*LIMIT THEOREMS
FOR RANDOMLY INDEXED SUMS OF RANDOM VECTORS*

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1. Introduction and notation. In the present paper we study certain limit theorems for randomly indexed sums of random vectors. Section 2 starts with some preliminary information concerning randomly indexed sequences of random elements in general metric spaces. Here we introduce the technique of quantiles which is crucial in the subject. Section 3 is devoted to the proof of the characterization theorem for weak convergence of randomly indexed arrays of symmetrically distributed random vectors in Banach spaces. This generalizes the earlier D. Szász's result for real random variables. The stability of randomly indexed sums of random vectors in normed linear spaces is investigated in Section 4 and then, in more detail (in the Hilbert space case), in Section 5. Section 6 gives some applications of the previous results and provides sufficient conditions for convergence to the infinitely divisible distribution of randomly indexed arrays of real random variables.

We introduce the notation and definitions that will be used throughout the paper. A *random element* on a probability system $(\Omega, \mathcal{F}, P)$ with values in a measurable space $(X, \mathcal{B})$ ($\mathcal{B}$ is a σ -algebra of subsets of X) is a mapping $X: \Omega \rightarrow X$ such that $X^{-1}(B) \in \mathcal{F}$ for each $B \in \mathcal{B}$. In the case where X is a metric space, $\mathcal{B}_X$ stands always for the σ -algebra of all Borel subsets of X . A *random index* ν is simply an N -valued random variable, where N is the set of positive integers. The Lebesgue measure on $\mathcal{B}_R$ is denoted by $|\cdot|$. Further on $\xrightarrow{w}$ denotes the weak convergence of distributions (cf. [6], p. 40), $\xrightarrow{P}$ the convergence in probability P , $\mathcal{M}(X)$ denotes the space of all probability measures on a metric space X , $\mathcal{L}(X)$ denotes the distribution of the random element X , and eventually conditionally compact means compact after closure.

2. Preliminaries: randomly indexed sequences of random elements in metric spaces. In this section we prove some results concerning randomly

indexed sequences of random elements in metric spaces that will be applied in the sequel.

Let $(\mathcal{X}, d)$ be a metric space equipped with the σ -algebra $\mathcal{B}_{\mathcal{X}}$ of Borel subsets of $\mathcal{X}$, let $Y_1, Y_2, \dots$ be a sequence of $\mathcal{X}$ -valued random elements, and let ν be a random index which is independent of the sequence $\{Y_n\}$.

In the present paper the t -quantile of the random index ν is defined as the greatest integer $l(t)$ that satisfies the inequality

$$P\{\nu < l(t)\} \leq t \leq P\{\nu \leq l(t)\}, \quad t \in [0, 1).$$

Now, let $\xi(t) = Y_{l(t)}$ be the $\mathcal{X}$ -valued stochastic process on $[0, 1)$.

PROPOSITION 2.1. *The stochastic process $\xi(t)$ for each $t \in [0, 1)$ is $(\mathcal{B}_{[0,1)} \times \mathcal{F}, \mathcal{B}_{\mathcal{X}})$ -measurable, and the $\mathcal{X}$ -valued random elements Y_{ν} and $\xi(\pi)$ are equidistributed, where π is a real random variable uniformly distributed on $[0, 1)$ and independent of $Y_1, Y_2, \dots$*

Proof. Put

$$t_0 = 0, \quad t_k = \sum_{i=1}^k P\{\nu = i\}, \quad 0 = t_0 \leq t_1 \leq \dots \leq 1.$$

Let $k_0 = 0, k_j, j \geq 1$, be the sequence (finite or infinite) of integers enjoying the property

$$t_{k_{j-1}+r} = t_{k_{j-1}}, \quad j \geq 1, \quad 0 \leq r < k_j - k_{j-1}.$$

Notice that, for $t \in [t_{k_{j-1}}, t_{k_j})$, we have $l(t) = k_j$. Now the measurability of the process $\xi(t)$ follows. To prove that $\xi(\pi)$ and Y_{ν} are equidistributed it is enough to make the observation that, for each $B \in \mathcal{B}_{\mathcal{X}}$ and $j = 1, 2, \dots$,

$$\sum_{k=k_{j-1}+1}^{k_j} P\{Y_k \in B\} P\{\nu = k\} = P\{Y_{k_j} \in B\} (t_{k_j} - t_{k_{j-1}}).$$

Thus

$$\begin{aligned} P\{\xi(\pi) \in B\} &= \int_0^1 P\{\xi(t) \in B\} dt = \sum_{j \geq 1} P\{Y_{k_j} \in B\} (t_{k_j} - t_{k_{j-1}}) \\ &= \sum_{k=1}^{\infty} P\{Y_k \in B\} P\{\nu = k\} = P\{Y_{\nu} \in B\}. \end{aligned}$$

The following proposition, in the case $\mathcal{X} = \mathbf{R}^n$, is due to Szász [9] who proved it utilizing the technique of characteristic functions which is not available in our general case.

PROPOSITION 2.2. *Let $\xi_n(t), \xi(t), t \in [0, 1)$, be $(\mathcal{B}_{[0,1)} \times \mathcal{F}, \mathcal{B}_{\mathcal{X}})$ -measurable stochastic processes with values in a metric space $\mathcal{X}$, and let τ be a real*

random variable with distribution μ concentrated on $[0, 1)$ and independent of ξ_n and ξ . If, for μ -almost each $t \in [0, 1)$,

$$\xi_n(t) \xrightarrow{w} \xi(t), \quad \text{as } n \rightarrow \infty,$$

then also

$$\xi_n(\tau) \xrightarrow{w} \xi(\tau), \quad \text{as } n \rightarrow \infty.$$

Proof. By Theorem II.6.1 of [6], for μ -almost all $t \in [0, 1)$ and for each closed $C \subset \mathcal{X}$, we have

$$\overline{\lim}_n P\{\xi_n(t) \in C\} \leq P\{\xi(t) \in C\}.$$

Now, by the independence assumption and the Fatou Lemma, we obtain

$$\begin{aligned} \overline{\lim}_n P\{\xi_n(\tau) \in C\} &= \overline{\lim}_n \int_0^1 P\{\xi_n(t) \in C\} \mu(dt) \\ &\leq \int_0^1 \overline{\lim}_n P\{\xi_n(t) \in C\} \mu(dt) \leq \int_0^1 P\{\xi(t) \in C\} \mu(dt) = P\{\xi(\tau) \in C\} \end{aligned}$$

for each closed $C \subset \mathcal{X}$ which, again in view of Theorem II.6.1 of [6], gives the desired conclusion.

PROPOSITION 2.3. Let Y and Y_{nk} ($n, k = 1, 2, \dots$) be random elements in a metric space $\mathcal{X}$, and let v_n be random indices with t -quantiles $l_n(t)$ such that, for each $n \in \mathbf{N}$, v_n is independent of $Y_{n1}, Y_{n2}, \dots$. If for any sequence $N_1 \subset \mathbf{N}$ there exists a sequence $N_2 \subset N_1$ such that, for almost all $t \in [0, 1)$ (Lebesgue measure),

$$Y_{nl_n(t)} \xrightarrow{w} Y \quad \text{as } n \in N_2, n \rightarrow \infty,$$

then

$$Y_{nv_n} \xrightarrow{w} Y \quad \text{as } n \rightarrow \infty.$$

In the case where Y is constant with probability 1 the inverse of the above-given statement is true.

Proof. Put $\xi_n(t) = Y_{nl_n(t)}$, $\xi(t) = Y$, $t \in [0, 1)$. By Proposition 1.1 the processes ξ_n are $(\mathcal{B}_{[0,1)} \times \mathcal{F}, \mathcal{B}_{\mathcal{X}})$ -measurable. Indeed, ξ is $(\mathcal{B}_{[0,1)} \times \mathcal{F}, \mathcal{B}_{\mathcal{X}})$ -measurable as well. Now, let N_1 be an arbitrary subsequence of $\mathbf{N}$. By our assumption there exists subsequence $N_2 \subset N_1$ such that, for almost each $t \in [0, 1)$,

$$\xi_n(t) \xrightarrow{w} \xi(t) \quad \text{as } n \in N_2, n \rightarrow \infty.$$

If π is a real random variable uniformly distributed on $[0, 1)$ and independent of $\{Y_{nk}\}$ and Y (and so independent of processes ξ_n and ξ),

by Proposition 2.2 we get

$$\xi_n(\pi) \xrightarrow{w} \xi(\pi) \quad \text{as } n \in N_2, n \rightarrow \infty.$$

However, by Proposition 2.1 for each $n \in N$ the random elements $\xi_n(\pi)$ and Y_{nv_n} are equidistributed and $\xi(\pi) = Y$, so that

$$Y_{nv_n} \xrightarrow{w} Y \quad \text{as } n \in N_2, n \rightarrow \infty.$$

Thus $Y_{nv_n} \xrightarrow{w} Y$ as $n \rightarrow \infty$ (use, e.g., Theorem II.6.1 of [6] in the case $\mathcal{X}$ is not separable).

To prove the converse assume $Y = a \in \mathcal{X}$ with probability 1. Notice that

$$Y_{nv_n} \xrightarrow{w} Y \quad \text{if and only if} \quad Y_{nv_n} \xrightarrow{P} a,$$

and so, by the assumption and Proposition 2.1,

$$\xi_n(\pi) \xrightarrow{P} a \quad \text{as } n \rightarrow \infty,$$

so that, for each $\varepsilon > 0$,

$$P\{d(\xi_n(\pi), a) \geq \varepsilon\} = \int_0^1 P\{d(\xi_n(t), a) \geq \varepsilon\} dt \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Therefore, for a fixed $k \in N$,

$$P\left\{d(\xi_n(t), a) \geq \frac{1}{k}\right\} \rightarrow 0, \quad t \in [0, 1),$$

in Lebesgue measure as $n \rightarrow \infty$. Let N_1 be an arbitrary subsequence of N . Applying the Cantor diagonal procedure, we are able to find a subsequence $N_2 \subset N_1$ and a subset $T \subset [0, 1)$ of Lebesgue measure 1 such that, for each $k \in N$ and each $t \in T$,

$$P\left\{d(\xi_n(t), a) \geq \frac{1}{k}\right\} \rightarrow 0 \quad \text{as } n \rightarrow \infty, n \in N_2.$$

But this exactly means that, for each $t \in T$,

$$Y_{nt_n(t)} \xrightarrow{P} a \quad \text{as } n \in N_2, n \rightarrow \infty,$$

which completes the proof.

COROLLARY 2.1. *Let X_n and X be random elements in $\mathcal{X}$, and let v_n be random indices such that, for each $n \in N$, v_n is independent of $X_1, X_2, \dots$ and*

$$v_n \xrightarrow{P} \infty \quad \text{as } n \rightarrow \infty.$$

If $X_n \xrightarrow{w} X$ as $n \rightarrow \infty$, then $X_{v_n} \xrightarrow{w} X$ as $n \rightarrow \infty$.

Proof. It is sufficient to notice that $v_n \xrightarrow{P} \infty$ if and only if, for each $t \in (0, 1)$, $l_n(t) \rightarrow \infty$ as $n \rightarrow \infty$, so that $X_n \xrightarrow{w} X$ implies $X_{l_n(t)} \xrightarrow{w} X$ for each $t \in (0, 1)$. Now one can apply Proposition 2.3 in the special case $Y_{nk} = X_k$ for $n, k = 1, 2, \dots$

COROLLARY 2.2. *Let $\{a_{nk}\}$ ($n, k = 1, 2, \dots$) be a sequence of real numbers such that $a_{n1} \leq a_{n2} \leq \dots$ for each $n \in \mathbf{N}$, and let $\{v_n\}$ be a sequence of random indices. Then $a_{nv_n} \xrightarrow{P} 0$ if and only if, for each $t \in (0, 1)$, $a_{nl_n(t)} \rightarrow 0$ as $n \rightarrow \infty$.*

Proof. By Proposition 2.1,

$$P\{|a_{nv_n}| > \varepsilon\} = |\{t: |a_{nl_n(t)}| > \varepsilon\}|.$$

If, for a certain $t_0 \in (0, 1)$, $a_{nl_n(t_0)} \not\rightarrow 0$ as $n \rightarrow \infty$, then

$$\lim_n \overline{|a_{nl_n(t_0)}|} > \varepsilon \quad \text{for some } \varepsilon > 0,$$

so that

$$\lim_n P\{|a_{nv_n}| > \varepsilon\} = \lim_n |\{t: |a_{nl_n(t)}| > \varepsilon\}| \geq \min\{t_0, 1 - t_0\} > 0.$$

The second implication follows directly from Proposition 2.3.

3. Limit theorems for random vectors in Banach spaces. Throughout this section $(\mathcal{X}, \|\cdot\|)$ stands for a separable Banach space, X_{nj} for $n, j = 1, 2, \dots$ are random vectors in $\mathcal{X}$ such that $X_{n1}, X_{n2}, \dots$ for each n are independent, symmetric and

$$\lim_{n \rightarrow \infty} \sup_{j \geq 1} P\{\|X_{nj}\| > \varepsilon\} = 0 \quad \text{for each } \varepsilon > 0,$$

and v_n ($n = 1, 2, \dots$) are random indices independent of $X_{n1}, X_{n2}, \dots$ for each n and such that $v_n \xrightarrow{P} \infty$ ($n \rightarrow \infty$). We introduce the notation

$$S_k^n = X_{n1} + \dots + X_{nv_n}.$$

THEOREM 3.1. *In order that there exists $\mu \in \mathcal{M}(\mathcal{X})$ such that the distributions of $S_{v_n}^n$ are weakly convergent to μ ($n \rightarrow \infty$) it is necessary and sufficient that for each sequence $N_1 \subset \mathbf{N}$ there exist a subsequence $N_2 \subset N_1$ and an $\mathcal{X}$ -valued $(\mathcal{B}_{[0,1)} \times \mathcal{F}, \mathcal{B}_{\mathcal{X}})$ -measurable stochastic process $\xi(t)$, $t \in [0, 1)$, such that*

(i) *for almost every $t \in [0, 1)$, $S_{l_n(t)}^n \xrightarrow{w} \xi(t)$, $n \in N_2$, $n \rightarrow \infty$;*

(ii) *the random vector $\xi(\pi)$, where π is a real random variable uniformly distributed on $[0, 1)$ and independent of $\{\xi(t)\}$, has the distribution μ .*

This theorem in the case $\mathcal{X} = \mathbf{R}^n$ is, essentially, due to Szász (cf. [9], Corollary 1) who worked (also in the non-symmetric case) with help of Doob's centers, the notion that is not readily available in our general case. In the proof of Theorem 3.1 we need several lemmas.

LEMMA 3.1. (a) *If there exists a sequence $t_k \in [0, 1)$, $t_k \rightarrow 1$ as $k \rightarrow \infty$, such that for each $k = 1, 2, \dots$ the sequence of distributions of sums $S_{l_n(t_k)}^n$ is conditionally compact, then the sequence of distributions of sums $S_{v_n}^n$ is also conditionally compact.*

(b) *Conversely, if the sequence of distributions of sums $S_{v_n}^n$ is conditionally compact, then for each $t \in [0, 1)$ the sequence of distributions of sums $S_{l_n(t)}^n$ is conditionally compact.*

Proof. (a) Assume to the contrary that the sequence of distributions of sums $S_{v_n}^n$ is not conditionally compact. The Prokhorov Theorem (cf. [6], p. 47) yields the existence of an $\varepsilon > 0$ such that, for each compact $K \subset \mathcal{X}$,

$$(1) \quad \sup_n P\{S_{v_n}^n \notin K\} \geq \varepsilon.$$

Take t_k such that $1 - t_k < \varepsilon/4$ and a compact $K \subset \mathcal{X}$ such that

$$P\{S_{l_n(t_k)}^n \notin K\} < \frac{\varepsilon}{4} \quad (n = 1, 2, \dots).$$

By Mazur's Theorem ([2], Theorem V.2.6), the set $\tilde{K} = \overline{\text{conv } K}$ is compact. Taking into account the inequality

$$(2) \quad P\{X \notin B\} \leq 2P\{X + Y \notin B\}$$

which holds true for any independent and symmetric random vectors in normed linear space $\mathcal{X}$ and arbitrary convex Borel $B \in \mathcal{B}_{\mathcal{X}}$, for each $i \leq l_n(t_k)$ we get

$$P\{S_i^n \notin \tilde{K}\} \leq 2P\{S_{l_n(t_k)}^n \notin \tilde{K}\}.$$

Thus

$$\begin{aligned} P\{S_{v_n}^n \notin \tilde{K}\} &\leq \sum_{i=1}^{l_n(t_k)} P\{S_i^n \notin \tilde{K}\} P\{v_n = i\} + \frac{\varepsilon}{4} \\ &\leq 2P\{S_{l_n(t_k)}^n \notin \tilde{K}\} + \frac{\varepsilon}{4} < \frac{3}{4}\varepsilon \end{aligned}$$

and, eventually,

$$\sup_n P\{S_{v_n}^n \notin \tilde{K}\} \leq \frac{3}{4}\varepsilon$$

which contradicts (1).

(b) Assume to the contrary that there exists a $t_0 \in [0, 1)$ such that the sequence of distributions of $S_{l_n(t_0)}^n$ is not conditionally compact. Thus we may find $\varepsilon > 0$ such that, for each compact $K \subset \mathcal{X}$,

$$(3) \quad \sup_n P\{S_{l_n(t_0)}^n \notin K\} \geq \varepsilon.$$

Now, let K be an arbitrary compact subset of $\mathcal{X}$ and let $\tilde{K} = \overline{\text{conv } K}$. In view of inequality (2) for $i \geq l_n(t_0)$ we get

$$P\{S_i^n \notin \tilde{K}\} \geq \frac{1}{2} P\{S_{l_n(t_0)}^n \notin \tilde{K}\},$$

so that

$$\begin{aligned} P\{S_{v_n}^n \notin K\} &\geq P\{S_{v_n}^n \notin \tilde{K}\} \geq \sum_{i=l_n(t_0)}^{\infty} P\{S_i^n \notin \tilde{K}\} P\{v_n = i\} \\ &\geq \frac{1-t_0}{2} P\{S_{l_n(t_0)}^n \notin \tilde{K}\}, \end{aligned}$$

which, by (3), implies that

$$\sup_n P\{S_{v_n}^n \notin K\} \geq \frac{1-t_0}{2} \varepsilon,$$

and this gives the necessary contradiction.

Now, let $\mathcal{X}^*$ be the dual space of $\mathcal{X}$ and let $\Theta \subset \mathcal{X}^*$ be countable and dense in $\mathcal{X}^*$ in the weak* topology. The existence of Θ is implied by the fact that the unit ball in $\mathcal{X}^*$ is metrizable and compact in the weak* topology.

We omit the instant proofs of the following two statements which are formulated as lemmas only for reference purposes.

LEMMA 3.2. *Let $\{X_n\}$ be a sequence of random vectors in $\mathcal{X}$. If the sequence $\{\mathcal{L}(X_n)\}$ is conditionally compact and if for each $x^* \in \Theta$ there exists $\lim_{n \rightarrow \infty} E \exp(ix^* X_n)$, then the sequence $\{\mathcal{L}(X_n)\}_{n \in \mathbb{N}}$ is weakly convergent.*

LEMMA 3.3. *Let $\{X_n\}$ be a sequence of random vectors in $\mathcal{X}$. If $\mathcal{L}(X_n) \xrightarrow{w} \mu \in \mathcal{M}(\mathcal{X})$ as $n \rightarrow \infty$, then for each $x^* \in \mathcal{X}^*$ there exists a limit distribution for the sequence of real random variables $x^* X_n$ (as $n \rightarrow \infty$).*

LEMMA 3.4. *Let X_n and Y_n for each $n = 1, 2, \dots$ be independent random vectors in $\mathcal{X}$. If the sequences $\{\mathcal{L}(X_n + Y_n)\}_{n \in \mathbb{N}}$ and $\{\mathcal{L}(X_n)\}_{n \in \mathbb{N}}$ are weakly convergent and if for each $x^* \in \Theta$ the limit distribution of the sequence of real random variables $x^* X_n$ is infinitely divisible, then also the sequence $\{\mathcal{L}(Y_n)\}_{n \in \mathbb{N}}$ is weakly convergent.*

Proof. By Lemma 3.3 the limit distribution of $\{x^* X_n\}$ exists for each $x^* \in \mathcal{X}^*$. Now, the independence assumption yields that

$$E \exp[ix^*(X_n + Y_n)] = E \exp[ix^* X_n] \cdot E \exp[ix^* Y_n],$$

and, by the infinite divisibility assumption, we have

$$\lim_{n \rightarrow \infty} E \exp[ix^* X_n] \neq 0 \quad \text{for each } x^* \in \Theta,$$

so that for each $x^* \in \Theta$ there exists the limit

$$\lim_{n \rightarrow \infty} E \exp[ix^* Y_n].$$

Moreover, Theorem III.2.1 of [6] implies that the sequence $\{\mathcal{L}(Y_n)\}_{n \in N}$ is conditionally compact and the application of Lemma 3.2 completes the proof.

Proof of Theorem 3.1. Sufficiency. The assumptions and Propositions 2.1 and 2.2 imply that the sequence of distributions of sums $S_{\nu_n}^n$ is conditionally compact and that μ is only the point of condensation. So the sequence of distributions of sums $S_{\nu_n}^n$ is weakly convergent to $\mu \in \mathcal{M}(\mathcal{X})$.

Necessity. The proof will be carried over in three steps. Let N_1 be an arbitrary but fixed throughout the proof subsequence of N .

Step I. *There exist a subsequence $N_2 \subset N_1$ and a subset $T \subset [0, 1)$, $|T| = 1$, such that for each $x^* \in \Theta$ there exists a real stochastic process $\xi_{x^*}(t)$, $t \in T$, with independent increments, which is stochastically continuous and such that, for each $t \in T$,*

$$x^* S_{l_n(t)}^n \xrightarrow{w} \xi_{x^*}(t) \quad \text{as } n \in N_2, n \rightarrow \infty.$$

Denote by D the set of dyadic numbers in $[0, 1)$. By Lemma 3.1, the sequence $\{\mathcal{L}(S_{l_n(t)}^n)\}_{n \in N}$ is conditionally compact for each $t \in [0, 1)$. Utilizing the Cantor diagonal procedure we may choose a subsequence $N_2 \subset N_1$ such that the sequence $\{\mathcal{L}(S_{l_n(t)}^n)\}_{n \in N_2}$ is convergent for each $t \in D$. By Lemma 3.4, the sequence $\{\mathcal{L}(S_{l_n(t_2)}^n - S_{l_n(t_1)}^n)\}_{n \in N_2}$ is also convergent for any $t_1, t_2 \in D$.

Let $x^* \in \Theta$. In view of Lemma 3.3, for each $t_1, t_2 \in D$ there exist limit distributions for real random variables $x^* S_{l_n(t_2)}^n - x^* S_{l_n(t_1)}^n$ as $n \in N_2$, $n \rightarrow \infty$. Szász has shown (cf. the proof of Theorem 2 in [9]) that, under these circumstances, there exists a real stochastic process $\xi_{x^*}(t)$, $t \in [0, 1)$, stochastically continuous at all except countably many $t \in [0, 1)$, that has independent increments and such that, at each point of continuity of ξ_{x^*} ,

$$x^* S_{l_n(t)}^n \xrightarrow{w} \xi_{x^*}(t) \quad \text{as } n \in N_2, n \rightarrow \infty.$$

To complete the proof of Step I it is enough to take as T the intersection of the sets of continuity of ξ_{x^*} , $x^* \in \Theta$.

Step II. *There exists a stochastic process $\tilde{\xi}(t)$, $t \in T$, with values in $\mathcal{X}$, stochastically continuous, with independent increments and such that, for each $t \in T$, $S_{l_n(t)}^n \xrightarrow{w} \tilde{\xi}(t)$ as $n \in N_2$, $n \rightarrow \infty$.*

Indeed, by Lemma 3.1 the sequence $\{\mathcal{L}(S_{l_n(t)}^n)\}_{n \in N_2}$ is conditionally compact and, by Step I and Lemma 3.2, it converges for each $t \in T$. In view of Lemma 3.4, for each $t_1, t_2 \in T$, the sequence $\{\mathcal{L}(S_{l_n(t_2)}^n - S_{l_n(t_1)}^n)\}_{n \in N_2}$ is also convergent. Now the Daniell-Kolmogorov Extension Theorem [6] yields the existence of the stochastic process $\tilde{\xi}(t)$, $t \in T$, with values in $\mathcal{X}$, with independent increments and such that, for each $t \in T$,

$S_{t_n(t)}^n \xrightarrow{w} \tilde{\xi}(t)$ as $n \in N_2, n \rightarrow \infty$. To prove that $\tilde{\xi}$ is stochastically continuous it is sufficient to notice that, for each $x^* \in \Theta$ and each $t \in T$, the random variables $x^* \tilde{\xi}(t)$ and $\xi_{x^*}(t)$ are equidistributed. Since ξ and $\tilde{\xi}_{x^*}$ are processes with independent increments and since the distributions of $x^* \tilde{\xi}(t)$ and $\xi_{x^*}(t)$ are infinitely divisible for each $t \in T$, the random variables $\xi_{x^*}(t_2) - \xi_{x^*}(t_1)$ and $x^* \tilde{\xi}(t_2) - x^* \tilde{\xi}(t_1)$ are equidistributed for each $t_1, t_2 \in T$ ($x^* \in \Theta$). Let $t_n \rightarrow t, t_n, t \in T$. Since the processes ξ_{x^*} are stochastically continuous on T , we have

$$x^* (\tilde{\xi}(t_n) - \tilde{\xi}(t)) \xrightarrow{P} 0 \quad \text{for each } x^* \in \Theta.$$

Now, to complete the proof of Step II it is sufficient to observe that the distributions $\{\mathcal{L}(\tilde{\xi}(t_n) - \tilde{\xi}(t))\}_{n \in N}$ are conditionally compact (here one has to apply inequality (2)) and to use Lemma 3.2.

Step III. *There exists a stochastic process $\xi(t), t \in [0, 1)$, which is $(\mathcal{B}_{[0,1)} \times \mathcal{F}, \mathcal{B}_{\mathcal{X}})$ -measurable, and such that, for $|\cdot|$ -almost $t \in [0, 1)$, $\xi(t) = \tilde{\xi}(t)$ with probability 1.*

Fix a sequence of partitions $\{t_k^n\}_{k=1}^n \subset T, n \in N$, such that

$$t_k^n < t_{k+1}^n \quad (k = 1, \dots, n-1), \quad \{t_k^n\}_{k=1}^n \subset \{t_k^{n+1}\}_{k=1}^{n+1}$$

and

$$\lim_{n \rightarrow \infty} \max_{0 \leq k \leq n} |t_{k+1}^n - t_k^n| = 0 \quad (t_0^n = 0, t_{n+1}^n = 1).$$

Now, we define a sequence of simple processes

$$\xi_n(t) = \begin{cases} \tilde{\xi}(t_1^n) & \text{for } t \in [0, t_1^n), \\ \tilde{\xi}(t_k^n) & \text{for } t \in [t_k^n, t_{k+1}^n), k = 1, \dots, n. \end{cases}$$

The processes ξ_n are evidently $(\mathcal{B}_{[0,1)} \times \mathcal{F}, \mathcal{B}_{\mathcal{X}})$ -measurable and, considered as functions on $[0, 1) \times \Omega$ into $\mathcal{X}$, they form a Cauchy sequence for convergence in measure. Let $\xi: [0, 1) \times \Omega \rightarrow \mathcal{X}$ be a limit in product measure of the sequence $\{\xi_n\}$. ξ is measurable and, for each $k \geq 1$,

$$\int_0^1 P \left\{ \|\xi_n(t) - \xi(t)\| \geq \frac{1}{k} \right\} dt \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Utilizing the Cantor diagonal procedure we are able to choose a subsequence $N^1 \subset N$ and a set $T_1, |T_1| = 1$, such that, for each $t \in T_1$ and each $k \in N$,

$$P \left\{ \|\xi_n(t) - \xi(t)\| \geq \frac{1}{k} \right\} \rightarrow 0 \quad \text{as } n \in N^1, n \rightarrow \infty,$$

which implies that $\xi_n(t) \xrightarrow{P} \xi(t)$ as $n \in N^1, n \rightarrow \infty$. Now, the stochastic

continuity of the process $\tilde{\xi}(t)$, $t \in T$, yields that

$$\xi_n(t) \xrightarrow{P} \tilde{\xi}(t) \quad \text{as } n \rightarrow \infty.$$

Thus for $t \in T \cap T_1$, $\xi(t) = \tilde{\xi}(t)$ with probability 1, which completes the proof of Theorem 3.1 in view of Propositions 2.1 and 2.2.

4. Stability of randomly indexed sums of random vectors in normed linear spaces. Throughout this section $(\mathcal{X}, \|\cdot\|)$ stands for a fixed normed linear space equipped with the natural norm topology.

Let X_{nj} ($n, j = 1, 2, \dots$) be random vectors in $\mathcal{X}$, and $\{v_n\}$ a sequence of random indices. We assume that random elements $v_n, X_{n1}, X_{n2}, \dots$ are independent for each $n \in \mathbf{N}$. As usual, $l_n(t)$ denotes the t -quantile of the random variable v_n ($t \in [0, 1]$). Put $S_k^n = X_{n1} + \dots + X_{nk}$.

Definition 4.1. The sequence $\{S_{v_n}^n\}$ is said to be *stable* if there exist vectors $B_k^n \in \mathcal{X}$ ($n, k = 1, 2, \dots$) such that

$$S_{v_n}^n - B_{v_n}^n \xrightarrow{P} 0 \quad \text{as } n \rightarrow \infty.$$

Definition 4.2. The sequence $\{S_{k_n}^n\}$ ($\{k_n\} \subset \mathbf{N}$, $n = 1, 2, \dots$) is said to *satisfy the (weak) law of large numbers* if there exist vectors $C_n \in \mathcal{X}$ ($n = 1, 2, \dots$) such that

$$S_{k_n}^n - C_n \xrightarrow{P} 0 \quad \text{as } n \rightarrow \infty.$$

The following theorem characterizes stable sequences $\{S_{v_n}^n\}$:

THEOREM 4.1. *The sequence $\{S_{v_n}^n\}$ is stable if and only if, for each $t \in (0, 1)$, the sequence $\{S_{l_n(t)}^n\}$ satisfies the law of large numbers.*

The proof of this theorem will rely on the following Lemmas 4.1 and 4.2.

LEMMA 4.1. *If the random vectors X_{nj} ($n, j = 1, 2, \dots$) are symmetric, then $S_{v_n}^n \xrightarrow{P} 0$ if and only if, for each $t \in (0, 1)$,*

$$S_{l_n(t)}^n \xrightarrow{P} 0 \quad \text{as } n \rightarrow \infty.$$

Proof. Fix $t \in (0, 1)$ and $\varepsilon > 0$. Applying inequality (2), we get

$$P\{\|S_{v_n}^n\| > \varepsilon\} \geq \sum_{k=l_n(t)}^{\infty} P\{\|S_k^n\| > \varepsilon\} P\{v_n = k\} \geq \frac{1-t}{2} P\{\|S_{l_n(t)}^n\| > \varepsilon\}$$

which proves the necessity. The sufficiency follows from Proposition 2.3.

LEMMA 4.2. *Let X_{nj} and Y_{nj} ($n, j = 1, 2, \dots$) be random vectors in $\mathcal{X}$ and let $\{v_n\}$ be a sequence of random indices. Assume that $v_n, X_{n1}, Y_{n1}, X_{n2}, Y_{n2}, \dots$ for each n are independent. Then, if*

$$\sum_{j=1}^{v_n} (X_{nj} - Y_{nj}) \xrightarrow{P} 0 \quad \text{as } n \rightarrow \infty,$$

then the sequence of sums $\{S_{v_n}^n\}$ is stable.

Proof. We define the functions $Q_n: \mathbf{R}^+ \rightarrow [0, 1]$ by the formula

$$Q_n(\varepsilon) = \sup \left\{ P \left(\left\| \sum_{k=1}^{r_n} X_{nk} - \sum_{k=1}^{r_n} y_k \right\| \leq \varepsilon \right) : \{y_k\}_{k=1}^{\infty} \subset \mathcal{X} \right\}, \quad \varepsilon \in \mathbf{R}^+.$$

By the independence argument and Proposition 2.1, we get

$$\begin{aligned} P \left\{ \left\| \sum_{j=1}^{r_n} (X_{nj} - Y_{nj}) \right\| \leq \varepsilon \right\} \\ &= \int_0^1 P \left\{ \left\| \sum_{j=1}^{l_n(t)} (X_{nj} - Y_{nj}) \right\| \leq \varepsilon \right\} dt \\ &= \int_0^1 P \times P \left\{ (\omega, \omega') : \left\| \sum_{j=1}^{l_n(t)} X_{nj}(\omega) - \sum_{j=1}^{l_n(t)} Y_{nj}(\omega') \right\| \leq \varepsilon \right\} dt \\ &= \int_{\Omega} P \left\{ \left\| \sum_{j=1}^{r_n} X_{nj} - \sum_{j=1}^{r_n} Y_{nj}(\omega') \right\| \leq \varepsilon \right\} P(d\omega') \leq Q_n(\varepsilon), \end{aligned}$$

so that, by our assumption, for each $\varepsilon > 0$, $Q_n(\varepsilon) \rightarrow 1$ as $n \rightarrow \infty$. Consequently, for each $r \in \mathbf{N}$ there exists $n_r \in \mathbf{N}$ ($n_1 < n_2 < \dots$) such that

$$Q_n\left(\frac{1}{r}\right) > 1 - \frac{1}{r} \quad \text{for } n \geq n_r.$$

Now, by virtue of the definition of Q_n , we are able to find, for each $n \in \mathbf{N}$, a sequence $\{b_{nj}\}_{j=1}^{\infty}$ such that if $n_r \leq n < n_{r+1}$, then

$$P \left\{ \left\| \sum_{j=1}^{r_n} X_{nj} - \sum_{j=1}^{r_n} b_{nj} \right\| \leq \frac{1}{r} \right\} > 1 - \frac{1}{r}.$$

Hence

$$\sum_{j=1}^{r_n} (X_{nj} - b_{nj}) \xrightarrow{P} 0 \quad \text{as } n \rightarrow \infty$$

which was to be shown.

Proof of Theorem 4.1. Let $\{Y_{nj}\}$ ($n, j = 1, 2, \dots$) be a sequence of random vectors in $\mathcal{X}$ such that $r_n, X_{n1}, Y_{n1}, \dots$ for each $n \in \mathbf{N}$ are independent and Y_{nj} and X_{nj} for each $n, j \in \mathbf{N}$ are equidistributed.

Assume $\{S_{r_n}^n\}$ is stable. Then

$$\sum_{j=1}^{r_n} (X_{nj} - Y_{nj}) \xrightarrow{P} 0$$

and, by virtue of Lemma 4.1, for each $t \in (0, 1)$,

$$\sum_{j=1}^{l_n(t)} (X_{nj} - Y_{nj}) \xrightarrow{P} 0 \quad \text{as } n \rightarrow \infty.$$

Fix $t \in (0, 1)$, and write a new sequence of indices $\tilde{v}_n = l_n(t)$. Clearly, $\{X_{nj}\}$, $\{Y_{nj}\}$ and $\{\tilde{v}_n\}$ satisfy the assumption of Lemma 4.2, so that there are vectors $C_k^n \in \mathcal{X}$ such that

$$S_{\tilde{v}_n}^n - C_{\tilde{v}_n}^n \xrightarrow{P} 0 \quad \text{as } n \rightarrow \infty$$

which proves the first part of the Theorem.

Assume that $S_{l_n(t)}^n$ for each $t \in (0, 1)$ satisfies the law of large numbers. Then, for each $t \in (0, 1)$,

$$\sum_{j=1}^{l_n(t)} (X_{nj} - Y_{nj}) \xrightarrow{P} 0 \quad \text{as } n \rightarrow \infty$$

and, by Lemma 4.1, also

$$\sum_{j=1}^{v_n} (X_{nj} - Y_{nj}) \xrightarrow{P} 0,$$

and the application of Lemma 4.2 completes the proof.

We conclude this section with the following observation. Making a comparison between the proofs of Lemmas 3.1 and 4.1, it is not difficult to show the following strengthening of one half of Lemma 4.1:

LEMMA 4.1.A. *Let X_{nj} ($n, j = 1, 2, \dots$) be symmetrically distributed. If there exists a sequence $\{t_k\} \subset (0, 1)$, $t_k \rightarrow 1$ ($k \rightarrow \infty$), such that, for each $k \in \mathbf{N}$, $S_{l_n(t_k)}^n \xrightarrow{P} 0$ as $n \rightarrow \infty$, then*

$$S_{v_n}^n \xrightarrow{P} 0 \quad \text{as } n \rightarrow \infty.$$

This, in turn, makes it possible to strengthen the assertion of one half of Theorem 4.1 as follows:

THEOREM 4.1.A. *If there exists a sequence $\{t_k\} \subset (0, 1)$, $t_k \rightarrow 1$ ($k \rightarrow \infty$), such that, for each $k \in \mathbf{N}$, the sequence $\{S_{l_n(t_k)}^n\}$ satisfies the law of large numbers, then the sequence $\{S_{v_n}^n\}$ is stable.*

5. Stability in Hilbert spaces. In this section we restrict our attention to the case $\mathcal{X} = H$, where H is a separable Hilbert space with the inner product $(\cdot, \cdot)$ and an orthonormal basis $\{e_j\} \subset H$ (as usual, $\|\cdot\| = \sqrt{(\cdot, \cdot)}$), and apply Theorem 4.1 to prove another, and perhaps more handy, characterization of stability of randomly indexed sums of random vectors in H . In the case $H = \mathbf{R}$, the sufficiency was proved originally in [1] under some additional restrictions, and the full characterization is due to Szász [10].

Let $X_{n1}, X_{n2}, \dots$ for each $n \in \mathbf{N}$ be independent random vectors in H such that

$$\limsup_{n \rightarrow \infty} \sup_{j \geq 1} \mathbf{P}\{\|X_{nj}\| > \varepsilon\} = 0 \quad \text{for each } \varepsilon > 0,$$

and let $\{\nu_n\}$ be a sequence of random indices such that, for each n , ν_n is independent of $X_{n1}, X_{n2}, \dots$ and $\nu_n \xrightarrow{P} \infty$ as $n \rightarrow \infty$.

THEOREM 5.1. *The sequence of sums $\{S_{\nu_n}^n\}$ is stable if and only if the set of following conditions is satisfied:*

For each $\varepsilon > 0$,

$$(A) \quad \sum_{j=1}^{\nu_n} \mu_{nj}(\|x - x_{nj}\| > \varepsilon) \xrightarrow{P} 0,$$

$$(B) \quad \sum_{j=1}^{\nu_n} \int_{\|x\| < \varepsilon} (x - x_{nj}, y)^2 \mu_{nj}(dx) \xrightarrow{P} 0, \quad y \in H,$$

$$(C) \quad \limsup_{M \rightarrow \infty} \sup_n P \left\{ \sum_{j=1}^{\nu_n} \int_{\|x\| \leq 1} \|x - x_{nj}\|^2 \mu_{nj}(dx) \geq M \right\} = 0,$$

$$(D) \quad \limsup_{m \rightarrow \infty} \sup_n P \left\{ \sum_{j=1}^{\nu_n} \int_{\|x\| \leq 1} r_m^2(x - x_{nj}) \mu_{nj}(dx) \geq \varepsilon \right\} = 0,$$

where μ_{nj} is the distribution of the random vector X_{nj} ,

$$x_{nj} = \int_{\|x\| \leq 1} x \mu_{nj}(dx) \quad \text{and} \quad r_m^2(x) = \sum_{j=m}^{\infty} (x, e_j)^2.$$

Proof. By Theorem 4.1, the sequence $\{S_{\nu_n}^n\}$ is stable if and only if $\{S_{l_n(t)}^n\}$ for each $t \in (0, 1)$ satisfies the law of large numbers. On the other hand, $\nu_n \xrightarrow{P} \infty$ if and only if, for each $t \in (0, 1)$, $l_n(t) \rightarrow \infty$ as $n \rightarrow \infty$, so that, by virtue of Theorem 6.3 of [6], p. 200, $\{S_{\nu_n}^n\}$ is stable if and only if, for each $t \in (0, 1)$ and $\varepsilon > 0$, the following conditions are satisfied:

$$(A_1) \quad \lim_{n \rightarrow \infty} \sum_{j=1}^{l_n(t)} \mu_{nj}(\|x - x_{nj}\| > \varepsilon) = 0,$$

$$(B_1) \quad \lim_{n \rightarrow \infty} \sum_{j=1}^{l_n(t)} \int_{\|x\| < \varepsilon} (x - x_{nj}, y)^2 \mu_{nj}(dx) = 0, \quad y \in H,$$

$$(C_1) \quad \sup_n \sum_{j=1}^{l_n(t)} \int_{\|x\| \leq 1} \|x - x_{nj}\|^2 \mu_{nj}(dx) < \infty,$$

$$(D_1) \quad \limsup_{m \rightarrow \infty} \sup_n \sum_{j=1}^{l_n(t)} \int_{\|x\| \leq 1} r_m^2(x - x_{nj}) \mu_{nj}(dx) = 0.$$

Corollary 2.2 implies the equivalences $(A) \Leftrightarrow (A_1)$ and $(B) \Leftrightarrow (B_1)$, so that the proof will be complete as soon as we show that $(C) \Leftrightarrow (C_1)$ and $(D) \Leftrightarrow (D_1)$.

$(C) \Rightarrow (C_1)$. In view of Proposition 2.1, condition (C) may be rewritten as follows:

$$(4) \quad \lim_{M \rightarrow \infty} \sup_n \left| \left\{ t: \sum_{j=1}^{l_n(t)} \int_{\|x\| \leq 1} \|x - x_{nj}\|^2 \mu_{nj}(dx) \geq M \right\} \right| = 0.$$

If, for some $t_0 \in (0, 1)$, the supremum in condition (C₁) were infinite, then, for each M , we would have the inequality

$$\sup_n \left| \left\{ t: \sum_{j=1}^{l_n(t)} \int_{\|x\| \leq 1} \|x - x_{nj}\|^2 \mu_{nj}(dx) \geq M \right\} \right| \geq 1 - t_0$$

which contradicts (4).

(C₁) $\Rightarrow$ (C). The Lebesgue Dominated Convergence Theorem yields

$$\begin{aligned} \lim_{M \rightarrow \infty} \sup_n \left| \left\{ t: \sum_{j=1}^{l_n(t)} \int_{\|x\| \leq 1} \|x - x_{nj}\|^2 \mu_{nj}(dx) \geq M \right\} \right| \\ \leq \lim_{M \rightarrow \infty} \left| \left\{ t: \sup_n \sum_{j=1}^{l_n(t)} \int_{\|x\| \leq 1} \|x - x_{nj}\|^2 \mu_{nj}(dx) \geq M \right\} \right| \\ = \left| \left\{ t: \sup_n \sum_{j=1}^{l_n(t)} \int_{\|x\| \leq 1} \|x - x_{nj}\|^2 \mu_{nj}(dx) = \infty \right\} \right| = 0. \end{aligned}$$

(D) $\Rightarrow$ (D₁). Assume to the contrary that, for some $t_0 \in (0, 1)$,

$$(5) \quad \lim_{m \rightarrow \infty} \sup_n \sum_{j=1}^{l_n(t_0)} \int_{\|x\| \leq 1} r_m^2(x - x_{nj}) \mu_{nj}(dx) = \varepsilon_1 > 0.$$

By Proposition 2.1, condition (D) can be restated as follows:

$$\lim_{m \rightarrow \infty} \sup_n \left| \left\{ t: \sum_{j=1}^{l_n(t)} \int_{\|x\| \leq 1} r_m^2(x - x_{nj}) \mu_{nj}(dx) \geq \varepsilon \right\} \right| = 0 \quad \text{for each } \varepsilon > 0.$$

However, (5) implies

$$\sup_n \left| \left\{ t: \sum_{j=1}^{l_n(t)} \int_{\|x\| \leq 1} r_m^2(x - x_{nj}) \mu_{nj}(dx) \geq \varepsilon_1 \right\} \right| \geq 1 - t_0 \quad \text{for each } m$$

which gives the necessary contradiction.

(D₁) $\Rightarrow$ (D). We have

$$\begin{aligned} \lim_{m \rightarrow \infty} \sup_n \left| \left\{ t: \sum_{j=1}^{l_n(t)} \int_{\|x\| \leq 1} r_m^2(x - x_{nj}) \mu_{nj}(dx) \geq \varepsilon \right\} \right| \\ \leq \lim_{m \rightarrow \infty} \left| \left\{ t: \sup_n \sum_{j=1}^{l_n(t)} \int_{\|x\| \leq 1} r_m^2(x - x_{nj}) \mu_{nj}(dx) \geq \varepsilon \right\} \right| \\ = \left| \left\{ t: \lim_{m \rightarrow \infty} \sup_n \sum_{j=1}^{l_n(t)} \int_{\|x\| \leq 1} r_m^2(x - x_{nj}) \mu_{nj}(dx) \geq \varepsilon \right\} \right| = 0, \end{aligned}$$

and the proof of Theorem 5.1 is complete.

6. Applications to real random variables. Applying the results of Section 2, one can easily obtain certain sufficient conditions for weak convergence of randomly indexed sums of random variables. We display this possibility on two examples.

Let $X_{n1}, X_{n2}, \dots$ for each n be independent real random variables such that

$$\limsup_{n \rightarrow \infty} \mathbf{P}\{|X_{nj}| > \varepsilon\} = 0 \quad \text{for each } \varepsilon > 0,$$

and let $\{\nu_n\}$ be a sequence of random indices such that ν_n for each $n \in N$ is independent of $X_{n1}, X_{n2}, \dots$ and $\nu_n \xrightarrow{\mathbf{P}} \infty$ as $n \rightarrow \infty$. Denote by F_{nj} the distribution function of X_{nj} , and

$$a_{nj} = \int_{|x| < 1} x dF_{nj}(x) \quad \text{and} \quad S_k^n = X_{n1} + \dots + X_{nk}.$$

THEOREM 6.1. *Let Y be a real random variable having the infinitely divisible distribution with Lévy-Khintchin representation (γ, G) . Assume that, for a sequence of constants $\{B_k^n\} \subset \mathbf{R}$ and for each point x of continuity of G and for $x = \pm \infty$, the following two conditions hold:*

$$(i) \quad \sum_{j=1}^{\nu_n} \int_{-\infty}^x \frac{y^2}{1+y^2} dF_{nj}(y+a_{nj}) \xrightarrow{\mathbf{P}} G(x) \quad \text{as } n \rightarrow \infty,$$

$$(ii) \quad \sum_{j=1}^{\nu_n} \left[a_{nj} + \int_{-\infty}^{\infty} \frac{x}{1+x^2} dF_{nj}(x+a_{nj}) \right] - B_{\nu_n}^n \xrightarrow{\mathbf{P}} \gamma \quad \text{as } n \rightarrow \infty.$$

Then $S_{\nu_n}^n - B_{\nu_n}^n \xrightarrow{\mathbf{w}} Y$ as $n \rightarrow \infty$.

Proof. Corollary 2.2 implies that condition (i) is equivalent to the following one: for each $t \in (0, 1)$ and for each point x of continuity of G and for $x = \pm \infty$,

$$(6) \quad \sum_{j=1}^{l_n(t)} \int_{-\infty}^x \frac{y}{1+y^2} dF_{nj}(y+a_{nj}) \rightarrow G(x) \quad \text{as } n \rightarrow \infty.$$

We introduce the notation

$$Y_{nk} = S_k^n - B_k^n, \quad Z_{nk} = \sum_{j=1}^k \left[a_{nj} + \int_{-\infty}^{\infty} \frac{x}{1+x^2} dF_{nj}(x+a_{nj}) \right] - B_k^n.$$

Let N_1 be an arbitrary sequence in N . By (ii),

$$Z_{\nu_n} \xrightarrow{\mathbf{P}} \gamma,$$

so that, by Proposition 2.3, we can find the subsequence $N_2 \subset N_1$ and a subset $T \subset [0, 1)$, $|T| = 1$, such that, for each $t \in T$,

$$\lim_{\substack{n \in N_2 \\ n \rightarrow \infty}} Z_{nl_n(t)} = \lim_{\substack{n \in N_2 \\ n \rightarrow \infty}} \sum_{j=1}^{l_n(t)} \left[a_{nj} + \int_{-\infty}^{\infty} \frac{x}{1+x^2} dF_{nj}(x + a_{nj}) \right] - B_{l_n(t)}^n = \gamma.$$

Since $v_n \xrightarrow{P} \infty$, for each $t \in (0, 1)$ we have $l_n(t) \rightarrow \infty$ as $n \rightarrow \infty$. Hence, by (6) and Theorem VI.22.3.B of [5], for each $t \in T \cap (0, 1)$

$$Y_{nl_n(t)} = S_{l_n(t)}^n - B_{l_n(t)}^n \xrightarrow{w} Y \quad \text{as } n \in N_2, n \rightarrow \infty,$$

and Proposition 2.3 implies that

$$Y_{nv_n} = S_{v_n}^n - B_{v_n}^n \xrightarrow{w} Y \quad \text{as } n \rightarrow \infty.$$

As the second application of our earlier results we provide an alternative (and simpler) proof of Theorem 1 of [8] which is a random version of Lindeberg's Central Limit Theorem.

Assume $X_1, X_2, \dots$ are independent real random variables with $\sigma_n^2 = EX_n^2 < \infty$, $EX_n = 0$. Write $s_n^2 = \sigma_1^2 + \dots + \sigma_n^2$ and $S_n = X_1 + \dots + X_n$. As before, $\{v_n\}$ is a sequence of random indices such that v_n for each $n \in N$ are independent of $X_1, X_2, \dots$ and $v_n \xrightarrow{P} \infty$ as $n \rightarrow \infty$.

THEOREM 6.2. *Assume that, for each $\varepsilon > 0$,*

$$(7) \quad \frac{1}{s_{v_n}^2} \sum_{j=1}^{v_n} \int_{|x| > \varepsilon s_{v_n}} x^2 dF_j(x) \xrightarrow{P} 0.$$

Then the distributions of random variables S_{v_n}/s_{v_n} are weakly convergent to the normal distribution $N(0, 1)$.

Proof. By Proposition 2.3, condition (7) is equivalent to the following one: for each $\varepsilon > 0$ and each sequence $N_1 \subset N$, there exists a subsequence $N_2 \subset N_1$ such that, for almost all $t \in [0, 1)$,

$$(8) \quad \frac{1}{s_{l_n(t)}^2} \sum_{j=1}^{l_n(t)} \int_{|x| > \varepsilon s_{l_n(t)}} x^2 dF_j(x) \rightarrow 0 \quad \text{as } n \in N_2, n \rightarrow \infty.$$

Write $Y_{nk} = S_k/s_k$ ($n, k = 1, 2, \dots$) and let Y be a random variable with distribution $N(0, 1)$. Given a fixed subsequence $N_1 \subset N$, utilizing (8) and the Cantor diagonal procedure, one can choose a subsequence $N_2 \subset N_1$ and a subset $T \subset [0, 1)$, $|T| = 1$, such that, for each $t \in T$ and each $r = 1, 2, \dots$,

$$\frac{1}{s_{l_n(t)}^2} \sum_{j=1}^{l_n(t)} \int_{|x| > s_{l_n(t)}/r} x^2 dF_j(x) \rightarrow 0 \quad \text{as } n \in N_2, n \rightarrow \infty.$$

Hence, for each $t \in T$ and each $\varepsilon > 0$,

$$\frac{1}{s_{l_n(t)}^2} \sum_{j=1}^{l_n(t)} \int_{|x| > \varepsilon s_{l_n(t)}} x^2 dF_j(x) \rightarrow 0 \quad \text{as } n \in N_2, n \rightarrow \infty,$$

and so, for each $t \in T \cap (0, 1)$, since the classical Lindeberg's conditions are satisfied (cf. [5], Theorem IV.20.2.B), we have

$$Y_{nl_n(t)} = \frac{S_{l_n(t)}}{s_{l_n(t)}} \xrightarrow{w} Y \quad \text{as } n \in N_2, n \rightarrow \infty.$$

This and Proposition 2.3 yield the assertion of Theorem 6.2.

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REFERENCES

- [1] А. П. Бикялис и Й. Модьороди, *Об одной задаче* Б. В. Гнеденко, *Acta Scientiarum Mathematicarum* 30 (1969), p. 241-245.
- [2] N. Dunford and J. T. Schwartz, *Linear operators*, Part I, New York 1958.
- [3] Д. Саас и Б. Фрайер, *Одна задача теории суммирования со случайным индексом*, *Литовский математический сборник* 11 (1971), p. 181-187.
- [4] Б. В. Гнеденко и Х. Фахим, *Об одной теореме переноса*, *Доклады Академии наук СССР* 187 (1969), p. 15-17.
- [5] M. Loève, *Probability theory*, Princeton 1955.
- [6] K. R. Parthasarathy, *Probability measures on metric spaces*, New York - London 1967.
- [7] H. Robbins, *The asymptotic distribution of the sums of a random number of random variables*, *Bulletin of the American Mathematical Society* 54 (1948), p. 1151-1161.
- [8] Z. Rychlik, *A central limit theorem for the sums of a random number of independent random variables*, *Colloquium Mathematicum* (in print).
- [9] D. Szász, *Limit theorems for the distributions of the sums of a random number of random variables*, *The Annals of Mathematical Statistics* 43 (1972), p. 1902-1913.
- [10] — *Stability and law of large numbers for sums of a random number of random variables*, *Acta Scientiarum Mathematicarum* 33 (1972), p. 269-274.

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