Normality and Martin's Axiom

by

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Abstract. It is known that Martin's Axiom plus the negation of the Continuum Hypothesis assures the existence of a whole variety of normal non-metrizable Moore spaces, e.g. complete separable ones, countable chain condition metacompact ones and complete metacompact ones. It also yields a paracompact space T such that T^2 is normal but not paracompact.

In this paper we prove that Martin's Axiom implies the normality of a certain class of spaces which admit one-to-one mappings onto metric spaces. The existence of the above mentioned examples is an easy consequence of this theorem. We deduce also that a locally compact separable Moore space T is normal if and only if T^ω is normal.

On the other hand, we show using Fleissner's theorem, that Axiom of Constructibility implies metrizability of every locally countable chain condition (or locally Lindelöf) normal Moore space.

In the last section we discuss the relation between normality and paracompactness in countable products.

1. Introduction. It is known that Martin's Axiom plus the negation of the Continuum Hypothesis implies the existence of a whole variety of normal non-metrizable Moore spaces, e.g. complete separable ones [25], countable chain condition metacompact ones [19] and complete metacompact ones [26]. It also yields a paracompact space T such that T^2 is normal but not paracompact [18].

In this paper we prove that Martin's Axiom implies the normality of a certain class of spaces which admit one-to-one mappings onto metric spaces. The existence of the above mentioned examples is an easy consequence of this theorem. Moreover, any such example S may be assumed to be non-locally metrizable at any point and to satisfy the condition $S = S^\omega$.

The above theorem in conjunction with the recent result of Reed and Zenor [21] shows that a locally compact separable Moore space T is normal if and only if T^ω is normal.

On the other hand, we prove using Fleissner's result [7] that Axiom of Constructibility implies metrizability of every locally countable chain condition (or locally Lindelöf) normal Moore space.

Finally, we show that Martin's Axiom plus the negation of the Continuum Hypothesis yields a paracompact space T such that T^ω is normal but not paracompact and a non-paracompact space T such that T^ω is perfectly normal.

All spaces are assumed to be Hausdorff. We often denote a topological space T by $(X, \mathcal{T})$, where X is the set on which the topology $\mathcal{T}$ is defined. By R and Q we denote respectively the spaces of real and of rational numbers.

A topological space is *countable chain condition* if every family of disjoint non-empty open sets is countable. For the definitions of Moore spaces, subparacompact, metacompact and p-spaces see e.g. [1] and [25]. All other undefined notions and symbols are as in [4].

DEFINITION. Let P be a set partially ordered by the relation $\leqslant$. We say that

1. elements p_1 and p_2 belonging to P are *compactible* if there exists a $p \in P$ such that $p \leqslant p_1$ and $p \leqslant p_2$.

2. $(P, \leqslant)$ satisfies the *countable chain condition* if every subset of P consisting of mutually incompatible elements is countable.

3. $D \subset P$ is *dense* if for every $p \in P$ there exists a $p' \in D$ such that $p' \leqslant p$.

4. $L \subset P$ is $\mathcal{D}$*-generic*, where $\mathcal{D}$ is a family of dense subsets of P, if:

(i) for every $p_1, p_2 \in L$ there exists a $p \in L$ such that $p \leqslant p_1$ and $p \leqslant p_2$,

(ii) if $p \in L$ and $p \leqslant p' \in P$, then $p' \in L$,

(iii) $L \cap D \neq \emptyset$, for every $D \in \mathcal{D}$.

MARTIN'S AXIOM (MA) [13]. *Let $(P, \leqslant)$ be a partial order satisfying the countable chain condition and let $\mathcal{D}$ be a family of less than continuum dense subsets. There exists a $\mathcal{D}$-generic subset of P.*

It is known [13] that Martin's Axiom is independent of the ZFC axioms of set theory. It is implied by the Continuum Hypothesis. Nevertheless, it is also consistent to assume Martin's Axiom and the negation of the Continuum Hypothesis. We denote the last system of axioms by MA$+\neg$CH. For topological consequences of Martin's Axiom see e.g. [27].

2. Main Theorem.

DEFINITION. Let $\mathcal{T}$ and $\mathcal{M}$ be two topologies defined on the set X. We say that $\mathcal{T}$ *is regular with respect to* $\mathcal{M}$ if for every $U \in \mathcal{T}$ and $x \in U$ there exists a $V \in \mathcal{T}$ such that $x \in V \subset \overline{V}^{\mathcal{M}} \subset U$, where $\overline{V}^{\mathcal{M}}$ denotes the closure of the set V in the topology $\mathcal{M}$.

THEOREM 1 (MA). *Let $T = (X, \mathcal{T})$ be a topological space which is the union of less than continuum compact subsets. If there exists a weaker metric separable topology $\mathcal{M}$ on X such that $\mathcal{T}$ is regular with respect to $\mathcal{M}$, then T^n is normal for every $n \in \omega$.*

Proof (¹). As for every $n \in \omega$ the space T^n also satisfies the conditions of the theorem, hence it suffices to prove that T is normal.

Let $T = \bigcup_{s \in S} F_s$, where F_s is compact and $|S| < 2^{\aleph_0}$. For a totally bounded metric ϱ in $M = (X, \mathcal{M})$ (see [4]; Theorem 4.3.5) let $B(x, \varepsilon)$ denote the ball with

(¹) We are grateful to J. Chaber for calling our attention to the idea used by Fleissner in ([8], Section 4), which we exploit in the proof.

the center at the point x and the radius ε; let $\mathcal{B}_n$ be a finite subcovering of the covering $\{B(x, 1/n)\}_{x \in X}$ of X and put $\operatorname{dist}(G, H) = \inf\{\varrho(x, y)\colon x \in G, y \in H\}$, for $G, H \subset X$. Choose two closed and disjoint subsets A, B of the space T.

Let $P = \{(G, H)\colon G$ and H are open in T, $\operatorname{dist}(G, H) > 0$, $\overline{G}^{\mathcal{M}} \cap B = \overline{H}^{\mathcal{M}} \cap A = \emptyset\}$ be partially ordered by the relation $(G, H) \leqslant (G', H')$ if $G \supset G'$ and $H \supset H'$.

For every $s \in S$ the sets $D_s = \{(G, H) \in P\colon F_s \cap A \subset G\}$ and $E_s = \{(G, H) \in P\colon F_s \cap B \subset H\}$ are dense in $(P, \leqslant)$. Let $s \in S$. There exists a neighbourhood G' of $F_s \cap A$ in T such that $\overline{G'}^{\mathcal{M}} \cap B = \emptyset$ and $G' \subset \bigcup_{x \in F_s \cap A} B(x, \frac{1}{2}\operatorname{dist}(F_s \cap A, \overline{H}^{\mathcal{M}}))$. Then we have $\operatorname{dist}(G \cup G', H) \geqslant \min(\operatorname{dist}(G, H), \frac{1}{2}\operatorname{dist}(F_s \cap A, \overline{H}^{\mathcal{M}})) > 0$ and $\overline{G \cup G'}^{\mathcal{M}} \cap B = \emptyset$, hence $(G \cup G', H) \in D_s$ and $(G \cup G', H) \leqslant (G, H)$.

To every pair $(G, H) \in P$ we can attach an $n \in \omega$ such that $\operatorname{dist}(G, H) \geqslant 5/n$ and two sets G_0, H_0 defined by

$$G_0 = \bigcup \{B \in \mathcal{B}_n\colon B \cap G \neq \emptyset\}, \quad H_0 = \bigcup \{B \in \mathcal{B}_n\colon B \cap H \neq \emptyset\}.$$

One easily sees that $G \subset G_0$, $H \subset H_0$ and $\operatorname{dist}(G_0, H_0) \geqslant 1/n$. As there are only countably many of so defined triples (n, G_0, H_0) and any two pairs belonging to P to which the same triple was attached are compatible, we conclude that $(P, \leqslant)$ satisfies the countable chain condition.

Let L be a $(\{D_s\}_{s \in S} \cup \{E_s\}_{s \in S})$-generic subset of P. Define

$$U = \bigcup \{G\colon (G, H) \in L\} \quad \text{and} \quad V = \bigcup \{H\colon (G, H) \in L\}.$$

It is easy to check that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$, which completes the proof. ■

THEOREM 2 (MA). *Let $T = (X, \mathcal{T})$ be a space of cardinality less than continuum. If there exists a weaker metric separable topology $\mathcal{M}$ on X such that $\mathcal{T}$ is regular with respect to $\mathcal{M}$, then T^ω is perfectly normal.*

Proof. By Theorem 1 and Katětov's result [12] it suffices to show that T^n is perfect (²) for every $n \in \omega$. As the cardinality of X is less than continuum, hence — by ([23], Lemma 3) — *every* subset of the space M^n, where $M = (X, \mathcal{M})$, is of F_σ-type and consequently *every* subset of the space T^n is of F_σ-type. ■

Remark. The assumption of $\aleph_1 < 2^{\aleph_0} < 2^{\aleph_1}$ implies the above theorems are false, so they are independent of the axioms of set theory (see the next section).

One might expect that Martin's Axiom implies every countable chain condition (or separable) regular space of cardinality less than continuum is normal. The following example shows there exists a completely regular separable and first countable non-normal space of cardinality $\aleph_1$.

EXAMPLE. Let αN be the compactification of the space N of natural numbers such that $\alpha N \setminus N$ is homeomorphic to the space $\omega_1 + 1$ of ordinals not greater than ω_1 [16]. By the theorem of Katětov [12], there exists a non-normal subspace A of $\omega_1 \times \omega_1$. The subspace $X = (N \times N) \cup A$ of $\alpha N \times \alpha N$ has the required properties. ■

(²) A space is called *perfect* if every open subset is of F_σ-type.

3. Normality of Moore spaces.

THEOREM 3 (MA). *Let $T = (X, \mathcal{T})$ be a Moore space which is the union of less than continuum compact subsets. If there exists a weaker metric separable topology $\mathcal{M}$ on X such that $\mathcal{T}$ is regular with respect to $\mathcal{M}$, then T^ω is normal.*

Proof. By Theorem 1 and Katětov's result [12] it suffices to show that T^n is perfect for every $n \in \omega$. Recall that every Moore space is perfect and that the countable product of Moore spaces is a Moore space. ■

To prove Corollary 1 we shall need the following theorem due to Reed and Zenor [21].

THEOREM 4 (Reed and Zenor). *If $T = (X, \mathcal{T})$ is a normal Moore space of cardinality not greater than continuum, then there exists a weaker metric separable topology $\mathcal{M}$ on X.* ■

The following question has often been raised: is the square T^2 of a normal Moore space T also normal. Our next result presents a partial answer to this question.

COROLLARY 1 (MA). *Let $T = (X, \mathcal{T})$ be a separable* (3) *locally compact Moore space. The following conditions are equivalent:*

(i) *T is normal,*

(ii) *T^ω is normal,*

(iii) *T is the union of less than continuum compact subsets and there exists a weaker metric topology $\mathcal{M}$ on X.*

Proof. (i)→(iii). Clearly $|T| \leqslant 2^{\aleph_0}$. By Theorem 4 it suffices to show that T is the union of less than continuum compact subsets. Let $\mathcal{U}$ be an open covering of T such that $\overline{U}$ is compact for every $U \in \mathcal{U}$ and find a σ-discrete closed refinement $\mathcal{F} = \bigcup_{n \in \omega} \mathcal{F}_n$ of $\mathcal{U}$. As the cofinality of $2^{\aleph_0}$ is greater than $\aleph_0$ it remains to show that every discrete family in T has cardinality less than continuum. Indeed, otherwise by the normality of T there would exist $2^{\mathfrak{c}}$ different continuous functions on T and on the other hand — by the separability of T — there are at most $2^{\aleph_0}$ of such functions on T.

(ii)→(i) is obvious and (iii)→(ii) follows from Theorem 3 and the observation that local compactness of T clearly implies $\mathcal{T}$ is regular with respect to $\mathcal{M}$. ■

Let us recall that in the model of Martin's Axiom plus the negation of the Continuum Hypothesis there exist separable locally compact Moore spaces which are normal but not metrizable (see e.g. the proof of Corollary 2 below).

(3) Martin's Axiom implies $2^\lambda = 2^{\aleph_0}$ for every infinite cardinal λ less than continuum. Using this fact, one can easily check that the assumption of separability of T may be replaced by the assumption that the density of T is less than $2^{\aleph_0}$. In particular, Corollary 1 is valid for locally compact Moore spaces of cardinality less than continuum, which was first observed by G. M. Reed [20]. We also gratefully acknowledge that the final form of Corollary 1 arosed from discussions with him.

Theorem 3 implies the existence of a whole variety of normal non-metrizable Moore spaces S additionally satisfying the condition $S = S^\omega$. Note that if S is a non-metrizable space such that $S = S^\omega$, then S is not locally metrizable at any point. Hence, if S is complete, then S cannot be represented as a countable union of its closed metrizable subsets (4).

Such spaces may but do not have to contain dense metrizable subspaces. Let us recall, that the results of Fitzpatrick [5], Fleissner [7] and Przymusiński and Tall [19] show that the existence of dense metrizable subspaces in normal Moore spaces is independent of the axioms of set theory.

The following corollary is a strengthening of the result of Silver (see Tall [25]) and Bing [3].

COROLLARY 2 (MA+ ¬CH). *There exists a separable complete normal non-metrizable Moore space S such that $S = S^\omega$.*

Proof. To illustrate the usefulness of Theorem 3 we shall give two examples:

A. Let P be any uncountable subset of R of cardinality less than continuum and let $T = (X, \mathcal{T})$ be a subspace of the Niemytzki space, where

$$X = \{(x, y) \in R^2 \colon y > 0 \quad \text{or} \quad y = 0 \quad \text{and} \quad x \in P\}.$$

It is clear that T is a complete separable non-metrizable Moore space and the conditions of Theorem 3 are satisfied. It suffices to put $S = T^\omega$. ■

B. Let P be any uncountable subset of the irrationals of cardinality less than continuum and let $X = (Q \times Q) \cup (P \times \{0\})$. For every $p \in P$ fix a sequence $\{z_n(p)\}_{n \in \omega}$ of points of $Q \times Q$ converging in the usual sense to the point $(p, 0)$ and define a topology $\mathcal{T}$ on X in such a way that the points of $Q \times Q$ are isolated and the family $\{U_n(p)\}_{n \in \omega}$ forms a base of neighbourhoods of the point $(p, 0)$, where $U_n(p) = \{(p, 0)\} \cup \{z_m(p)\}_{m=n}^{\infty}$ (cf. [1], p. 167). It is well-known that $T = (X, \mathcal{T})$ is a locally compact, separable non-metrizable Moore space. As the condition (iii) of Corollary 1 is obviously satisfied, it suffices to put $S = T^\omega$. ■

The next Corollary improves the result of Przymusiński and Tall [19].

COROLLARY 3 (MA+ ¬CH). *There exists a metacompact countable chain condition normal non-metrizable Moore space S such that $S = S^\omega$.*

Proof. For an uncountable subset of the unit interval I of cardinality less than continuum let $T = (X, \mathcal{T})$ be the subspace of the space defined by Pixley and Roy in [17], where $X = \{F\colon F \subset P \text{ and } |F| < \aleph_0\}$. As in [19] we show that T is a countable chain condition metacompact non-separable Moore space. Consider the relative topology $\mathcal{M}$ on X induced by the Vietoris topology on 2^I (see [4], Problem 2.7.20). The space $M = (X, \mathcal{M})$ is metric separable (see [4], Problem

(4) Normal Moore spaces which are not metrizable at any point has been studied e.g. by Fitzpatrick and Traylor in [6] and [29].

4.5.21), $\mathscr{M}$ is weaker than $\mathscr{T}$ and $\mathscr{T}$ is regular with respect to $\mathscr{M}$. It follows from ([11], Theorem 5.5) and Theorem 3 that $S = T^{\omega}$ is countable chain condition and normal. To complete the proof it suffices to recall that a countable product of metacompact Moore spaces is a metacompact Moore space. ■

Note that the last space is not complete [19].

Corollary 4 (MA+$\neg$CH). *There exists a metacompact complete normal non-metrizable Moore space S such that $S = S^{\omega}$.*

Proof. Take any uncountable subset P of R of cardinality less than continuum. For every $p \in P$ define

$$A_p = \{(x, y) \in R^2 \colon x = p+y\}, \quad B_p = \{(x, y) \in R^2 \colon x = p-y\},$$
$$X = \bigcup \{A_p \cap B_q \colon p, q \in P\}$$

and let $M = (X, \mathscr{M})$ be the subspace of R^2. We shall introduce a stronger topology $\mathscr{T}$ on X in such a way that every point $(x, y) \in X$ such that $y \neq 0$ is isolated and if $p \in P$ then the family $\{U_n(p)\}_{n \in \omega}$ forms a base of neighbourhoods of the point $(p, 0)$, where

$$U_n(p) = (A_p \cup B_p) \cap \{(x, y) \in X \colon -1/2^n < y < 1/2^n\}.$$

It is easy to verify that the space $T = (X, \mathscr{T})$ is a metacompact, complete, (locally metrizable), non-metrizable Moore space. Theorem 3 implies $S = T^{\omega}$ is normal. ■

It is known that the assumption of $2^{\aleph_0} < 2^{\aleph_1}$ contradicts the existence of spaces considered in Corollaries 2 and 3 (Šapirovskiĭ [24], Theorem 2.5′; see also Tall [25], Corollaries II.0.14 and II.0.15). Using the result of Fleissner [7] we can prove a stronger consistency result. Here V = L denotes the Gödel's Axiom of Constructibility (see [10]).

Theorem 5 (V = L). *Every locally countable chain condition (or locally Lindelöf) normal Moore space is metrizable.*

As every Moore space is subparacompact it remains to prove the following lemma.

Lemma 1 (V = L). *Paracompactness, metacompactness and subparacompactness are equivalent in the class of normal locally countable chain condition (or locally Lindelöf) first countable spaces.*

Proof. Note first that V = L implies CH [10]. It suffices to prove that metacompactness (resp. subparacompactness) implies paracompactness. We can restrict ourselves to the "locally Lindelöf" case. Indeed, let U be a countable chain condition subset of a metacompact or subparacompact space. Then $\overline{U}$ is also countable chain condition and metacompact or subparacompact. From ([24], Theorem 2.3′) we infer that $\overline{U}$ is collectionwise normal, hence paracompact, hence Lindelöf. Our conclusion follows from ([26], Lemma 4) and Lemma 2 below.

Lemma 2 (V = L). *Let $\{F_s\}_{s \in S}$ be a discrete collection of closed Lindelöf subsets of a normal first countable space. Then there exists a disjoint collection $\{U_s\}_{s \in S}$ of open sets such that $F_s \subset U_s$, for $s \in S$.*

Proof. By Arhangel'skiĭ's theorem [2] we have $|F_s| \leqslant 2^{\aleph_0}$. First countability and Lindelöfness of F_s imply that the character of the sets F_s in the entire space is less or equal to $2^{\aleph_0} = \aleph_1$. Identifying the sets F_s to points and making use of the theorem due to Fleissner [7] we complete the proof. ■

4. Normality of products. Zenor proved [30] that if (i) T^n is paracompact for every $n \in \omega$ and (ii) T^{ω} is normal, then T^{ω} is paracompact.

A natural question arises whether (i) can be replaced by the assumption of paracompactness of T. Our next corollary improves the result of Przymusiński [18] and gives a (partial) negative answer to this question.

Corollary 5 (MA+$\neg$CH). *There exists a (separable, first countable) paracompact space T such that T^{ω} is perfectly normal but not paracompact.*

Proof. Take any uncountable subset X of R of cardinality less than continuum satisfying: $1-x \in X$ iff $x \in X$. Let $T = (X, \mathscr{T})$ be the subspace of the Sorgenfrey line. It follows from Theorem 2 that T^{ω} is perfectly normal. One easily checks that T^2 is not even collectionwise normal (cf. [18]). ■

Nagami asked ([15], Problem 3) if normality of S^{ω} implies paracompactness of S. The answer is negative — it suffices to take the Σ-product of uncountably many unit intervals (5).

Assuming MA+$\neg$CH we can give a better example.

Corollary 6 (MA+$\neg$CH). *There exists a (separable, first countable) non-paracompact space S such that S^{ω} is perfectly normal.*

Proof. It is a straightforward consequence of Corollary 2. ■

It follows from [18] or from our Corollary 5 that it is consistent to assume the existence of a paracompact space T such that $T \times T$ is normal but not paracompact. On the other hand, we have the following theorem which is a simple consequence of the results of Tamano [28], Morita (6) and Starbird and Rudin [22].

Theorem 6 (Tamano, Morita, Starbird and Rudin). *If S is a paracompact p-space and T is paracompact, then:*

$S \times T$ is normal $\Leftrightarrow$ $S \times T$ is paracompact $\Leftrightarrow$ $S \times T$ is countably paracompact.

Proof. Let $f \colon S \to M$ be a perfect mapping of S onto a metric space M. We may assume M is non-discrete. If $S \times T$ is normal then $M \times T$ is normal as a perfect

(5) This remark is due to R. Pol. Also N. Noble noticed that the countable product of the space of countable ordinals is normal.

(6) As Professor Morita kindly informed us, the proof presented by Tamano in ([28], Theorem 3) contains some difficult to repair errors. A correct proof of Tamano's theorem has been given by Morita in ([14], Theorem 2.4) and ([9], Theorem 1.3).

image of a normal space. Hence, by [22] $M \times T$ is countably paracompact and consequently $S \times T$ is countably paracompact as a perfect preimage of a countably paracompact space. If $S \times T$ is countably paracompact then so is $M \times T$. By [28, 14, 9] $M \times T$ is paracompact. It follows that $S \times T$ is paracompact as a perfect preimage of a paracompact space. ■

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