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THE STRONG AMALGAMATION PROPERTY

BY

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0. Introduction. A model A of a theory T is called a *strong amalgamation base* if any two extensions of A in T overlapping only in A have a common extension in T. We shall investigate the uses of this notion in Model Theory and prove the following Main Theorem which reduces this concept to known ones:

If T is any theory and A is a model of T, then A is a strong amalgamation base just if A is an amalgamation base and is algebraically closed.

As this paper continues the investigation of [1] into algebraic elements, we shall employ the (fairly standard) notation of that paper, and some of the results.

The lay-out of this paper is as follows. In Section 1 we show that, for a universal theory with the amalgamation property (AP), the notion of strong amalgamation base provides a good explication of the notion of algebraical closedness (in that it has all the expected properties), and is sandwiched between the notions of existential closedness and algebraical closedness in the usual (Robinson-Jónsson) sense. In particular, we obtain the easy (necessity) direction of the Main Theorem.

Then in Section 2 we show that if T has a model-completion, then every algebraically closed model of T is a strong amalgamation base. This gives a "non-combinatorial" proof of a theorem of Park [7].

In Section 3 we use Park's result and some facts from [1] to prove the Main Theorem, and in Section 4 we discuss some related results. Finally, in Section 5 we give a rather straightforward "algebraic" proof of the Main Theorem for the special case of equational theories; in fact, this was the author's original proof.

It will help if we introduce some new terminology. We call a pair $A \to B$, $A \to C$ of injections a *wedge at A* and write it as $B \leftarrow A \to C$; such a wedge has a *strong amalgam* $B \to D$, $C \to D$ if the square of injections

$$\begin{tikzcd} & B \arrow[dr] & \\ A \arrow[ur] \arrow[dr] & & D \\ & C \arrow[ur] & \end{tikzcd}$$

commutes, and $B' \cap C' = A'$, where B' is the image of B in D (and similarly for C, A). Sometimes we call the whole square a strong amalgam. A model A of T is called a ***strong amalgamation base*** (SA base) for T if every wedge at A has a strong amalgam (with all structures models of T). A theory T has the ***strong amalgamation property*** (SAP) if every model of T is a strong amalgamation base.

Note that if we leave out the intersection condition, all these properties become the well-known amalgamation ones. Note also that for a model A of T the following conditions depend only on the ***universal part*** $T \cap \forall_1$ of T (as any model of $T \cap \forall_1$ has an extension to a model of T):

A is a strong amalgamation base,

A is an amalgamation base,

A is algebraically closed (in every extension which is a model of T).

Thus throughout we need only consider universal theories.

I should like to thank Ed Fisher and Paul Eklof for their helpful conversations and correspondence on this subject. I first proved the Main Theorem for the two special cases "T is universal Horn" and "T has a model-completion". This work appeared in a preprint *Strong amalgamation bases*. Somewhat later I proved it for all countable T by a rather unpleasant method. Shortly afterwards I received an early draft of Eklof's preprint [3] proving it for the case "T has AP" by an inductive approach. Spurred on by this, I managed to prove the general result in this paper (which supersedes the previous one).

1. A new notion of algebraical closedness. Let T be a universal theory with AP. Several authors, among them Jónsson [4], Robinson [9] and [10] (in two ways), and Simmons [11], have investigated general notions of algebraical closedness. It seems to us that any class $\mathscr{A}$ of algebraically closed models of T should satisfy the following axioms:

(1) Every existentially closed (e.c.) model of T belongs to $\mathscr{A}$.

(2) $\mathscr{A}$ is inductive.

(3) If $B \prec_1 A \in \mathscr{A}$, then $B \in \mathscr{A}$.

(4) $\mathscr{A}$ is *convex*, i.e. if $B_1 \leqslant C$, $B_2 \leqslant C$ and $B_1, B_2 \in \mathscr{A}$, then $B_1 \cap B_2 \in \mathscr{A}$.

We shall now show that the class $\mathscr{A}$ of SA bases for T satisfies these four axioms.

To begin with, (1) follows from a lemma due to Ed Fisher (oral communication) which, for completeness, we shall prove here.

LEMMA 1.1 (Fisher). *Let T be any universal theory. Then each e.c. model of T is an SA base.*

Proof. Let $B \leftarrow A \rightarrow C$ be a wedge of $\mathscr{M}(T)$. We can assume that $A \rightarrow B$, $A \rightarrow C$ are inclusions and $B \cap C = A$. It is easy to see that this has a strong amalgam just if

$$(*) \qquad T + \Delta_0(B) + \Delta_0(C) + \{b \neq c \colon\ b \in B \setminus A \ \&\ c \in C \setminus A\}$$

is consistent in $L(B \cup C)$.

Now suppose that $A \prec_1 C$ and that the theory $(*)$ is inconsistent. Then there are open formulae $\theta_1(\bar{z}, \bar{x})$ and $\theta_2(\bar{z}, \bar{y})$ and lists $\bar{a} \in A$, $\bar{b} \in B \setminus A$ and $\bar{c} \in C \setminus A$ such that

$$T + \theta_1(\bar{a}, \bar{b}) + \theta_2(\bar{a}, \bar{c}) + \bigwedge_{i,j} b_i \neq c_j$$

is inconsistent, and

$$B \models \theta_1(\bar{a}, \bar{b}), \qquad C \models \theta_2(\bar{a}, \bar{c}).$$

Hence

$$T \vdash \theta_1(\bar{z}, \bar{x}) \ \&\ \theta_2(\bar{z}, \bar{y}) \to \bigvee_{i,j} x_i = y_j,$$

and

$$A \models \theta_2(\bar{a}, \bar{d}) \qquad \text{for some } \bar{d} \in A$$

(as $A \prec_1 C$). Thus

$$B \models \theta_1(\bar{a}, \bar{b}) \ \&\ \theta_2(\bar{a}, \bar{d}),$$

and so $b_i = d_j \in A$ for some i, j. Since $b_i \notin A$, we obtain a contradiction. If A is e.c., then $A \prec_1 C$ for each $C \geqslant A$, and the result follows.

There are two remarks worth making.

1. We shall often use the stronger result proved within Lemma 1.1.

2. An alternative proof has been given by Eklof: he has constructed a diagram of injections

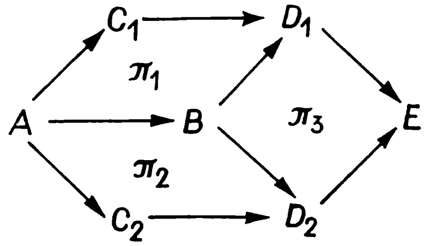
$$\begin{tikzcd}
 & B \arrow[rr] \arrow[dr, phantom, "\pi"] & & B^I/D \\
A \arrow[ur] \arrow[rr] \arrow[dr] & & A^I/D \arrow[ur] & \\
 & C \arrow[ur] & &
\end{tikzcd}$$

for a suitable set I and an ultrafilter D on I, and observed that the square π is a strong amalgam by well-known results on ultrapowers. (In fact, such results were used in Park's thesis [7], which we shall come to later.) However, the proof of Lemma 1.1 seems more natural and is the one that leads on to further considerations.

Since axiom (2) follows by an easy compactness argument, using the theory $(*)$ of the proof of Lemma 1.1, we move on to (3).

LEMMA 1.2. *Let T be any universal theory, let $A \prec_1 B \models T$, and let B be an SA base for T. Then A is an SA base for T.*

Proof. Let $C_1 \leftarrow A \to C_2$ be a wedge in $\mathscr{M}(T)$. As $A \prec_1 B$, in the following diagram we can construct π_1 and π_2 and then, B being an SA base, also π_3 such that each π_i is a strong amalgam:

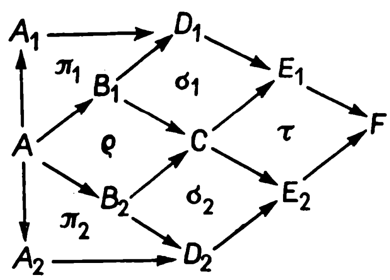


Then it is easy to see that $C_1 \to E$, $C_2 \to E$ is a strong amalgam of $C_1 \leftarrow A \to C_2$.

By using a more complicated diagram, we can prove (4) as follows:

LEMMA 1.3. *Let T be a universal theory with models C, B_1 and B_2, where $B_i \leqslant C$ $(i = 1, 2)$ and B_i is an SA base for T. Let $A = B_1 \cap B_2$. If A is an amalgamation base, then A is, in fact, an SA base for T.*

Proof. By extending C to an e.c. model, we can assume that C is also an SA base. Given a wedge $A_1 \leftarrow A \to A_2$, we construct a strong amalgam by building up the following diagram of injections in the sequence π_1, π_2, σ_1, σ_2, τ:



In this diagram σ_1, σ_2 and τ are strong amalgams (as B_1, B_2 and C are SA bases), π_1 and π_2 are amalgams (as A is an amalgamation base), and ϱ is a pre-existing strong amalgam (by the definition of A). It is easy to check that $D_1 \to F$, $D_2 \to F$ is a strong amalgam of $A_1 \leftarrow A \to A_2$.

In fact, $\mathscr{A}$ satisfies one more condition which determines it uniquely. Namely, it is the smallest class satisfying (1) and (4). This follows from

LEMMA 1.4. *Let T be any universal theory and let A be an SA base for T.*

(i) *There are e.c. models B_1, B_2 and C of T such that $B_1, B_2 \leqslant C$ and $A = B_1 \cap B_2$.*

(ii) *A is algebraically closed.*

Proof. Let $A \to A'$ be any inclusion and let $A' \leqslant B$, where B is e.c. There are $C \models T$ and an injection f: $B \to C$ such that $A = B \cap fB$, A being an SA base, and, by a further extension, we may assume that C is e.c. Clearly, fB is e.c. and $A = B \cap fB$. This proves (i).

For (ii) it suffices to note that A is algebraically closed in C (as so are B and fB), and thus in A'. Since A' was an arbitrary extension of A, A is algebraically closed.

We have proved a little more in these four lemmas than we originally wanted. Firstly, if we add to axiom (4) the condition "$B_1 \cap B_2$ is an amalgamation base" and call the resulting axiom (4′), then the lemmas, in fact, show that, for any universal theory (not necessarily having AP), the class of SA bases satisfies axioms (1)-(3) and (4′) and is the smallest such class.

Secondly, part (ii) of Lemma 1.4 establishes the easier direction of the Main Theorem: every strong amalgamation base is an algebraically closed amalgamation base.

One could derive this in a slightly different way, as it is not very difficult to prove directly that the class of algebraically closed amalgamation bases satisfies (1)-(3) and (4′), and so contains $\mathscr{A}$.

2. Park's theorem. Now let us consider a topic which looks rather different — Park's work on convexity properties (see [7]). He used a notion of "algebraic" which we can describe as follows. Given an L-structure D, a subset Y of D and an element $a \in D$, we say that a is *Park-algebraic* over Y in D if there is an L-formula $\theta(x, \bar{z})$, a list $\bar{b} \in Y$, and an integer n such that

$$D \models \theta(a, \bar{b}) \;\&\; \exists^{\leqslant n} x \theta(x, \bar{b}).$$

Then we call Y *Park-algebraically closed* (Park - a.c.) in D if no element of $D \setminus Y$ is Park-algebraic over Y in D.

As the key to Chapter 2 of his thesis, Park proved the following remarkable theorem (Theorem 2.3 of [7]):

If A is Park - a.c. in B, there exist $C \succ B$ and $B' \prec C$ such that $A = B \cap B'$.

His proof used a rather ingenious combinatorial result. We shall give a model-theoretic proof avoiding combinatorial considerations; in this proof we shall use the fact that the theorem of Park is essentially equivalent to a special case of our Main Theorem.

THEOREM 2.1. *Let T be a universal theory with AP and a model-companion K. Then every algebraically closed model of T is an SA base.*

The proof uses three lemmas.

LEMMA 2.2. *Under the hypotheses of Theorem 2.1, the class $\mathscr{A}$ of SA bases for T is elementary.*

Proof. Let P, Q and R be unary predicates and let K^* be the theory with axioms

$$K + K^P + K^Q + \forall x \;\; (Rx \leftrightarrow Px \;\&\; Qx) + T^R.$$

Then, by Lemmas 1.3 and 1.4,

$$\mathscr{A} = \{D\colon \; (A, B, C, D) \models K^*\}.$$

Hence $\mathscr{A}$ is closed under ultraproduct. But $\mathscr{A}$ is also closed under elementary substructures (by Lemma 1.2). The result follows.

LEMMA 2.3 (Kueker [5], Lemma 2.3). *Let G be a special L-structure and let $A \leqslant G$ be Park-a.c. in G with $|G| > |L| + |A|$. Then*

$$A = \bigcap\{C\colon A \leqslant C \prec G\}.$$

The following assertion is essentially Theorem 5.2 of [1] (with $B = G$ there):

LEMMA 2.4. *Let T be any universal theory, let G be a generic model of T, and let $A \leqslant G$ be algebraically closed in G. Then A is Park-a.c. in G.*

Proof of Theorem 2.1. Let A be an algebraically closed model of T, let $B \geqslant A$ be e.c. (hence a model of K), and let $G \succ B$ be a special L-structure with $|G| > |L| + |B|$. Thus $G \models K$, and so G is also generic for T. Now A is algebraically closed in G, so Park-a.c. in G; thus $A = \bigcap\{C\colon A \leqslant C \prec G\}$. But if $C \prec G$, then C is e.c., and so $C \in \mathscr{A}$. Also $\mathscr{A}$ is convex and elementary, and so closed under arbitrary intersection. Hence $A \in \mathscr{A}$.

COROLLARY 2.5 (Park). *Let A be Park-a.c. in B. Then there are $C \succ B$ and $B' \prec C$ with $A = B \cap B'$ and also B' isomorphic to B.*

Proof. Let B^* be the full expansion of B in the sense of Morley and let T_B be the theory with axioms all universal sentences of $\mathrm{Th}(B^*)$. Then, by Lemma 1.2 of [6], T_B is a universal theory with AP which has a model-companion $K = \mathrm{Th}(B^*)$. It is easy to check that the substructure of B^* built on A is algebraically closed in B^*, and so algebraically closed (as B^* is e.c.). Hence A is an SA base for T_B by Theorem 2.1. Thus we have a strong amalgam

$$\begin{tikzcd}[column sep=small] & B^* \arrow[dr] & \\ A \arrow[ur] \arrow[dr] & & C^* \\ & B^* \arrow[ur] & \end{tikzcd}$$

in $\mathscr{M}(T_B)$, where $C^* \models \mathrm{Th}(B^*)$. The result follows by taking reducts to the language L of T.

In fact, it is possible to prove a variant of Kueker's lemma with "special" replaced by "homogeneous universal for T", "Park-a.c." replaced by "algebraically closed", and $\prec$ replaced by $\prec_1$. This would replace the use of Lemmas 2.3 and 2.4 in the proof of Theorem 2.1.

By looking hard at this new proof, one can also eliminate the hypothesis of having a model-companion. This would give an alternative proof of Theorem 2.3 of [3]. But in the next section we shall prove a still more general result.

3. The Main Theorem. The proof of the Main Theorem is now fairly easy.

THEOREM 3.1. *Let T be any theory and let A be an algebraically closed amalgamation base for T. Then A is a strong amalgamation base for T.*

Proof. As mentioned in the Introduction, we can assume that T is universal. Now let G be a generic model of T such that $A \leqslant G$. Then A is Park-a.c. in G (by Lemma 2.4), and so (by Corollary 2.5) there are $H \succ G$ and $G' \prec H$ isomorphic to G such that $A = G \cap G'$. Since G, and so G', are SA bases for T, A is an SA base for T by Lemma 1.3.

There is another representation of SA bases that we can give. This uses a combinatorial lemma extracted from a construction of Park.

LEMMA 3.2. *Consider the diagram of injections*

$$(+) \qquad \begin{array}{ccccccccc} A & \longrightarrow & B_0 & \longrightarrow & B_1 & \longrightarrow & \ldots & \longrightarrow & B \\ \downarrow & & \downarrow{\scriptstyle u_0} & & \downarrow{\scriptstyle u_1} & & & & \downarrow{\scriptstyle u} \\ B_0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & \ldots & \longrightarrow & B \end{array}$$

where the unlabelled arrows are inclusions, $B = \bigcup_{n<\omega} B_n$, and u is the direct limit of the u_n. Then

$$A = \bigcap_{n<\omega} u^n B.$$

Proof. Note first that $u_n = u|B_n$. Now $A = B_0 \cap uB_0$ and $uB_n = B_{n+1} \cap uB_{n+1}$. Thus we have $A = B_0 \cap uB_n$ by induction on n, and so $A = B_0 \cap uB$. Similarly, $uB_n = B_{n+1} \cap uB$.

We can now prove by induction that

$$A = B_n \cap \bigcap_{s=1}^{n+1} u^s B.$$

Hence $A \leqslant B \cap \bigcap_{s=1}^{n+1} u^s B$ for all n, and so

$$A \leqslant \bigcap_{s<\omega} u^s B.$$

But if $b \in \bigcap_{s<\omega} u^s B$, then $b \in B_n$ for some n, and so

$$b \in B_n \cap \bigcap_{s=1}^{n+1} u^s B = A.$$

Using this, we can prove that every SA base is the intersection of a decreasing sequence of e.c. models. More precisely, the following is true:

THEOREM 3.3. *Let T be a universal theory. If A is a strong amalgamation base for T, then there are an e.c. model B of T and an injection u: $B \to B$ such that*

$$A \leqslant B \quad \text{and} \quad A = \bigcap_{n<\omega} u^n B.$$

The converse is also true provided A is an amalgamation base.

Proof. Let $B_0 \geqslant A$ be e.c. Since A is an SA base, there are $B_1 \geqslant B_0$ and an injection u_0: $B_0 \to B_1$ with $u_0(a) = a$ for $a \in A$, and we can take

B_1 e.c. (by a further extension). Now build up the diagram (+); this is possible as each B_n is an SA base. Let

$$B = \bigcup_{n<\omega} B_n.$$

Then B is an e.c. model of T. The other results follow from Lemma 3.2.

Conversely, if

$$A = \bigcap_{n<\omega} u^n B$$

with B e.c., then $A \to B$ is antialgebraic (as each $u^n B \to B$ is); thus A is algebraically closed, and so an SA base by the Main Theorem.

In fact, one can prove the converse without using the Main Theorem by employing a certain compactness argument.

4. Applications of strong amalgamation to convexity properties. In this section we show how an explicit use of strong amalgamation can simplify or at least reduce to more standard considerations results obtained by Robinson, Park and others on convexity and related properties of theories.

We shall use the standard preservation theorem asserting that a theory T is inductive (i.e. closed under chain unions) just if $A \models T$ whenever $A \prec_1 B \models T$.

The following slight strengthenings of Lemma 1.1 will be useful (they can be proved by similar compactness arguments):

LEMMA 4.1. *Let $B \leftarrow A \rightarrow C$ be a wedge of inclusions with $B \cap C = A$.*

(i) *If $A \prec_1 B$ and $A \leqslant C$, then there is D with $B \leqslant D$ and $C \prec D$.*

(ii) *If $A \prec B$ and $A \prec C$, we can make $B \prec D$ also.*

Now we can prove

LEMMA 4.2 (Robinson). *If T is convex, then T is inductive.*

Proof. Let $A \prec_1 B \models T$. By Lemma 4.1, there are $D \succ B$ and an injection $f\colon B \to D$ with $A = B \cap fB$. Now B, fB, $D \models T$, and so $A \models T$ if T is convex.

Before we can proceed, we need two more definitions.

We call T *descending-chain-closed* (DC-closed) if whenever $(A_n)_{n<\omega}$ is a descending chain of models of T and $B = \bigcap_{n<\omega} A_n \neq \emptyset$, then $B \models T$. (This is the *nested intersection property* of Park [7].) And we call T *hypo-convex* if whenever A, $B \prec C \models T$ and $A \cap B \neq \emptyset$, then $A \cap B \models T$.

Clearly, a convex theory is hypo-convex, and it is also DC-closed, by a result of Robinson.

LEMMA 4.3 (Park). *If T is DC-closed, then T is inductive.*

Proof. Let $A \prec_1 B \models T$. Then there are $B_1 \succ B_0$ and an injection $u_0\colon B_0 \to B_1$ with $B_0 \cap u_0 B_0 = A$. Repeating this, we obtain the diagram $(+)$, where $B_m \prec B_n$ if $m < n$. Hence $B \models T$, and so $u^n B \models T$. Since

$$A = \bigcap_{n<\omega} u^n B,$$

it follows that $A \models T$ if T is DC-closed.

THEOREM 4.4 (Park). *If T is DC-closed, then T is hypo-convex.*

Proof. Rather than use Park's characterization of hypo-convex theories, it is simpler to use the above methods. Let T be DC-closed.

(a) We first prove an apparently weaker form of hypo-convexity. Let $A \leqslant B_0 \prec B_1 \models T$ and let $u_0 \colon B_0 \to B_1$ be an elementary injection such that $u_0(a) = a$ for $a \in A$ and $A = B_0 \cap u_0 B$. By repeated uses of Lemma 4.1 (ii), we build up the diagram $(+)$ with $B_m \prec B_n$ if $m < n$ and u_n elementary. Thus $B \models T$, and so

$$A \models T \quad \text{as } A = \bigcap_{n<\omega} u^n B.$$

(b) Now we prove that this weaker form actually implies hypo-convexity. Let $A = B_1 \cap B_2$, where $B_1, B_2 \prec C \models T$. Let C_i be an isomorphic copy of C such that $C_i \cap C = B_i$ for $i = 1, 2$, and $C_1 \cap C_2 = A$. We can find D_i such that, by Lemma 4.1 (ii), $C, C_i \prec D_i$ for $i = 1, 2$, and then we can take $G \models T$ with $D_1, D_2 \prec G$. Thus $C_1, C_2 \prec G \models T$ and $A = C_1 \cap C_2$, and C_1 is isomorphic to C_2. By part (a), it follows that $A \models T$.

However, we have not been able to give a simpler proof of the converse of Park's main theorem asserting that an inductive hypo-convex theory is DC-closed.

5. An algebraic proof and some remarks. It is possible to give a proof of the Main Theorem for universal Horn (and a fortiori equational) T, which requires a minimum of model theory. This uses the following lemma:

LEMMA 5.1. *Let $B \leftarrow A \rightarrow C$ be a wedge of inclusions of $\mathcal{M}(T)$ with $B \cap C = A$. This has a strong amalgam just if for each $b \in B \setminus A$ there are an extension D_b of C and an injection $f_b \colon B \to D_b$, the identity on A, with $f_b(b) \notin C$.*

Proof. The necessity is obvious. Conversely, let

$$D = \prod_{b \in B \setminus A} D_b$$

and define $f \colon B \to D$ by $f(x)(b) = f_b(x)$ for $b \in B \setminus A$, and $x \in B$, and $u \colon C \to D$ by $u(x)(b) = x$ for $b \in B \setminus A$, and $x \in C$. Clearly, f and u are injections with $f|A = u|A$. If $f(b) = u(c)$, where $b \in B \setminus A$ and $c \in C \setminus A$, then $f_b(b) = f(b)(b) = u(c)(b) = c \in C$, contrary to hypothesis.

THEOREM 5.2. *Let T be a universal Horn theory, and A an algebraically closed amalgamation base for T. Then A is a strong amalgamation base for T.*

Proof. Suppose not. Then we can find extensions B and C of A with $B \cap C = A$ and (by Lemma 5.1) $b \in B \setminus A$ such that for every extension D' of C and injection $w\colon B \to D'$, the identity on A, we have $w(b) \in C$. We shall show that b is algebraic, of degree 1, over A in B.

Suppose not. Then, by an amalgamation, we obtain an extension D of C and injections $u, v\colon B \to D$, the identity on A, with $u(b) \neq v(b)$. Let $\varkappa$ be a cardinal such that $\varkappa > |C|$. For each $j < \varkappa$, we define an injection $w_j\colon B \to D^\varkappa$ by

$$w_j(b)(i) = u(b) \text{ if } i \neq j \quad \text{and} \quad w_j(b)(j) = v(b).$$

Now let $e\colon C \to D^\varkappa$ be given by $e(c)(i) = c$. Then $w_j|A = e|A$, and so, by hypothesis, $w_j(b) \in uC$. But if $i < j < \varkappa$, then $w_i(b) \neq w_j(b)$. This gives a contradiction, as $|uC| < \varkappa$.

It would be of interest to extend the Main Theorem to non-elementary classes, but our knowledge here is fragmentary. Note, however, that Theorem 5.2 does not use compactness, and so applies to any class of structures closed under product and substructures, e.g. metric spaces of diameter not greater than 1. Note also that the trivial direction always holds.

There are two particular non-elementary classes where the theorem does hold.

(a) Let $\mathscr{C}$ be the category of metric spaces and contractions. Any complete metric space is a strong amalgamation base (by a direct argument), and any strong amalgamation base is algebraically closed in its completion, and so complete.

(b) Let $\mathscr{B}$ be the category of complete Boolean algebras and complete embeddings. Ronald Jensen has observed (oral communication) that any complete Boolean algebra is a strong amalgamation base, by using Boolean-valued models to reduce the problem to a known joint embedding result.

Some categorical observations on strong amalgamation can be found in [2] and [8].

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