Weiter rechnet man nach:

$$\varrho(u_1, v_1) = 725.$$

Damit bekommt man

$$\Sigma(\sqrt{219}) = 29 \equiv \begin{cases} u_1 v_1 = 740 \bmod 3, \\ 725 \bmod 8, \end{cases}$$

wie es nach Satz 3 sein muß.

Literatur

[1] S. I. Borevič und I. R. Šafarevič, *Zahlentheorie*, Basel–Stuttgart 1966.
[2] P. Chowla and S. Chowla, *Problems on periodic simple continued fractions*, Proc. Nat. Acad. Sci. USA 69 (1972), S. 3745.
[3] — — *On Hirzebruch sums and a theorem of Schinzel*, Acta Arith. 24 (1973), S. 223–224.
[4] H. Hasse, *Vorlesungen über Zahlentheorie*, 2. Aufl., Berlin–Göttingen–Heidelberg–New York 1964.
[5] F. Hirzebruch and D. Zagier, *Class-numbers, continued fractions and the Hilbert modular group*, Erscheint demnächst.
[6] H. Lang, *Über die Klassenzahlen eines imaginären bizyklischen biquadratischen Zahlkörpers und seines reell-quadratischen Teilkörpers I*, J. Reine Angew. Math. 262/263 (1973), S. 18–40.
[7] H. Lang und R. Schertz, *Kongruenzen zwischen Klassenzahlen quadratischer Zahlkörper*, Erscheint im J. Number Theory.
[8] C. Meyer, *Die Berechnung der Klassenzahl abelscher Körper über quadratischen Zahlkörpern*, Berlin 1957.
[9] — *Über einige Anwendungen Dedekindscher Summen*, J. Reine Angew. Math. 198 (1957), S. 143–203.
[10] O. Perron, *Die Lehre von den Kettenbrüchen*, Bd. 1, 3. Aufl., Stuttgart 1954.
[11] H. Rademacher, *Zur Theorie der Modulfunktionen*, J. Reine Angew. Math. 167 (1932), S. 312–336.
[12] A. Schinzel, *On two conjectures of P. Chowla and S. Chowla concerning continued fractions*, Annali di Matematica Pura ed Applicata 98 (1974), S. 111–117.
[13] C. L. Siegel, *Lectures on Advanced Analytic Number Theory*, Bombay 1961.

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The distribution of sequences modulo one

by

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It can easily be seen that if f is a real function such that $f(x) \to +\infty$ and $f'(x) \to 0$ as $x \to +\infty$, then the fractional parts of the numbers $f(1)$, $f(2), \ldots$ are dense in the unit interval. (Indeed, if also $xf'(x) \to +\infty$, then they are uniformly distributed there [2].) This result can be applied to the sequence $\log 1, \log 2, \ldots$, but not to $1\log 1, 2\log 2, \ldots$, since the derivative of $x\log x$ does not approach zero.

We prove a more general result covering functions like the latter with second derivatives approaching zero. We also give conditions assuring the denseness in the unit square of the points $(\langle f(n)\rangle, \langle f(n-1)\rangle)$, and also the points $(\langle f(n)\rangle, \langle f'(n)\rangle)$, where $\langle t\rangle$ denotes the fractional part of t.

THEOREM 1. *Suppose f is defined for $x \geqslant 0$, $f(x)$ tends monotonically to 0 as x tends to $+\infty$, and $\int_0^t f(x)\,dx = h(t)$ tends to $+\infty$ as t tends to $+\infty$. Then as $j = 1, 2, \ldots$*

1) *$\langle h(j)\rangle$ is dense in the unit interval,*
2) *$\langle l(j)\rangle$ is dense in the unit interval and $(\langle h(j)\rangle, \langle l(j)\rangle)$ is dense in the unit square, where $l(x) = \int_0^x h(t)\,dt$.*

Proof. Result 1) is well known [2] but we will give our proof since we prove 2) in a similar manner. By hypothesis, for any $\varepsilon > 0$ there exists N_0 such that if $x > N_0$ then $0 < f(x) < \varepsilon$, and hence if $k > N_0$,

$$|h(k+1) - h(k)| = \int_k^{k+1} f(t)\,dt < \varepsilon.$$

But since $h(k)$ tends to $+\infty$ it follows that $\langle h(k)\rangle$ is dense in the unit interval. This proves 1).

To prove 2) we let I be an open interval in $[0, 1)$. Choose n and k, k odd, such that $I_k = (k/2^n, (2^{2n}k+1)/2^{3n}) \subset I$. By part 1) there exists N_0

such that if $N_0 \leqslant t \leqslant N_0+2^n$, then $\langle h(t)\rangle \in I_k$ and $[h(t)]$ is a constant, say M.

Now suppose $0 \leqslant r < 2^n$, r an integer. Then

$$l(N_0+r) = \int_0^{N_0} h(t)\,dt + \int_{N_0}^{N_0+r} M\,dt + \int_{N_0}^{N_0+r} \langle h(t)\rangle\,dt,$$

so

$$l(N_0+r) - l(N_0) - Mr = \int_{N_0}^{N_0+r} \langle h(t)\rangle\,dt \in \left(rk/2^n,\ r(2^{2n}k+1)/2^{3n}\right)$$
$$\subset \left(rk/2^n,\ (rk+1)/2^n\right).$$

Since Mr is an integer

$$\langle l(N_0+r) - l(N_0)\rangle \in (s/2^n,\ (s+1)/2^n),$$

where $0 \leqslant s < 2^n$ and $rk \equiv s \pmod{2^n}$.

But since n can be arbitrarily large, since r runs through a complete residue system modulo 2^n independently of N_0, and since $\langle h(N_0+r)\rangle \in I$, we see that the points $(\langle h(j)\rangle, \langle l(j)\rangle)$ are dense in the unit square. This completes the proof of 2).

Analogous results can be stated for sequences of real numbers, where the integral sign is replaced by the summation sign. Indeed we have

THEOREM 2. *Let $a_1, a_2, \ldots$ be a sequence of real numbers such that $a_{n+1}+a_{n-1}-2a_n$ approaches 0 and changes sign only finitely-many times. Then the sequence $\langle a_1\rangle, \langle a_2\rangle, \ldots$ is either dense in $[0,1)$ or else has exactly the limit points $\langle s+nr\rangle$, $n = 1, 2, \ldots$, where s is some real and r some rational number. Indeed one of the following three cases holds:*

(1) *There are rational r and real s such that $a_n - nr \to s$;*

(2) *The points a_n are dense in $[0,1)$ and $a_n - a_{n-1}$ converges;*

(3) *For any open intervals I and J in $[0,1)$ there exists n such that $\langle a_n\rangle \in I$ and $\langle a_{n-1}\rangle \in J$.*

The proof will be by Lemmas 1 through 4 in which the sequence $a_1, a_2, \ldots$ is assumed to obey the hypotheses of Theorem 2. First we note that it is not hard to prove that (3) is equivalent to

(3′) *Given open intervals I and J in $[0,1)$ there exists n such that $\langle a_n\rangle \in I$ and $\langle a_n - a_{n-1}\rangle \in J$.*

LEMMA 1. *If t is an irrational limit point of $\langle a_2-a_1\rangle, \langle a_3-a_2\rangle, \ldots$ and if $\varepsilon > 0$, then the points $\langle a_n\rangle$ satisfying $\langle |a_n - a_{n-1} - t|\rangle < \varepsilon$ are dense in $[0,1)$.*

Proof. Let $\delta > 0$. By a *δ-net* we mean a subset of $[0,1)$ having distance less than δ from each point in $[0,1]$. Since t is irrational, by Kronecker's Theorem there exists k such that $\langle t\rangle, \langle 2t\rangle, \ldots, \langle kt\rangle$ is a δ-net ([1]). Choose α, $0 < \alpha < \varepsilon$, such that if $\langle |d_j - jt|\rangle < \alpha$, $j = 1, 2, \ldots, k$, then $\langle d_1\rangle, \ldots, \langle d_k\rangle$ is a δ-net. By hypothesis $(a_{n+1}-a_n)-(a_n-a_{n-1}) \to 0$ and (since t is a limit point) given any integer $k > 0$ we can choose N such that

$$\langle |a_{N+j} - a_{N+j-1} - t|\rangle < \alpha k^{-1} \quad \text{for} \quad j = 1, 2, \ldots, k.$$

Then

$$\begin{aligned}\langle |a_{N+j} - a_N - jt|\rangle &= \langle |(a_{N+j} - a_{N+j-1} - t) + (a_{N+j-1} - a_{N+j-2} - t) + \ldots + (a_{N+1} - a_N - t)|\rangle \\ &\leqslant \langle |a_{N+j} - a_{N+j-1} - t|\rangle + \ldots + \langle |a_{N+1} - a_N - t|\rangle < k(\alpha k^{-1}) = \alpha < \varepsilon\end{aligned}$$

for $j = 1, 2, \ldots, k$. Thus $\langle a_{N+j} - a_N\rangle$ is a δ-net, and hence $\langle a_{N+j}\rangle$ is a $2\,\delta$-net. This proves the lemma.

Note that $a_n - a_{n-1}$ is eventually monotone, and thus either converges or diverges to $\pm\infty$.

LEMMA 2. *If $a_n - a_{n-1} \to \pm\infty$, then* (3) *holds.*

Proof. From $(a_{n+1}-a_n)-(a_n-a_{n-1}) \to 0$ we see that $\langle a_n - a_{n-1}\rangle$ crosses $[0,1)$ infinitely many times in smaller and smaller steps and so is dense in $[0,1)$. Given intervals I and J in $[0,1)$, choose an irrational t and $\varepsilon > 0$ such that $(t-\varepsilon, t+\varepsilon)$ is a subset of J. By Lemma 1 there exists n such that

$$\langle |a_n - a_{n-1} - t|\rangle < \varepsilon \quad \text{and} \quad \langle a_n\rangle \in I.$$

Thus $\langle a_n - a_{n-1}\rangle \in J$ and we have (3′).

LEMMA 3. *If $a_n - a_{n-1}$ converges to an irrational number, then* (2) *holds.*

Proof. Lemma 1 implies that the points $\langle a_n\rangle$ are dense in $[0,1)$.

LEMMA 4. *If $a_n - a_{n-1}$ converges to a rational number r, then* (1) *or* (2) *holds.*

Proof. First suppose r is an integer. We assume $a_n - a_{n-1}$ is nondecreasing for $n \geqslant N$, the other argument being analogous. Then $\langle a_n - a_{n-1}\rangle$ approaches 1 from the left. The sequence

$$\langle a_{N+1} - a_N\rangle,\ \langle a_{N+2} - a_N\rangle = \langle (a_{N+2} - a_{N+1}) + (a_{N+1} - a_N)\rangle,$$
$$\langle a_{N+3} - a_N\rangle = \langle (a_{N+3} - a_{N+2}) + (a_{N+2} - a_N)\rangle, \ldots$$

is nonincreasing except when it jumps from the left to the right end of $[0,1)$, and the distance between two consecutive terms tends to zero except when there is a jump. Either (I) this sequence crosses $[0,1)$ an infinite number of times, in which case the sequence is dense and (2) holds, or else (II) it converges to a real number k in $[0,1)$ from the right. In the latter case set $\delta_j = a_{N+j} - a_N - k - jr$. Then $\langle \delta_j\rangle \to 0$ while $\delta_{j+1} - \delta_j$

$= a_{N+j+1} - a_{N+j} - r \to 0$. It follows that δ_j converges (to an integer) and so $a_{N+j} - (N+j)r = \delta_j + a_N + k - Nr$ converges as $j \to \infty$. Thus we have (1).

Now suppose r is not an integer. Choose a positive integer M such that Mr is an integer. Set $b_n = a_{Mn}$. Note that

$$b_n - b_{n-1} = (a_{Mn} - a_{Mn-1}) + (a_{Mn-1} - a_{Mn-2}) + \ldots + (a_{Mn-M+1} - a_{Mn-M}),$$

which eventually converges monotonically to Mr; and so $b_{n+1} + b_{n-1} - 2b_n$ converges to zero and changes sign only finitely many times. Thus the hypotheses of the first case of this lemma apply to $b_1, b_2, \ldots$ If (II) holds for this sequence, then by the proof above $b_n - n(Mr) = a_{Mn} - (nM)r \to s$; so also for fixed j

$$a_{Mn+j} - (Mn+j)r = \sum_{i=1}^{j} (a_{Mn+i} - a_{Mn+i-1}) - jr + a_{Mn} - Mnr \to s \quad \text{as} \quad n \to \infty.$$

On the other hand if (I) and thus (2) hold for $b_1, b_2, \ldots$, then (2) clearly holds also for $a_1, a_2, \ldots$

Some sequences to which Theorem 1 applies are n^w $(w < 2)$, $(\log n)^w$ (any w), $n(\log n)^w$ (any w), $(\arctan n)^w$ (any w), $n(\arctan n)^w$ (any w), $\int_1^n (t + \sin t)^w dt$ $(w < 1)$, and $\int_1^n (t + \cos t)^w dt$ $(w < 1)$.

Note added in proof. P. Csillag in his paper *Über die Verteilung iterierter Summen von Positiven Nullfolgen* mod 1, Acta Litt. Sci. Szeged 4 (1929), pp. 151–154, has shown that if $f^k(x) \to \infty$ while $f^{(k+1)}(x) > 0$ and $f^{(k+1)}(x) \to 0$ then $\langle f(k) \rangle$ is dense in $(0, 1)$. This contains our result that if $k = 1$, then $\langle f(n) \rangle$ is dense in $(0, 1)$ but says nothing about density in the unit square. Recently John Daily in his Ph.D. dissertation has shown that under the above conditions $(\langle f(n) \rangle, \ldots, \langle f^k(n) \rangle)$ is dense in the n cube.

References

[1] G. H. Hardy and E. M. Wright, *Introduction to the Theory of Numbers*, 4th ed., Cambridge 1959, Chapter 23.

[2] G. Polya and G. Szego, *Aufgaben und Lehrsätze aus der Analysis*, vol. 1, 2nd ed., Berlin 1954, p. 72.

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Eine Anwendung des Selbergschen Siebes auf algebraische Zahlkörper

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1. Mittels der Selbergschen Siebmethode bewiesen Jurkat und Richert [2] unter anderem folgende Resultate:

(a) Für $\varepsilon > 0$ und $x \geqslant x_0(\varepsilon)$ existiert mindestens eine ganze Zahl n in dem Intervall

$$x - x^{\frac{14}{25}+\varepsilon} < n \leqslant x,$$

die höchstens zwei Primfaktoren besitzt.

(b) Für $\varepsilon > 0$, $k \geqslant k_0(\varepsilon)$ und $(k, l) = 1$ existiert mindestens eine ganze Zahl n,

$$n \equiv l \bmod k, \quad 1 \leqslant n \leqslant k^{\frac{25}{11}+\varepsilon},$$

die höchstens zwei Primfaktoren besitzt.

Diese Ergebnisse wurden von Schaal [7] mit etwas abweichenden Methoden auf algebraische Zahlkörper übertragen. Richert [5] erreicht eine weitere Verschärfung der Ergebnisse aus [2], indem er neue Gewichtsfunktionen einführt ([5], Theorem 1). Zudem wird der Bereich der zugelassenen Mengen, auf die der Siebprozeß angewendet wird, erweitert ([5], Theorem A, Theorem B, Einführung der Funktion γ).

Als Anwendungen werden unter anderem folgende Resultate bewiesen:

(c) Für $x \geqslant x_0$ existiert mindestens eine ganze Zahl n in dem Intervall

$$x - x^{6/11} < n \leqslant x,$$

die höchstens zwei Primfaktoren besitzt.

Das Intervall ist hier etwas kürzer als unter (a).

(d) Sei $F(x)$ ein irreduzibles Polynom vom Grade $g \geqslant 1$ mit ganzen Koeffizienten und ohne festen Primteiler.

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Herrn Prof. Dr. W. Schaal, der dieses Thema anregte, danke ich für seine Unterstützung.