi_n)^0 BC(X_n)$ and we write

$$L(f) = L_n\big[\big((\pi_n)^0\big)^{-1}(f)\big].$$

It is clear that L is linear as soon as it is well defined. That it is well defined follows from the following self-explanatory computation:

$$\begin{aligned} L_n\big[\big((\pi_n)^0\big)^{-1}(f)\big] &= L_m L_m^n (\pi_m^n)^0 \big((\pi_m^n)^0\big)^{-1} \big((\pi_m^n)^0\big)^{-1} f \\ &= L_m \big((\pi_m^n)^0\big)^{-1} \big((\pi_n)^0\big)^{-1} f = L_m \big((\pi_m)^0\big)^{-1} f, \end{aligned}$$

where $n > m$ and $L_n = L_m L_m^n$. We have used here the fact that

$$\big((\pi_m)^0\big)^{-1} = \big((\pi_m^n)^0\big)^{-1} \big((\pi_n)^0\big)^{-1}$$

and that $L_m^n (\pi_m^n)^0$ is the identity on $BC(X_m)$.

By the localization lemma, all the maps L_m^n and L_n are strict continuous and have norm one. The same thing is then true for L, by definition. Also $L\pi^0$ is the identity map on $BC(X)$.

We would like to extend now L from M to $BC(X_\infty)$. The subset M of $BC(X_\infty)$ is a self-adjoint subalgebra which contains the constants and separates the points of X: if α and β are two different points in X_∞, there is a first n such that $\alpha(n) \neq \beta(n)$. Consider a function f in $BC(X_n)$ which is equal to one on $A_{\alpha(1)\ldots\alpha(n)}$ and equal to zero on $A_{\alpha(1)\ldots\alpha(n-1)\beta(n)}$. Then the function $(\pi_n)^0 f$ is equal to one on α and equal to on zero on β.

Now we are faced with some problems in extending L to $BC(X_\infty)$ because the Stone–Weierstrass theorem is not valid for the norm topology on $BC(X_\infty)$ (M may not be norm dense in $BC(X_\infty)$, see the example after the end of the proof). In any case, we can apply the Stone–Weierstrass theorem (3.2) for the strict topology, to conclude that M is strict dense in $BC(X_\infty)$. Since the strict topology is complete (3.4), L has then a unique strictly continuous linear extension to $BC(X_\infty)$; we also denote this map by L. By the strict continuity of all three maps in the relation $L\pi^0 = \mathrm{Id}_{BC(X)}$ this relation also holds for the extended L. There is no reason to expect that L is an operator of norm one (i.e., regular). However, the extended map turns out to be norm continuous. This last point is a consequence of the following very general fact about linear operators.

LEMMA. *Let E and F be Banach (or Fréchet) spaces; let α and β be weaker Hausdorff topologies on E and F, respectively. If a linear map L: $E \to F$ is continuous when E (resp. F) has the topology α (resp. β), then L is also norm continuous (continuous in the metric topologies in the Fréchet situation).*

Proof. Since β is weaker than the norm topology on E, L is also norm-to-β continuous. To prove that L is norm continuous, it is enough to verify that the graph of L is closed in the (product) norm topology of $E \times F$. Assume that (x_n) is a sequence in E, converging to zero in norm and such that Lx_n converges in norm to some element y in F. Then also x_n converges to zero for the topology α and, by the α-to-β continuity of L, Lx_n converges to both 0 and y. Since β is a Hausdorff topology, y must be equal to zero. This completes the proof of the lemma.

(g) *In case* (a) *in the theorem* X_∞ *can be taken to be* N^∞. We are not claiming that X_∞ is N^∞ or even homeomorphic to it, due to the possible presence of isolated points in X_∞. However, if we replace X_∞ by the topological space $X_\infty \times N^\infty$, this new space satisfies the hypotheses of the previously-mentioned characterization of N^∞; that is, $X_\infty \times N^\infty$ is homeomorphic to N^∞. Finally, it is enough to define an averaging operator from $BC(X_\infty \times N^\infty)$ onto $BC(X_\infty)$, since then we can obtain the desired averaging operator by composition.

Let $p\colon X_\infty \times N^\infty \to X_\infty$ be the natural projection onto the first factor. Let $s\colon X_\infty \to X_\infty \times N^\infty$ be any continuous section for p; then it is immediate that $s^0\colon BC(X_\infty \times N^\infty) \to BC(X_\infty)$ is a (regular) averaging operator for p. (This argument is adapted from Ditor [5].)

Observe that this argument can be avoided, since all we need at the end is the fact that the space $BC(X)$ is isomorphic to a complemented subspace of $BC(N^\infty)$. Indeed, the theorem proves that $BC(X_\infty)$ is isomorphic to a complemented subspace of $BC(X_\infty)$, and, X_∞ being a closed subset of N^∞, $BC(X_\infty)$ itself is isometric to a complemented subspace of $BC(N^\infty)$, by the linear extension theorem.

(h) *In case* (b) *in the theorem* X_∞ *can be taken to be* $\Delta \times N$. Observe first that if X is compact, then so is X_∞, so, using $\Delta \times X_\infty$ instead of X_∞, we can assume as in the last section that X_∞ is the Cantor set.

In the locally compact case we can assume that the sets A_a of the covering are compact. Observe that we have, by the localization lemma, an intermediate regular averaging operator $C(X_1) \to C(X)$ for π_0^1. Now, on each of the compact sets A_a we do as in the first paragraph to get a regular averaging operator $C(\Delta_a) \to C(A_a)$. This gives a well-defined map from the space of bounded continuous functions on the disjoint union of the Cantor sets Δ_a onto $BC(X_1)$. This is also a regular averaging operator. We now obtain the desired regular averaging operator from $BC(X_\infty)$ onto $BC(X)$ by composition. Observe, finally, that the collection of indices a for which A_a is non-empty is countable, so X_∞ is indeed the countable disjoint union of Cantor sets.

We remark that in this case, as in the compact case, the use of the strict topology is not necessary because the Stone–Weierstrass theorem applies to the norm topology for compact spaces (or the strict topology coincides with the norm topology in the compact case). This ends the proof of Theorem C.

Remarks. 1. If, in case (a) in the theorem, all the sets $A_{a(1)\ldots a(n)}$ are non-empty, then X_∞ coincides with N^∞, without having to take products.

2. If X is not compact, then the subset M will not, in general, be norm dense in $BC(X_\infty)$. For example, if X is N^∞ itself then this is the case. The characteristic function of the subset $N_1 \cup N_{22} \cup N_{333} \cup \ldots$ is clearly not in the norm closure of M.

5. Some variants of the main theorem.

(A) The non-separable case. We consider the case of metrizable topologically complete (but not necessarily separable) X. All the proofs remain essentially unchanged; however, we need a slight strenghtening of the hypotheses.

A subset Y of a metric space (X, d) is called *metrically discrete* if there is a positive ε such that $d(x, y) \geqslant \varepsilon$, for every pair of distinct points of Y. Clearly, such a set is also discrete. A metrizable, topologically complete space X is called *w-totally unbounded* (w being an infinite cardinal), if X is non-empty and each non-empty open subset of X has a subset of cardinality w which is metrically discrete in X, for some choice of a complete compatible metric on X. (Both the ε and the metric depending on the open set.) In case w is countable, we say that X is *totally unbounded*. Observe that this condition is slightly stronger than the one used in Proposition 2.2. Examples of spaces satisfying these conditions are following: The space N^∞ is totally unbounded. Indeed, in the basic open set $N_{a(1)\ldots a(n)}$, the sequence of points $x(m) = (a(1), \ldots, a(n), m, \ldots)$, $m = 1, 2, \ldots$, forms a metrically discrete countable subset (with the usual metric on N and $\varepsilon = 1/(n+1)$). Similarly, W^∞, W a discrete space of infinite cardinality w, is w-totally unbounded. The space $\boldsymbol{R}^\infty$, the countable product of lines, is totally unbounded. Every infinite dimensional separable Banach space is totally unbounded, and so is every non-empty open subset of such a space. This is also true for separable, infinite dimensional Fréchet spaces. We mention that it is known (see Anderson and Bing [1] for part of the proof and further references) that all separable infinite dimensional Fréchet spaces are homeomorphic; this makes the corresponding BC spaces isometric. All separable topological manifolds modeled on the above spaces are also totally unbounded. The Banach space l^∞ is c-totally unbounded (c the cardinal of the reals), and so is the Hilbert space $l^2(R)$.

With these definitions we can give modified versions of propositions (2.2) and (2.3). In proposition (2.2) assume that X is a w-totally unbounded topologically complete metrizable space. Then X has a closed subset homeomorphic to W^∞, W a discrete space of cardinality w. In (2.3) assume that X is locally compact, metrizable and topologically complete. Assume that every open non-empty subset of X has at least two points, and that X has a metrically discrete subset of cardinality w. Then X contains a closed subset homeomorphic to $\Delta \times W$, with W a discrete space of cardinality w.

The proofs of these propositions need only minor modifications with respect to the separable case. For example, in the modified version of (2.3), we now take a family $\{V_a\}_{a \in W}$ of compact balls of diameters at most $\varepsilon/3$ (the ε of the metrically discrete subset), to assure that the set $\bigcup \{V_a\colon a \in W\}$ is closed. These balls are centered at the points of the metrically discrete set assumed in the hypotheses.

Observe that, applied to the separable case, these modified propositions have stronger hypotheses than the previous ones. This is due to the fact that in the original propositions we made strong use of the countability of the discrete sets.

The following is now the final theorem in the non-separable case. The rest of the proof is the same as in the separable case, replacing N by W.

5.1. THEOREM. *Let X be metrizable, topologically complete and of infinite weight w.*

(a) *Let X contain a closed subset which is w-totally unbounded. Then the Banach spaces $BC(X)$ and $BC(W^\infty)$ are isomorphic.*

(b) *Let X be locally compact and contain a closed subset which has a metrically discrete subset of cardinality w, every non-empty open subset of which has at least two points. Then the Banach spaces $BC(X)$ and $BC(\Delta\times W)$ are isomorphic.*

(B) The case of the strict topology. Theorems A and (5.1) also hold for the strict topology. That is, the corresponding BC spaces are also strict isomorphic. This, in fact, implies isomorphism in the norm topologies.

It has already been observed that Theorem C is true for the strict topology, we had to prove this to do the norm topology case. The fact that Theorem B also holds for the strict topology is an immediate consequence of the fact that the linear extension operator in the linear extension theorem is both an isometry and continuous in the compact-open topology (see 2.1 and 3.1). It remains to see how to modify the argument in Section 1. For simplicity we deal only with the separable N^∞-case. Let E be a Banach space with a weaker locally convex topology τ. We put on $BC(N;E)$ the norm topology and also the (weaker) strict topology β defined in Section 3. This was the finest locally convex topology on $BC(N;E)$ coinciding, on norm bounded sets of $BC(N;E)$, with the topology τ-CO. The topology τ-CO is, in this particular case, just the topology of pointwise τ-convergence on N. The symbol "$\sim$" will stand here for strict isomorphism.

If $(E,\|\cdot\|,\tau)$ and $(F,\|\cdot\|',\tau')$ are as above and $l\colon E\to F$ is an isomorphism, both in the norm and τ-τ' sense, then relation (1) in the proof becomes

$$BC(N;E)_\beta\sim BC(N;F)_\beta. \tag{1'}$$

Indeed, define a map $L\colon BC(N;E)\to BC(N;F)$ by

$$[L(e_1,e_2,\ldots)=(l(e_1),l(e_2),\ldots)].$$

By symmetry, and because L is a norm bounded linear bijection, it suffices to check that: if $E(\alpha)$ is a net in $BC(N;E)$, with $\|E(\alpha)\|\leqslant 1$ and $E(\alpha)\to 0$ τ-pointwise (i.e., if each $e_1(\alpha)\to 0$ with respect to τ in E), then $LE(\alpha)\to 0$

τ-pointwise. This is now clear. Relation (2) becomes

$$BC(N;E\times F)_\beta\sim BC(N;E)_\beta\times BC(N;F)_\beta. \tag{2'}$$

This is again clear since both sides are isometric (when the right-hand side has the max norm). Relation (3) becomes

$$BC(N;E)_\beta\sim E_\tau\times BC(N;E)_\beta. \tag{3'}$$

Again this follows as above (i.e., by an application of (3.1)), by considering the map

$$(e_1,e_2,\ldots)\mapsto(e_1,(e_2,e_3,\ldots)).$$

Writing $E=BC(N^\infty)$ and $\tau=\beta$, and observing that $N\times N^\infty$ is homeomorphic to N^∞, we have

$$BC(N^\infty)_\beta\sim BC(N\times N^\infty)_\beta\sim BC(N;BC(N^\infty))_\beta. \tag{4'}$$

Indeed, let $L\colon BC(N\times N^\infty)\to BC(N;BC(N^\infty))$ be the natural map, which is clearly an isometry. We again use (3.1) to prove that L is strict continuous. Let $\|f_\alpha\|\leqslant 1$ and let f_α converge uniformly on compact subsets of $N\times N^\infty$. It suffices to show that, for each m in N, $f_\alpha(m,\cdot)\to 0$ in $BC(N^\infty)_\beta$. For this, in turn, it suffices to show convergence in the CO-topology of $BC(N^\infty)$; but this is clear by hypothesis. Similarly, L^{-1} is strict continuous.

From (3′) and (4′) we get

$$BC(N^\infty)_\beta\sim BC(N^\infty)_\beta\times BC(N;BC(N^\infty))_\beta \tag{5'}$$

(product topology).

As before we have

$$BC(N^\infty)_\beta\sim BC(X)_\beta\times U_\beta, \tag{6'}$$

$$BC(X)_\beta\sim V_\beta\times BC(N^\infty)_\beta, \tag{7'}$$

where U and V have the relative β topology. In (5′) and (6′) we have used the easy to check fact that, if E and F are closed subspaces of a topological vector space G, with $G=E\oplus F$ a topological direct sum (i.e., there is a continuous projection from G onto E, which annihilates F), then $G\sim E\times F$.

The decomposition process now runs as before, except that we must change all symbols $\simeq$ to $\sim$. This completes the argument in the strict case.

(C) Other topologies. We mention briefly the cases of the compact-open and uniform topologies. In both of these cases we have to consider unbounded functions to have completeness, that is, we work with the space $C(X)$ instead of $BC(X)$.

The main theorem is true for the space $C(X)$ with the compact-open topology. The proof of this is just a streamlined version of the arguments already given, with obvious changes. Observe only two points: First, the locally convex space $(C(X), \mathrm{CO})$ is complete if X is a k_f-space; the proof of this is similar to that of (3.4), but simpler. Second, in the proof of Milutin's lemma we define the subalgebra M of $BC(X)$ exactly as before; since M is strictly dense in $BC(X)$, it is also dense in the CO-topology (which is weaker), and then observe that $BC(X)$ in dense in $(C(X), \mathrm{CO})$.

A similar conclusion can be stated for $B(X)$ with the uniform topology (the topology of uniform convergence on all of X). However, this is only a metrizable topologically complete abelian group, and the isomorphism result is only for this topological group structure. Secondly, we have to restrict ourselves to the case of separable X. The reason for this is that then we can apply, in the proof of the corresponding Milutin lemma, the following theorem of Stone–Weierstrass type: *Let X be a Hausdorff Lindelöf space. Let A be a subalgebra of $C(X; \boldsymbol{R})$ which contains the constants, separates points and closed sets, and is closed under uniform convergence and inversion in* $C(X; \boldsymbol{R})$ (that is, if f is in A and does not vanish at any point, then $1/f$ is also in A). *Then A is uniformly dense in $C(X; \boldsymbol{R})$.*

This result is due to Henriksen and Johnson [8] and Mrówka [16].

In relation to the proof, we observe the following points: (i) The localization lemma fails, in general, in this setting. However, all we need is the case where each operator L_a is the identity operator and, in this case the proof of that lemma works. To prove the uniform continuity of L, write $Lf_n - Lf$ instead of Lf in the inequality for $\|Lf\|$. Observe that $\|f-g\|$ makes sense in $C(X)$ as soon as f is uniformly near g. (ii) In the proof of the Milutin lemma define $M = \bigcup_{n=1}^{\infty} \pi_n^0 C(X_n)$ and use the Stone–Weierstrass theorem discussed above.

(D) Quotient spaces. We consider here the Banach spaces $BC(X)/C_0(X)$. Only the locally compact case is of interest since, for example, $C_0(N^\infty) = \{0\}$. Observe also that in the locally compact polish case all the spaces $C_0(X)$ are isomorphic to $C(\Delta)$. Indeed, the one point compactification of $\Delta \times N$, for example, is homeomorphic to Δ. Since $C_0(\Delta \times N)$ is isometric to a closed hyperplane in $C(\Delta)$, $C_0(\Delta \times N)$ is isomorphic to $C(\Delta)$ (see, e.g., Semadeni [20], 21. 5.8, p. 378).

5.2. Theorem. *Let X be a locally compact polish space. If X has a non-compact closed subset with the property that every non-empty (relatively) open subset of it has at least two points. Then there is an isomorphism from $BC(X)$ onto $BC(\Delta \times N)$* (norm topology) *such that its restriction to $C_0(X)$ is an isomorphism from $C_0(X)$ onto $C_0(\Delta \times N)$.*

5.3. Corollary. *With the same hypotheses as in the theorem, the Banach spaces $C(\beta X - X)$ and $C(\beta(\Delta \times N) - \Delta \times N)$ are isomorphic.*

Remarks. 1. Since X is sigma-compact, the topological spaces $\beta X - X$ and $\beta(\Delta \times N) - \Delta \times N$ are compact (and non-metrizable). This corollary thus gives an isomorphism result for some (very special) compact non-separable spaces.

2. All the above extends to the locally compact non-separable case by the structure theorem for paracompact locally compact spaces (see Dugundji [6]). This says that such a space is the topological disjoint union of a family of sigma-compact spaces.

3. One could do something similar for $BC(X)$ and $C_0(X)$ with the strict and compact-open topologies.

We describe below the modifications in the proof of the norm topology case that are needed to obtain this theorem.

(a) *In the linear extension theorem* (2.1). The following modification in the proof will give a map T from $BC(Y)$ into $BC(X)$, which is still of norm one and such that $TC_0(Y)$ is contained in $C_0(X)$. We will only lose the property $T(1) = 1$. Let (K_n) be an increasing sequence of compact subsets of Y, whose union is Y. For every point in K_1 choose a relatively compact open ball with center at this point and radius at most 1. Select a finite subcovering of K_1 from these balls, and write V_1 for the (finite) union of them. Then V_1 is an open, relatively compact, neighborhood of K_1, consisting of points at distance at most one from K_1. Do the same thing for K_2, with balls of radii at most $\frac{1}{2}$, to get an open relatively compact neighborhood V_2 of K_2, consisting of points at distance at most $\frac{1}{2}$ from K_2. Proceed by induction. Then $V = \bigcup_1^\infty V_n$ is an open neighborhood of Y. There is a continuous function F_0, with $0 \leqslant F_0 \leqslant 1$ on X, $F_0 = 1$ on Y and $F_0 = 0$ on $X - V$. In the proof of (2.1) replace then Tf by $F_0 \cdot Tf$; this will be the new linear extension map T'.

To conclude that $T'f$ is in $C_0(X)$ whenever f is in $C_0(Y)$, it is enough to prove that if f is in $C_0(Y)$ then (the old) Tf, restricted to the closure V^- of V, is in $C_0(V^-)$. For this it suffices in turn to prove that if (say) x^n is a point in $V^- - Y$ for $n = 1, 2, \ldots$, and $x^n \notin V_n^-$, then $Tf(x^n) \to 0$ for f in $C_0(Y)$.

The proof of the linear extension theorem 2.1 gives the following (see [11] or [20]): For each point $x \in X - Y$, an open ball V_x with center x and radius $d(x, Y)/3$; a locally finite open refinement $\{W_t : t \in T\}$ of this covering of $X - Y$; points $w_t \in W_t$ and points $v_t \in Y$ with $d(v_t, w_t) < 3d(w_t, Y)$. Further, $Tf(x)$ is some convex combination of the $f(v_t)$, for those t such that x is in W_t. Specializing all this to the x^n and changing the notation slightly, we obtain corresponding W_i^n, w_i^n, v_i^n, for $i = 1, \ldots$

$\ldots, k(n)$ and $n = 1, 2, \ldots$ Also choose, for each n and i, a ball V_i^n such that $W_i^n \subset V_i^n$, and let x_i^n be its center. Then V_i^n has radius $d(x_i^n, Y)/3$, and $d(v_i^n, w_i^n) < 3d(w_i^n, Y)$. The scalar $Tf(x^n)$ is a convex combination of the $f(v_i^n)$, $i = 1, \ldots, k(n)$, and we want to prove that $Tf(x^n)$ tends to zero.

To prove that $Tf(x^n)$ tends to zero, it suffices to verify that, given an integer N, there is an integer n_0 such that, for all $n \geqslant n_0$ and $i = 1, \ldots$ $\ldots, k(n)$, we have $v_i^n \notin K_N$.

To see this, observe first that, by hypothesis, $a_n = d(x^n, Y) \to 0$. Then we prove that, for $i = 1, \ldots, k(n)$, we have

$$(*) \qquad d(v_i^n, x^n) < 7a_n.$$

Indeed,

$$\begin{aligned} d(v_i^n, x^n) &\leqslant d(v_i^n, w_i^n) + d(w_i^n, x^n) \\ &< 3d(w_i^n, Y) + d(w_i^n, x^n) \\ &< 3[d(w_i^n, x^n) + d(x^n, Y)] + d(w_i^n, x^n) \\ &= 4d(w_i^n, x^n) + 3d(x^n, Y) \\ &< 4\cdot 2/3 \cdot d(x_i^n, Y) + 3a_n, \end{aligned}$$

(because x^n and w_i^n are in W_i^n, which is contained in V_i^n). Now

$$d(x_i^n, Y) < 3a_n/2.$$

Putting these inequalities together we get inequality $(*)$. The importance of $(*)$ is that, for all i, $d(v_i^n, x^n)$ is close to zero for large n. From this, letting $d(x^n, V_{\bar{N}}) = \varepsilon > 0$ and using $K_N \subset V_{\bar{N}}$, we have the main claim. Indeed, choose n_0 such that $a_n < \varepsilon/14$ for all $n \geqslant n_0$. Then $d(v_i^n, K_n) > \varepsilon/2$ for $i = 1, \ldots, k(n)$ and all $n \geqslant n_0$. In particular, $v_i^n \notin K_N$ for $n \geqslant n_0$ and $i = 1, \ldots, k(n)$.

Even if $T'(1) = F_0 \neq 1$, T' is still an isometry into $BC(X)$. Thus, both $T'BC(Y)$ and $T'C_0(Y)$ are still norm closed. This shows that the corresponding version (for BC and C_0) of Theorem B (case b) is still true.

(b) It is not true in general that $\pi^0 C_0(Y)$ is contained in $C_0(X)$. However, in the case at hand we proved that the fibers $\pi^{-1}(y)$ of π are compact, and this implies the above stability property.

(c) *In the localization lemma.* In the present situation we may assume that the sets X_a and Y_a^- are compact. Then, if f is a function in $C_0(X)$, given $\varepsilon > 0$, choose a compact subset K of X, in such a way that $|f|$ is less than ε off K. Then only finitely many of the X_a intersect K, say $\{X_a\colon a \in I\}$, I a finite subset of the index set in the localization lemma. We can write

$$L(f) = \sum \lambda_a L_a(f|X_a) = \sum_{a\in I} + \sum_{a\notin I}.$$

The first summand is a C_0 function, being a finite linear combination of such functions. The second summand is a function which vanishes on K and is at most ε off K. This proves the C_0 version of the localization lemma in the case of compact X_a.

(d) *In the Milutin lemma.* In general the maps L_n (see part (f) in the proof) do not preserve C_0, as is easily seen by example. However, we can again assume here that all the sets $A_{a(1)\ldots a(n)}$ are compact. The modified version of the localization lemma then gives maps L_n' that preserve C_0. There is no problem at the level of the subalgebra M of $BC(X)$; the map L defined in part (f) of the proof sends the subalgebra $M_0 = \bigcup_1^\infty (\pi_n)^0 C_0(X_n)$ onto $C_0(X)$. Another application of the modified localization lemma then gives an extension of L to a map from $BC(X_\infty)$ onto $BC(X)$ such that the image of $C_0(X_\infty)$ is $C_0(X)$ (as in part (h) of the proof).

(e) In Theorem B we use the modified version of the linear extension theorem.

(f) *In the Milutin theorem.* We are only dealing with the locally compact case here. The subspace $C_0(X; E)$ of $BC(X; E)$ has the obvious definition. The C_0 versions of formulas (1) through (7) follow through restriction and using the previous modifications. The decomposition argument that ends the proof also follows by restricting everything to the corresponding C_0 subspaces.

6. Applications of the main theorem. In this section we discuss a few applications of the isomorphism theorem. We begin with a result on bounds for the norms of the isomorphisms.

(A) Uniform bound for the norms of the isomorphisms. We could obtain a numerical estimate, except that, by using the strict topology in the proof the Milutin lemma, we have lost control of the norms of the operators (see [17] for a numerical bound in the compact case).

For the sake of concreteness we discuss only the N^∞ case. We claim that there is a positive number M such that, if X is any polish space with an uncountable closed subset which is not locally compact at any point, then there is an isomorphism T between $BC(X)$ and $BC(N^\infty)$ with $\|T\|\cdot\|T^{-1}\| \leqslant M$. Otherwise there would be polish spaces X_n as above such that, for any isomorphism T_n between $BC(X_n)$ and $BC(N^\infty)$, we would have $\|T_n\|\cdot\|T_n^{-1}\| \geqslant n$. Observe now that the space

$$(BC(X_1) \times BC(X_2) \times \ldots)_{l^\infty}$$

is isometric to $BC(X)$, X being the topological disjoint union of the X_n. But X is of the same type as the X_n, so $BC(X)$ is isomorphic to $BC(N^\infty)$. Let $T\colon BC(X) \to BC(N^\infty)$ be an isomorphism; let also i_n be the natural injection of $BC(X_n)$ in $BC(X)$ and P_n the natural projection from $BC(X)$ onto $BC(X_n)$, both of these maps have norm one. The map Ti_n is an embedding of $BC(X_n)$ into $BC(N^\infty)$, and the map $Ti_nP_nT^{-1}$ is a projection onto its image, of norm at most $\|T\|\cdot\|T^{-1}\| \leqslant M$ (independent of n). Since N^∞ is homeomorphic to a closed subset of X, the decomposition method, as in the proof of the main theorem, gives an isomorphism from $BC(X_n)$ onto $BC(N^\infty)$ of norm independent of n. This is a contradiction.

(B) Contractibility of the corresponding linear groups. In Mityagin [15] it is proved that if X is an uncountable compact metrizable space and $E = C(X)$, then $GL(E)$, the general linear group of the Banach space E, is contractible to a point. We observe that the same result holds for the Banach spaces $E = BC(\Delta\times N)$ and $E = BC(N^\infty)$, the proof being a trivial adaptation of that in [15]. The same thing is true for the non-separable cases.

(C) Extension of a theorem of Pełczyński. In [18], Pełczyński proved that if X is compact and metrizable and if E is a complemented subspace of $C(X)$, containing a further subspace isomorphic to $C(X)$, then E itself is isomorphic to $C(X)$. There is a somewhat simpler proof of this result in the 1972 Berkeley thesis of James Hagler. We extend this theorem and slightly simplify the argument given by Hagler, by using the Bartle–Graves selection theorem. For simplicity we state the theorem in the polish case only. Also, since we have simple criteria for a polish space to contain a closed copy of Δ, $\Delta\times N$ or N^∞, we do not include these conditions in the statement of the theorems.

The result is based on the following.

6.1. THEOREM. *Let X be a polish space and E a Banach space. Let T be an isomorphism from the Banach space $BC(X)$ into the Banach space E such that $\|T\|\cdot\|T^{-1}\|\leqslant\lambda$. Assume that X is either countable or contains a closed subset homeomorphic to either Δ, $\Delta\times N$ or N^∞. Then there is a closed subspace F of $BC(X)$, F isometric to $BC(X)$, such that the subspace TF is λ-complemented in E.*

Proof. The adjoint map $T^*\colon E^*\to BC(X)^*$ is surjective so, by the Bartle–Graves selection theorem (see, e.g., Michael [12]), there is a continuous (not necessarily linear) function $S\colon BC(X)^*\to E^*$ such that $T^*S = \mathrm{Id}_{BC(X)^*}$; the map S is then a homeomorphism of $BC(X)^*$ with a closed subset of E^*.

Let us identify X with the corresponding set of point masses in $BC(X)^*$. Let now Y be a closed subset of X (or its image in $BC(X)^*$) which is all of X if X is countable, and Δ, $\Delta\times N$, or N^∞, in the other cases. Let $Z = S(Y)$. Let also $L\colon BC(Y)\to BC(X)$ be a linear extension; S^0 the map $f\mapsto fS$; $(T^*|Z)^0$ the map $g\mapsto g(T^*|Z)$; R' the restriction map $x\mapsto (Jx)|Z$ from E into the restriction of E^{**} to Z (J being the canonical map from E into the second conjugate E^{**} of E); R the restriction map from $BC(X)$ onto $BC(Y)$ (i.e., i^0, with i the injection of Y into X). It is then easy to check that the following diagram commutes (i.e., $R'T = (T^*|Z)^0R$).

$$\begin{tikzcd}
BC(X) \arrow[rr, "T"] \arrow[d, shift left, "R"] & & E \\
BC(Y) \arrow[u, dashed, shift left, "L"] \arrow[rr, shift left, "(T^*|Z)^0"] & & BC(Z) \arrow[u, "R'"'] \arrow[ll, dashed, shift left, "S^0"]
\end{tikzcd}$$

Then $P = TLS^0R'$ is a projection from E onto TF, where $F = LBC(Y)$. Indeed, $P^2 = TLS^0(R'T)LS^0R'$, which is the same as

$$TLS^0(T^*|Z)^0(RL)S^0R',$$

which collapses to $TLS^0\big((T^*|Z)^0S^0\big)R'$, which in turn collapses to $TLS^0R' = P$. Also this projection has the correct image and the correct estimate on the norm. This ends the proof of the theorem.

The next result is the extension of Pełczyński's theorem. It is obtained from the corresponding isomorphism theorem and a decomposition argument. The countable compact case follows from an isomorphism theorem of Bessaga and Pełczyński [3] for this case. We leave out the countable non-compact cases because they would need an extension of the above-mentioned result in [3], an extension we do not have.

6.2. COROLLARY. *Let X be a polish space which is either countable and compact, or locally compact and contains a closed subset homeomorphic to $\Delta\times N$, or contains a closed subset homeomorphic to N^∞. If E is a closed complemented subspace of $BC(X)$, containing a further subspace isomorphic to $BC(X)$; then E itself is isomorphic to $BC(X)$.*

7. Some open problems. It remains to be shown whether there are further isomorphisms between the classes of spaces considered in the previous sections. The most interesting problem left open in this direction seems to be the following one.

PROBLEM 1. *Are the Banach spaces $BC(\Delta\times N)$ and $BC(N^\infty)$ isomorphic?* Equivalently, *are the Banach spaces $BC(\boldsymbol{R})$ and $BC(l^2)$ isomorphic?*

The usual invariants coincide. For example (Rosenthal [19], Remark 3, p. 242), these spaces have (up to isometry) the same dual, namely $(l^\infty)^*$. On the other hand, these two classes are indeed non-isomorphic for the strict topology. This is so because the minimal number of seminorms necessary to describe the topology is an isomorphic invariant, and this cardinal numbers is easily seen to be countable for the first class and uncountable for the second. Similarly, other cases can be distinguished by this or similar techniques (like l^∞ and the above Banach spaces, and cases of different weights).

Bessaga and Pełczyński [3] have classified all the Banach spaces $C(X)$, with X compact metrizable and countable. Such an X is homeomorphic to a countable compact ordinal by a classical theorem of Mazurkiewicz and Sierpiński (see, e.g., Semadeni [20], 8.6.10). If, now, X is a countable polish space, a theorem of Knaster and Urbanik (cf. Kuratowski, vol. II, [10], p. 25) implies that X is homeomorphic to a subset of the countable ordinals. Thus the situation is rather similar, if more complicated, to the case considered in [3].

PROBLEM 2. *Classify up to isomorphism all the Banach spaces $BC(X)$, with X countable and polish.*

Typical spaces arising here are l^∞, c and $l^\infty \oplus c$.

In the proof of the Milutin lemma we do not get, at least in principle, a *regular* averaging operator in the N^∞ case.

PROBLEM 3. *Is the averaging operator in the proof of the Milutin lemma always of norm one? If not, is there always an averaging operator of norm one?*

One can also consider some related questions. For example: (a) *Are the Banach spaces $BC(\beta N^\infty - N^\infty)$ and $BC(\beta X - X)$ isomorphic, for X polish uncountable and not locally compact at any point?* (b) *Are the Banach spaces $BC(\beta N^\infty - N^\infty)$ and $C(\beta \boldsymbol{R} - \boldsymbol{R})$ isomorphic?* (c) *Is $C(\beta N - N)$ isomorphic to $C(\beta \boldsymbol{R} - \boldsymbol{R})$?* (The first space is just l^∞/c_0 and the second is $BC(\boldsymbol{R})/C_0(\boldsymbol{R})$.) One could also try to deal with other compactifications, besides the Stone–Čech compactification. Equivalently, one could consider other subalgebras of $BC(X)$ besides the full algebra. For example, the bounded uniformly continuous functions on some (uniform) spaces X. In general there are, however, problems with partition of unity arguments and with the linear extension theorem.

References

[1] R. D. Anderson and R. H. Bing, *A completely elementary proof that Hilbert space is homeomorphic to the countable product of lines*, Bull. Amer. Math. Soc. 74 (1968), pp. 771–792.

[2] W. G. Bade, *The Banach space $C(S)$*, Aarhus University Lecture Notes, Aarhus, 1971.

[3] C. Bessaga and A. Pełczyński, *Spaces of continuous functions IV*, Studia Math. 19 (1960), pp. 53–62.

[4] J. B. Cooper, *The strict topology and spaces with mixed topologies*, Proc. Amer. Math. Soc. 30 (1971), pp. 583–592.

[5] S. Z. Ditor, *On a lemma of Milutin concerning averaging operators in continuous function spaces*, Trans. Amer. Math. Soc. 149 (1970), pp. 443–452.

[6] J. Dugundji, *Topology*, Allyn and Bacon, Boston 1966.

[7] D. H. Fremlin, D. J. H. Garling and R. G. Haydon, *Bounded measures on topological spaces*, Proc. London Math. Soc. 25 (1972), pp. 115–136.

[8] M. Henriksen and D. G. Johnson, *On the structure of a class of archimedean lattice-ordered algebras*, Fundamenta Math. 50 (1961), pp. 73–94.

[9] J. Hoffmann-Jørgensen, *A generalization of the strict topology*, Math. Scand. 30 (1972), pp. 313–323.

[10] K. Kuratowski, *Topology*, Vol. I (1966), Vol. II (1968), New York, London, Warszawa.

[11] E. A. Michael, *Some extension theorems for continuous functions*, Pacific J. Math. 3 (1953), pp. 789–806.

[12] — *Selected selection theorems*, Amer. Math. Monthly, 63 (1956), pp. 233–238.

[13] A. A. Milutin, *On space of continuous functions*, Dissertation, Moscow State University, 1952 (Russian).

[14] — *Isomorphism of spaces of continuous functions on compacta of power of continuum*, Teor. Funct., Funct. Anal. i Pril. (Kharkov) 2 (1966), pp. 150–156. (Russian).

[15] B. S. Mityagin, *The homotopy structure of the linear group of a Banach space*, Russian Math. Surveys 25 (1970), pp. 59–103.

[16] S. Mrówka, *On some approximation theorems*, Nieuw Archief voor Wiskunde 16 (1968), pp. 94–111.

[17] A. Pełczyński, *Linear extensions, linear averagings, and their applications to linear topological classification of spaces of continuous functions*, Dissertationes Math. (Rozprawy Mat.) 58 (1968), pp. 1–89.

[18] — *On $C(S)$ subspaces of separable Banach spaces*, Studia Math. 31 (1968), pp. 513–522.

[19] H. P. Rosenthal, *On injective Banach spaces and the spaces $L^\infty(\mu)$ for finite measures μ*, Acta Math. 124 (1970), pp. 205–248.

[20] Z. Semadeni, *Banach spaces of continuous functions*, Warszawa, 1971.

[21] F. D. Sentilles, *Bounded continuous functions on a completely regular space*, Trans. Amer. Math. Soc. 168 (1972), pp. 311–336.

[22] A. H. Stone, *Non-separable Borel sets*, Dissertationes Math. (Rozprawy Mat.) 28 (1962), pp. 1–41.

[23] W. H. Summers, *Separability in the strict and substrict topologies*, Proc. Amer. Math. Soc. 35 (1972), pp. 507–514.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA
and
DEPARTAMENTO DE MATEMATICAS
UNIVERSIDAD DE CHILE, SANTIAGO, CHILE

Received September 13, 1973 (730)