On three types of simplicial objects

by

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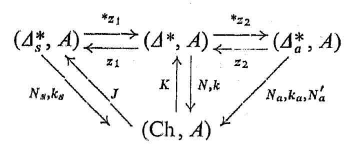
Abstract. Let Δ, Δ_s, Δ_a denote categories with the same set of objects consisting of sets $[n] = \{0, 1, \ldots, n\}$, for $n = 0, 1, \ldots$ and the sets of morphisms consisting of increasing, strictly increasing or all functions, respectively. We consider three categories of simplicial objects (Δ_s^*, A), (Δ^*, A), (Δ_a^*, A) of all contravariant functors with values in an Abelian category A and we present some results concerning functors between these categories and the category of complexes over A. Moreover we study the homotopy of maps of simplicial objects of three types and prove several theorems on a preservation of the homotopy by some functors.

Let us denote by Δ (resp., Δ_s, resp., Δ_a) a category with the set of objects consisting of sets $[n] = \{0, 1, \ldots, n\}$, $n = 0, 1, \ldots$ and sets of maps $\alpha\colon [m] \to [n]$ consisting of all weakly increasing functions (resp., all strictly increasing functions, resp., all functions). Thus we have $\Delta_s \subset \Delta \subset \Delta_a$. All contravariant functors defined on Δ_s, Δ or Δ_a with values in a fixed category M and with natural transformations as maps form categories (Δ_s^*, M), (Δ^*, M), (Δ_a^*, M). Categories (Δ^*, M) and specially (Δ^*, Set) play an important role in algebra and topology. Categories (Δ_s^*, M) and (Δ_a^*, M) are considered rather seldom (see [5], [7], [4]).

In the first part of this paper we present some results concerning functors (defined below) in the following diagram:



z_1, z_2 and $z\colon (\Delta_a^*, A) \to (\Delta_s^*, A)$ are forgetful functors induced by inclusions $\Delta_s \subset \Delta$, $\Delta \subset \Delta_a$, $\Delta_s \subset \Delta_a$; thus $z = z_1 z_2$. If a category A has finite colimits, then there exist left adjoint functors $^*z_1$, $^*z_2$, $^*z$ of functors z_1, z_2, z. For each integer $n \geqslant 0$ we have functions $\varepsilon^i = \varepsilon_n^i\colon [n-1] \to [n]$, $\eta^i = \eta_n^i\colon [n+1] \to [n]$, $i = 0, 1, \ldots, n$, defined as follows: $\varepsilon^i(j) = j$ for $j < i$ (resp., $j+1$ for $j \geqslant i$), $\eta^i(j) = j$ for $j \leqslant i$ (resp., $j-1$ for $j > i$). If X is an object in (Δ_s^*, A), then we write $X_n = X([n])$, $d_i = \tilde{\varepsilon}^i = X(\varepsilon^i)$ (face operator), and if X is in (Δ^*, A), then we write $s_i = \tilde{\eta}^i = X(\eta^i)$ (degeneracy operator). Let A be an Abelian category, then we denote by (Ch, A) the category of all left chain complexes over A. N_s, N, N_a denote normalization functors which associate with an object X a chain complex with the nth component equal to $\bigcap_{i=1}^{n} \operatorname{Ker}(d_i\colon X_n \to X_{n-1})$ and the nth differential is induced by $d_0\colon X_n \to X_{n-1}$. Functors k_s, k, k_a associate with X a chain complex with the nth component X_n and with the nth differential equal to $\sum_{i=0}^{n}(-1)^i d_i$. Let X be an object in (Δ_a^*, A); then we denote by $D_n(X)$, $n = 0, 1, \ldots$ a subobject $\sum_{\pi} \operatorname{Im}(X(\pi) - \operatorname{sgn}(\pi) 1_{X_n}) + \sum_{\gamma} \operatorname{Im} X(\gamma)$ of X_n, where π runs over $S_n = \operatorname{Aut}_{\Delta_a}([n])$ and γ runs over all such maps $\gamma\colon [n] \to [p]$ in Δ_a that $p < n$. The subobjects $D_n(X)$ determine a subcomplex $D(X)$ of $k_a(X)$ and we denote by $N_a'(X)$ the complex $\operatorname{Coker}(D(X) \to k_a(X))$ (see [4]). Let C be a chain complex in (Ch, A) with differentials $\partial_n\colon C_n \to C_{n-1}$; then the nth component of an object $J(C)$ in (Δ_s^*, A) is equal to C_n, face operators d_i are zero for $i > 0$ and $d_0 = \partial_n$. The Kan-Dold functor K (see [2]) associates with a chain complex C in (Ch, A) an object KC in (Δ^*, A) such that $(\mathrm{KC})_n = \bigsqcup_{\eta} C_q$ where $\eta\colon [n] \to [q]$ runs over all epimorphic maps in Δ, $q = 0, 1, \ldots, n$. We denote by $w_\eta\colon C_q \to (\mathrm{KC})_n$ the corresponding imbedding; then, for any map $\alpha\colon [m] \to [n]$ in Δ, an induced map $\tilde{\alpha}\colon (\mathrm{KC})_n \to (\mathrm{KC})_m$ is defined as follows. Let $\eta\alpha = \varepsilon\eta'$ where η' is epimorphic and ε is monomorphic, then $\tilde{\alpha} \circ w_\eta$ is equal to $w_{\eta\alpha}$ (resp., $w_{\eta'} \circ \partial_q$, resp., 0) if $\varepsilon = 1_{[q]}$ (resp., $\varepsilon = \varepsilon^0$, resp., $\varepsilon \neq 1_{[q]}, \varepsilon^0$). Values of all these functors on maps of objects are defined in a natural way.

In the second part we study the homotopy of maps of simplicial objects of three types. In the third part we prove several theorems on a preservation of the homotopy of maps of simplicial objects of three types by some functors.

We give only sketchy proofs and omit all computations; details may be found in [1].

§ 1. Three types of simplicial objects and chain complexes

1. Adjoints of forgetful functors. It is clear that the values of a left adjoint $^*z_1$ of z_1 on a functor $X\colon \Delta_s^* \to A$, i.e., $^*z_1(X)\colon \Delta^* \to A$, is a Kan extension of X along $\Delta_s^* \subset \Delta^*$ and similarly for $^*z_2$ and $^*z$. Using the well known method of computing Kan extensions (see [6]), we get

THEOREM 1. *Suppose that a category A has finite colimits.*

(i) *If X is an object of (Δ_s^*, A), then*

$$(^*z_1(X))_n = \bigsqcup_{\eta} X_q, \quad n = 0, 1, \ldots,$$

where $\eta\colon [n] \to [q]$ runs over epimorphisms in Δ and for a map $\alpha\colon [m] \to [n]$ in Δ the induced map $\tilde{\alpha}$ is defined by the formula $\tilde{\alpha} \circ w_\eta = w_{\eta'} X(\varepsilon)$, if $\eta\alpha = \varepsilon\eta'$ and η' is an epimorphic map in Δ, ε is a map in Δ_s.

(ii) *If X is an object in (Δ_s^*, A), then*

$$(^*z(X))_n = \bigsqcup_{\eta} X_q, \quad n = 0, 1, \ldots,$$

where $\eta\colon [n] \to [q]$ runs over epimorphisms in Δ_a and for a map $\alpha\colon [m] \to [n]$ in Δ_a the induced map $\tilde{\alpha}$ is defined by the formula $\tilde{\alpha} \circ w_\eta = w_{\eta'} \circ X(\varepsilon)$, if $\eta\alpha = \varepsilon\eta'$ and η' is an epimorphic map in Δ_a, ε is a map in Δ_s.

(iii) *If X is an object in (Δ^*, A), then*

$$(^*z_2(X))_n = \operatorname{Coker}\Big(\bigsqcup_{(\beta,\pi_1,\pi_2)} X_q \overset{\lambda_n}{\underset{\mu_n}{\rightrightarrows}} \bigsqcup_{\pi \in S_n} X_n\Big), \quad n = 0, 1, \ldots,$$

where (β, π_1, π_2) runs over all such triples that $\beta\colon [n] \to [q]$ is a map in Δ, $\pi_1, \pi_2 \in S_n$ and $\beta\pi_1 = \beta\pi_2$. Moreover, λ_n, μ_n are defined by the formula $\lambda_n w_{(\beta,\pi_1,\pi_2)} = w_{\pi_1} X(\beta)$, $\mu_n w_{(\beta,\pi_1,\pi_2)} = w_{\pi_2} X(\beta)$ and let $\nu_n\colon \bigsqcup X_n \to (^*z_2(X))_n$ denote the natural map. For a map $\alpha\colon [m] \to [n]$ in Δ_a we define a map $\bar{\alpha}\colon \bigsqcup_{\pi \in S_n} X_n \to \bigsqcup_{\varrho \in S_m} X_m$ by the formula $\bar{\alpha} \circ w_\pi = w_{\varrho_1} X(\beta)$ if $\pi\alpha = \beta\varrho_1$, β is a map in Δ and ϱ_1 (not unique) is in S_m. Thus $\nu_m \bar{\alpha} \lambda_n = \nu_m \bar{\alpha} \mu_n$; hence $\bar{\alpha}$ induces a unique map $\tilde{\alpha}\colon (^*z_2(X))_m \to (^*z_2(X))_n$ such that $\tilde{\alpha}\nu_m = \nu_n \bar{\alpha}$.

For example, let L be a simplicial complex with vertices ordered by a relation $\leqslant$. Then the sets $L_n' = \{\langle v_0, \ldots, v_n\rangle\}$, where $v_0, \ldots, v_n$ are vertices of a simplex in L and $v_0 < \ldots < v_n$, with obvious face operators $d_i\colon L_n' \to L_{n-1}'$, determine an object L' in $(\Delta_s^*, \mathrm{Set})$. Similarly we define objects L'' in (Δ^*, Set) and L''' in $(\Delta_a^*, \mathrm{Set})$ with components consisting of sequences $\langle v_0, \ldots, v_n\rangle$ of vertices of a simplex in L, such that $v_0 \leqslant \ldots \leqslant v_n$, or of all such sequences, respectively. It is easy to see that $L'' = {}^*z_1(L')$, $L''' = {}^*z_2(L'') = {}^*z(L')$.

2. Direct decompositions of components of objects in (Δ^*, A). In the sequel we denote by A an Abelian category. It is known (see [2]) that the functor $K \circ N$ is equivalent to the identity functor; thus there exist natural isomorphisms $X_n \approx \bigsqcup_{\eta} (NX)_q$ where $\eta\colon [n] \to [q]$ runs over epimorphisms in Δ and X is an object in (Δ^*, A). We give an effective description of this decomposition. For this purpose we denote

$$p_q = (1 - s_0 d_1) \ldots (1 - s_{q-1} d_q)\colon X_q \to X_q,$$

$$f_{i,q} = d_{i+1}(1 - s_{i+1} d_{i+2}) \ldots (1 - s_{q-1} d_q)\colon X_q \to X_{q-1}$$

for $q = 0, 1, \ldots, i = 0, 1, \ldots, q-1$. It is known and easy to see that

$$p_q p_q = p_q, \quad \operatorname{Im} p_q = (NX)_q, \quad \operatorname{Ker} p_q = \sum_{j=0}^{q-1} \operatorname{Im}(s_j\colon X_{q-1} \to X_q);$$

thus there exists a decomposition

$$p_q\colon X_q \overset{p_q'}{\to} (NX)_q \overset{w_q}{\to} X_q$$

with p'_q epimorphic and w_q monomorphic. Each epimorphic map $\eta\colon [n]\to[q]$ in Δ may be uniquely represented as $\eta = \eta_{n-t}^{j_1} \dots \eta_{n-1}^{j_t}$ with $0\leqslant j_1 < \dots < j_t < n$, $t+q = n$. For such η we define maps

$$i_\eta = X(\eta)w_q\colon (NX)_q\to X_q\to X_n,$$
$$p_\eta = p'_q f_{j_1,q+1} \cdots f_{j_t,n}\colon X_n\to X_q\to(NX)_q.$$

THEOREM 2. *Let A be an Abelian category an let X be an object in (Δ^*, A). For each $n = 0, 1, \dots$ the family of maps $\{i_\eta, p_\eta\}$, where η runs over all epimorphic maps $\eta\colon [n]\to[q]$ in Δ, represents X as a direct sum $X_n \approx \coprod_\eta (NX)_q$.*

Proof. We prove the formulas $1_{X_n} = \sum_\eta i_\eta p_\eta$, $p_\eta i_{\eta'} = 0$ for $\eta\neq\eta'$, $p_\eta i_\eta = 1_{(NX)_q}$, using the following relations:

$$s_i f_{i,n} s_j f_{j,n} = \begin{cases} 0 & \text{for } i\neq j,\\ s_i f_{i,n} & \text{for } i = j,\end{cases}$$

$$f_{j,n} s_k = \begin{cases} (1-s_j d_{j+1})\dots(1-s_{n-2}d_{n-1}) & \text{for } k = j,\\ 0 & \text{for } k>j,\\ s_k f_{j-1,n-1} & \text{for } k<j,\end{cases}$$

$$1_{X_n} = p_n + \sum_{j=0}^{n-1} s_j f_{j,n}, \qquad f_{j,n}p_n = 0.$$

3. Functors on Δ_s, Δ and chain complexes. It is known that functors N, k are homotopically equivalent, i.e., for each object X in (Δ^*, A) the chain complexes $N(X)$, $k(X)$ are naturally homotopically equivalent (see [2]). The functors N_s, k_s are not homotopically equivalent. To show this let A be the category Z-Mod and let l be a fixed integer different from 0 and ±1. We define an object X in $(\Delta_s^*, Z\text{-Mod})$ as follows: $X_n = Z$ for $n = 0, 1, \dots$ and $d_i(x) = lx$ for all $x\in Z$, $i = 0, 1, \dots$ Then $H_{2m+1}(kX) = Z/lZ$ for all m but $(NX)_n = 0$ for all $n>0$; thus $H_n(NX) = 0$ for all $n>0$, whence $N_s(X)$ and $k_s(X)$ are not homotopically equivalent.

THEOREM 3. *Let A be an Abelian category.*

(i) *The functors k_s and $N\circ {}^*z_1$ are equivalent.*

(ii) $k_s\circ J = N_s\circ J = 1_{(\mathrm{Ch},A)}$.

(iii) *The functors ${}^*z_1\circ J\circ N$ and $1_{(\Delta^*,A)}$ are equivalent.*

(iv) $K = {}^*z_1\circ J$.

(v) (*Kan-Dold Theorem*) *The functors $N\circ K$, $K\circ N$ are equivalent to the identity functors.*

(vi) *Functors $K\circ k_s$ and ${}^*z_1$ are equivalent.*

Proof. (i) Let X be an object in (Δ_s^*, A); then we define the equivalences φ and ψ as compositions

$$\varphi(X)_n\colon (k_s(X))_n = X_n \xrightarrow{w_{1_{[n]}}} ({}^*z_1(X))_n \xrightarrow{p'_n} ((N{}^*z_1)(X))_n,$$
$$\psi(X)_n\colon ((N{}^*z_1)(X))_n \xrightarrow{w_n} ({}^*z_1(X))_n \xrightarrow{p''_n} X_n = (k_s(X))_n,$$

where p''_n denotes a projection on a direct summand. A standard computation, in which we use the formula $d_0 p_q = p_{q-1}\partial_q$, shows that $\varphi\psi = 1$ and $\psi\varphi = 1$.

(ii) and (iv) follow by the definition of functors k_s, N_s, z_1, K and J.

(iii) In the proof we check that for each object X in (Δ^*, A) the isomorphisms described in Theorem 2

$$(({}^*z_1\circ J\circ N)(X))_n = \coprod_\eta ((J\circ N)(X))_q = \coprod_\eta (NX)_q \approx X_n$$

determine the natural isomorphism of objects $({}^*z_1\circ J\circ N)(X)\approx X$ in (Δ^*, A).

(v) From (i) and (iii) it follows that $NK = N{}^*z_1 J \approx k_s J = 1$, $KN = {}^*z_1 JN \approx 1$.

(vi) Follows from (i) and (v).

4. Functors on Δ_a and chain complexes. It is not known whether are functors k_a and N'_a homotopically equivalent or not (see [4]). A partial answer is contained in the following theorem.

THEOREM 4. *Let A be an Abelian category.*

(i) *The functors k_s and $N'_a{}^*z = N'_a{}^*z_2{}^*z_1$ are equivalent.*

(ii) *The functors k_s and $k_a{}^*z$ are homotopically equivalent.*

(iii) *The functors N and $N'_a{}^*z_2$ are equivalent.*

(iv) *The functors $k_a{}^*z$ and $N'_a{}^*z$ are homotopically equivalent.*

(v) *The functors k and $N'_a{}^*z_2$ are homotopically equivalent.*

Proof. (i) Let X be an object in (Δ_s^*, A). We have the formulas

$$({}^*z(X))_n = X_n \sqcup D_n({}^*z(X)),$$
$$D_n({}^*z(X)) = \sum_{\pi\in S_n} \mathrm{Im}\{(X(\pi)-\mathrm{sgn}(\pi)1_{X_n})\circ w_{1_{[n]}}\} \sqcup \coprod_\eta X_q,$$

where $\eta\colon [n]\to[q]$ runs over epimorphisms in Δ and $q<n$; thus we get the isomorphisms

$$(k_s(X))_n = X_n \approx \mathrm{Coker}\big(D_n({}^*z(X))\to({}^*z(X))_n\big) = (N'_a{}^*z(X))_n,$$

and it is easy to check that they are natural and commute with differentials.

(ii) Let X be an object in (Δ_s^*, A); then we define maps of chain complexes

$$f(X)\colon k_s(X)\to(k_a\circ{}^*z)(X), \quad g(X)\colon (k_a\circ{}^*z)(X)\to k_s(X)$$

as follows:

$$f_n(X)\colon (k_s(X))_n = X_n \xrightarrow{w_{1_{[n]}}} \coprod_\eta X_q = ({}^*z(X))_n = ((k_a\circ{}^*z)(X))_n,$$
$$g_n(X)\colon ((k_a\circ{}^*z)(X))_n = \coprod_\eta X_q \xrightarrow{\Sigma} X_n = (k_s(X))_n,$$

where $\sum = \sum_{\pi\in S_n} \operatorname{sgn}(\pi)p''_\pi$, p''_π denotes a projection onto a direct summand corresponding to π and $\eta\colon [n]\to[q]$ runs over epimorphisms in Δ_a. A standard but lengthy computation shows that $f(X)$ and $g(X)$ are, in fact, maps of complexes. It is clear that $g(X)f(X) = 1_{k_s(X)}$.

We prove that $f(X)g(X)$ is homotopic to the identity map of the complex $(k_a\circ {}^*z)(X)$ by means of acyclic models. In the categories $(\Delta_s^*, Z\text{-Mod})$, $(\Delta_a^*, Z\text{-Mod})$ the objects $C(p)$, $\bar{C}(p)$ corresponding to a standard p-dimensional simplex are defined as follows:

$$C(p)_n = \bigsqcup_{\varepsilon} Z_\varepsilon, \quad \bar{C}(p)_n = \bigsqcup_{\gamma} Z_\gamma, \quad n = 0, 1, \ldots,$$

where $\varepsilon\colon [n]\to[p]$ runs over maps in Δ_s, $\gamma\colon [n]\to[p]$ runs over maps in Δ_a and Z_ε, Z_γ are free Z-modules on free generators ε, γ. Maps induced by a map $[m]\to[n]$ in Δ_s or Δ_a are obvious. We know that $\bar{C}(p) = {}^*z(C(p))$ and the chain complex $k_a\bar{C}(p)$ is homotopically trivial.

For each object X in (Δ_s^*, A) and for each epimorphism $\eta\colon [n]\to[q]$ in Δ_a we denote by $w_{X,\eta}$ the imbedding $w_{X,\eta}\colon X_q\to({}^*z(X))_n$. We define by induction such natural maps $h_n(X)\colon ({}^*z(X))_n\to({}^*z(X))_{n+1}$, $n = 0, 1, \ldots$ that

(n) $$f_n(X)g_n(X) - 1_{k_a{}^*z(X)_n} = \partial_{n+1}^X h_n(X) + h_{n-1}(X)\partial_n^X,$$

where ∂_n^X denotes the nth differential of a complex $k_a{}^*z(X)$. We put $h_0(X) = 0$ and let us assume that the maps $h_0(X), \ldots, h_{n-1}(X)$ are defined for all objects X in (Δ_s^*, A) and an arbitrary Abelian category A and that they satisfy $(0), \ldots, (n-1)$. Then we have

$$\partial_n^{\bar{C}(p)}[f_n(C(p))g_n(C(p)) - 1_{\bar{C}(p)_n} - h_{n-1}(C(p))\partial_n^{\bar{C}(p)}] = 0$$

and for each epimorphic map $\eta\colon [n]\to[p]$ in Δ_a we have $w_{C(p),\eta}(1_{[p]})\in\bar{C}(p)_n$; thus there exists such an element $b_\eta\in(\bar{C}(p))_{n+1} = \bigsqcup_{\eta'}\bigsqcup_{\varepsilon'} Z_{\varepsilon'}$ (where $\eta'\colon [n+1]\to[q]$ runs over epimorphic maps in Δ_a and $\varepsilon'\colon [q]\to[p]$ runs over maps in Δ_s) that

$$\partial_{n+1}^{\bar{C}(p)} b_\eta = [f_n(C(p))g_n(C(p)) - 1_{\bar{C}(p)_n} - h_{n-1}(C(p))\partial_n^{\bar{C}(p)}]\, w_{C(p),\eta}(1_{[p]})$$

and b_η is of the form

$$b_\eta = \sum_{\eta'} w_{C(p),\eta'}\Big(\sum_{\varepsilon'} \alpha(\eta, \eta', \varepsilon')\varepsilon'\Big),$$

where $\alpha(\eta, \eta', \varepsilon')\in Z$. For any object X in (Δ_a^*, A) we define $h_n(X)$ as follows: let $\eta\colon [n]\to[p]$ be an epimorphic map in Δ_a; then

$$h_n(X)w_{X,\eta} = \sum_{\eta'} w_{X,\eta'}\Big(\sum_{\varepsilon'} \alpha(\eta, \eta', \varepsilon')X(\varepsilon')\Big).$$

It is easy to verify that $h_n(X)$ is natural and the formula (n) follows by a standard but lengthy computation.

(iii) We define maps $t_n(X)\colon \bigsqcup_{\pi\in S_n} X_n\to(NX)_n$ by the conditions $t_n(X)w_\pi = \operatorname{sgn}(\pi)p'_n$ and it is easy to prove that the maps $t_n(X)$ induce the maps $\bar{t}_n(X)\colon (N_a'{}^*z_2(X))_n\to(NX)_n$. Let maps $\bar{u}_n(X)\colon (NX)_n\to(N_a'{}^*z_2(X))_n$ be compositions

$$(NX)_n \xrightarrow{w_n'} X_n \xrightarrow{w_{1_{[n]}}} \bigsqcup_{\pi\in S_n} X_n \to ({}^*z_2X)_n \to (N_a'{}^*z_2(X))_n.$$

Then $\bar{u}_n(X)$ are inverses of $\bar{t}_n(X)$ and commute with differentials.

(iv) follows by (i) and (ii), (v) follows by (iii).

§ 2. Homotopies in categories of simplicial objects

1. A standard triangulation of prisms. We denote the vertices of a standard n-dimensional simplex Δ_n by $0, 1, \ldots, n$. For any map $\alpha\colon [m]\to[n]$ in the category Δ_a we denote by $|\alpha|\colon \Delta_m\to\Delta_n$ such an affine map that $|\alpha|(i) = \alpha(i)$, $i = 0, 1, \ldots, n$ and by $\bar{\alpha}\colon \Delta_m\to\Delta_n$ such a simplicial map of barycentric subdivisions that a barycenter of a face σ of Δ_m is mapped onto a barycenter of a face $|\alpha|(\sigma)$ of Δ_n. If α is monomorphic, then $|\alpha| = \bar{\alpha}$. We denote by $\Delta_s[n]$, (resp., $\Delta[n]$, resp., $\Delta_a[n]$) a contravariant functor represented by an object $[n]$ and defined on the category Δ_s (resp., Δ, resp., Δ_a). Let $\varphi_{j,n}\colon \Delta_{n+1}\to\Delta_n\times I$, for $n = 0, 1, \ldots$, $j = 0, 1, \ldots, n$, be such affine maps that $\varphi_{j,n}(i) = (i, 0)$ for $i\leqslant j$ (resp., $(i-1, 1)$ for $i>j$). The maps $\varphi_{j,n}$, for $j = 0, 1, \ldots, n$, determine a standard triangulation of a prism $\Delta_n\times I$. To this triangulation (with the usual ordering of vertices) correspond objects $P_{n+1,s}$ in the category $(\Delta_s^*, \mathrm{Set})$, $P_{n+1} = {}^*z_1(P_{n+1,s})$ in the category (Δ^*, Set) and $P_{n+1,a} = {}^*z(P_{n+1,s}) = {}^*z_2(P_{n+1})$ in the category $(\Delta_a^*, \mathrm{Set})$. It is well known that $P_{n+1}\approx\Delta[n]\times\Delta[1]$. It is easy to see that similar formulas do not hold for $P_{n+1,s}$ and $P_{n+1,a}$. Essential properties of standard triangulations of prisms are collected in the following proposition:

Proposition 1. *Maps* $\varphi_{j,n}\colon \Delta_{n+1}\to\Delta_n\times I$, $n = 0, 1, \ldots$, $j = 0, 1, \ldots, n$, *satisfy the following relations*:

(1) $$\varphi_{n,n}\circ|\varepsilon^{n+1}| = i_0, \quad \varphi_{0,n}\circ|\varepsilon^0| = i_1$$

where $i_\delta\colon \Delta_n\to\Delta_n\times I$, $\delta = 0, 1$ *and* $i_\delta(x) = (x, \delta)$ *for* $x\in\Delta_n$,

(2) $$\varphi_{j,n}\circ|\varepsilon^i| = (|\varepsilon^i|\times 1)\circ\varphi_{j-1,n-1} \quad \textit{for} \quad i<j,$$

(3) $$\varphi_{i,n}\circ|\varepsilon^{i+1}| = \varphi_{i+1,n}\circ|\varepsilon^{i+1}| \quad \textit{for} \quad i = 0, 1, \ldots, n-1,$$

(4) $$\varphi_{j,n}\circ|\varepsilon^i| = (|\varepsilon^{i-1}|\times 1)\circ\varphi_{j,n-1} \quad \textit{for} \quad i>j+1,$$

(5) $$\varphi_{j,n}\circ|\eta^i| = (|\eta^i|\times 1)\circ\varphi_{j+1,n+1} \quad \textit{for} \quad i\leqslant j,$$

(6) $$\varphi_{j,n}\circ|\eta^i| = (|\eta^{i-1}|\times 1)\circ\varphi_{j,n+1} \quad \textit{for} \quad i>j.$$

Each isomorphism $\pi\colon [n]\to[n]$ in the category Δ_a induces the simplicial map $|\pi|\colon \Delta_n\to\Delta_n$ but the maps $|\pi|\times 1\colon \Delta_n\times I\to\Delta_n\times I$ are not simplicial unless $\pi = 1_{[n]}$. Thus to define a homotopy in a category (Δ_a^*, M) which is "consistent with models" we have to consider another triangulation of prisms.

2. Homotopy in categories (Δ^*, M). Let M be an arbitrary category. We recall a well-known definition of homotopy in (Δ^*, M).

DEFINITION 2. A homotopy of maps of an object X in Y in a category (Δ^*, M) is a *family of maps* $\{h_{j,n}\}$, $n = 0, 1, \ldots, j = 0, 1, \ldots, n$, where $h_{j,n}\colon X_n\to Y_{n+1}$ which satisfy the relations

(7) $$d_i h_{j,n} = h_{j-1,n-1}d_i \quad \text{for} \quad i<j,$$

(8) $$d_{i+1}h_{i+1,n} = d_{i+1}h_{i,n} \quad \text{for} \quad i = 0, 1, \ldots, n-1,$$

(9) $$d_i h_{j,n} = h_{j,n-1}d_{i-1} \quad \text{for} \quad i>j+1,$$

(10) $$s_i h_{j,n} = h_{j+1,n+1}s_i \quad \text{for} \quad i\leqslant j,$$

(11) $$s_i h_{j,n} = h_{j,n+1}s_{i-1} \quad \text{for} \quad i>j.$$

Homotopy $\{h_{j,n}\}$ joins maps $f_0, f_1\colon X\to Y$ where

(12) $$(f_0)_n = d_{n+1}h_{n,n}, \qquad (f_1)_n = d_0 h_{0,n}.$$

It is well known that there exists a natural one-to-one correspondence between the set of all homotopies of maps of object X in Y and the set of all maps of $X\times\Delta[1]$ in Y (see [2]).

Let us suppose for a moment that $M = \text{Set}$. Then each element $x_n\in X_n$ determines a unique map $\tilde{x}_n\colon \Delta[n]\to X$ such that $\tilde{x}_n(1_{[n]}) = x_n$; thus for a map $h\colon X\times\Delta[1]\to Y$ corresponding to the homotopy $h_{j,n}$ we have

$$h_{j,n}(x_n) = h_{n+1}(s_j x_n, \sigma_j) = h_{n+1}(s_j x_n(1_{[n]}), \sigma_j) = h_{n+1}(\tilde{x}_n(\eta^j), \sigma_j)$$
$$= (h\circ(\tilde{x}_n\times 1))_{n+1}(\eta^j, \sigma_j) = (h\circ(\tilde{x}_n\times 1)\circ\varphi'_{j,n})_{n+1}(1_{[n+1]}),$$

where $\varphi'_{j,n}\colon \Delta[n]\to P_{n+1}$ denotes such a map that $\varphi_{j,n} = |\varphi'_{j,n}|$ and $\sigma_j = \sigma_{j,n}\colon [n]\to[1]$ satisfies $\sigma_j(i) = 0$ for $i\leqslant j$ (resp., 1 for $i>j$). Now it is easy to check that all the formulas (7)–(11) follow from Proposition 1; for instance, we obtain formula (7) as follows:

$$d_i h_{j,n}(x_n) = d_i(h\circ(\tilde{x}_n\times 1)\circ\varphi'_{j,n})_{n+1}(1_{[n+1]})$$
$$= (h\circ(\tilde{x}_n\times 1)\circ\varphi'_{j,n})_n d_i(1_{[n+1]})$$
$$= (h\circ(\tilde{x}_n\times 1)\circ\varphi'_{j,n})_n(\varepsilon^i)$$
$$= (h\circ(\tilde{x}_n\times 1)\circ\varphi'_{j,n})_n\circ\Delta[\varepsilon^i]_n(1_{[n]})$$
$$= (h\circ(\tilde{x}_n\times 1)\circ\varphi'_{j,n}\circ\Delta[\varepsilon^i])_n(1_{[n]})$$

$$= (h\circ(\tilde{x}_n\times 1)\circ(\Delta[\varepsilon^i]\times 1)\circ\varphi'_{j-1,n-1})_n(1_{[n]})$$
$$= (h\circ(\tilde{x}_n\circ\Delta[\varepsilon^i])\times 1\circ\varphi'_{j-1,n-1})_n(1_{[n]})$$
$$= (h(\widetilde{d_i x_n}\times 1)\circ\varphi'_{j-1,n-1})_n(1_{[n]})$$
$$= h_{j-1,n-1}(d_i x_n)$$

since $\varphi'_{j,n}\circ\Delta[\varepsilon^i] = (\Delta[\varepsilon^i]\times 1)\circ\varphi'_{j-1,n-1}$ and $\tilde{x}_n\circ\Delta[\varepsilon^i](1_{[n-1]}) = \tilde{x}_n(\varepsilon^i) = \tilde{x}_n d_i(1_{[n]}) = d_i\tilde{x}_n(1_{[n]}) = d_i x_n = \widetilde{d_i x_n}(1_{[n-1]})$.

Thus relations (7)–(11) reflect properties of the standard triangulation of prisms.

3. Homotopy in categories (Δ_s^*, M).

DEFINITION 3. A homotopy of maps of an object X in Y in a category (Δ_s^*, M) is a family of maps $\{h_{j,n}\}$, $n = 0, 1, \ldots, j = 0, 1, \ldots, n$, where $h_{j,n}\colon X_n\to Y_{n+1}$, which satisfy relations (7)–(9). Homotopy $\{h_{j,n}\}$ joins the maps $f_0, f_1\colon X\to Y$ defined by (12).

Let us assume that the category M is closed with respect to finite coproducts. For each object X of the category (Δ_s^*, M) we define an object $X\times\Delta_s[1]$ in (Δ_s^*, M) as follows:

$$(X\times\Delta_s[1])_n = \coprod_{k=0}^{n-1} X_{k,n-1}\sqcup\coprod_{\kappa=-1}^{n} X_{k,n},$$

where $X_{k,n} = X_n$, and we denote the corresponding imbeddings by $w_{k,n}\colon X_{k,n-1}\to(X\times\Delta_s[1])_n$, $w'_{k,n}\colon X_{k,n}\to(X\times\Delta_s[1])_n$. Face operators are defined by the conditions

$$d_i w_{k,n} = \begin{cases} w_{k-1,n-1}d_i & \text{for} \quad i<k, \\ w'_{k-1,n-1} & \text{for} \quad i = k, \\ w'_{k,n-1} & \text{for} \quad i = k+1, \\ w_{k,n-1}d_{i-1} & \text{for} \quad i>k+1, \end{cases}$$

$$d_i w'_{k,n} = \begin{cases} w'_{k-1,n-1}d_i & \text{for} \quad i\leqslant k, \\ w'_{k,n-1}d_i & \text{for} \quad i>k. \end{cases}$$

Each map $f\colon X\to X'$ induces a map $f\times 1\colon X\times\Delta_s[1]\to X'\times\Delta_s[1]$ and maps $i_0, i_1\colon X\to X\times\Delta_s[1]$ are defined by $(i_0)_n = w'_{n,n}$, $(i_1)_n = w'_{-1,n}$.

It is easy to see that if X is a scheme of vertices of a polyhedron $|X|$, then $X\times\Delta_s[1]$ is a scheme of vertices of a polyhedron $|X|\times I$.

THEOREM 4. *There exists a natural one-to-one correspondence between the set of all homotopies of maps of an object X in Y in the category (Δ_s^*, M) and the set of all maps of $X\times\Delta_s[1]$ in Y. If a map h corresponds to a homotopy which joins f_0 and f_1, then we have $h\circ i_0 = f_0$, $h\circ i_1 = f_1$.*

Proof. If $\{h_{j,n}\}$ is a homotopy, then we define a corresponding map $h\colon X\times\Delta_s[1]\to Y$ by the conditions $h_n w_{k,n} = h_{k,n-1}$, $h_n w'_{n,k} = d_{k+1}h_{k,n}$ for $k = 0, 1, \ldots, n$ and $h_n w'_{-1,n} = d_0 h_{0,n}$. Conversely, if $h\colon X\times\Delta_s[1]\to Y$ is any map, then we define a corresponding homotopy $\{h_{j,n}\}$ as $h_{j,n} = h_{n+1}w_{j,n+1}$.

It is easy to see that in the case $M = \mathrm{Set}$ we have $X\times\Delta_s[1] \approx X\otimes\Delta_s[1]$, where $\otimes$ is defined in [7], and in this case our homotopy is identical with that defined in [7].

4. Godement homotopy in categories (Δ_a^*, M). The definition of homotopy in the category $(\Delta_a^*, \mathrm{Set})$ given by Godement in [4] admits an obvious generalization for categories (Δ_a^*, M). For each object X in (Δ_a^*, M) we define an object $X\times\Delta_a[1]$ in (Δ_a^*, M) as follows:

$$(X\times\Delta_a[1])_n = \coprod_{\sigma} X_{\sigma,n}$$

where $X_{\sigma,n} = X_n$, σ runs over all maps $\sigma\colon [n]\to[1]$ in Δ_a and any map $\alpha\colon [m]\to[n]$ in Δ_a induces a map $\tilde{\alpha}\colon (X\times\Delta_a[1])_n\to(X\times\Delta_a[1])_m$ determined by the conditions $\tilde{\alpha}\circ w_\sigma = w_{\sigma\alpha}X(\alpha)$ for all σ ($w_\sigma\colon X_{\sigma,n}\to(X\times\Delta_a[1])_n$ denotes an imbedding). Each map $f\colon X\to X'$ induces a map $f\times 1\colon X\times\Delta_a[1]\to X'\times\Delta_a[1]$ and maps $i_0, i_1\colon X\to X\times\Delta_a[1]$ are imbeddings corresponding to two constant maps $[n]\to[1]$.

DEFINITION 5. A G-homotopy of maps of an object X in Y in a category (Δ_a^*, M) is a *map* $h\colon X\times\Delta_a[1]\to Y$ and it joins maps $h\circ i_0$ and $h\circ i_1$.

If $M = \mathrm{Set}$ then a standard computation shows that the functor $\cdot\times\Delta_a[1]\colon (\Delta_a^*, \mathrm{Set})\to(\Delta_a^*, \mathrm{Set})$ is a Kan extension of a functor $Q\colon \Delta_a\to(\Delta_a^*, \mathrm{Set})$ along the Yoneda map $\Delta_a\to(\Delta_a^*, \mathrm{Set})$ where $Q([n]) = \Delta_a[2n+1] \approx \Delta_a[n]\times\Delta_a[1]$ and for $\alpha\colon [m]\to[n]$ in Δ_a a map $Q(\alpha)\colon \Delta_a[2m+1]\to\Delta_a[2n+1]$ is defined as follows. For each map $\gamma\colon [p]\to[2m+1]$ there exist unique maps $\gamma_1\colon [p]\to[m]$, $\gamma_2\colon [p]\to[1]$ such that $\gamma = \gamma_1+(m+1)\gamma_2$ and we define $(Q(\alpha))_p(\gamma) = \alpha\circ\gamma_1+(n+1)\gamma_2$. It is easy to see that Q is equivalent to a functor $[n]\mapsto\Delta_a[n]\times\Delta_a[1]$.

5. τ**-homotopy in categories** (Δ_a^*, M). We have observed that in the case of categories $(\Delta_s^*, \mathrm{Set})$, $(\Delta_a^*, \mathrm{Set})$ (and similarly for (Δ^*, Set)) we can identify a homotopy of maps of an object X in Y with a map of one of objects $X\times\Delta_s[1]$, $X\times\Delta[1]$, $X\times\Delta_a[1]$ in Y and that functors $\cdot\times\Delta_s[1]$, $\cdot\times\Delta[1]$, $\cdot\times\Delta_a[1]$ are Kan extensions of one of the functors P_s, P (which is defined similarly as P_s), Q, along a Yoneda map. Using formulas which express values of Kan extension on an object X, we can define a homotopy in categories (Δ_s^*, M), (Δ^*, M) and (Δ_a^*, M), where M is assumed to be closed with respect to direct limits (this assumption is not essential). A choice of one of the functors P_s, P, Q and of transformations i_0, i_1 of a Yoneda map in P_s, P, Q determines a "model" for homotopy. The functors P_s, P and the transformations correspond to traditional "prismatic models" $\Delta_n\times I$ with maps of Δ_n into lower and upper faces of $\Delta_n\times I$. The functor Q corresponds to "simplicial models" Δ_{2n+1} with maps of Δ_n into faces $\{0, 1, \dots, n\}$, $\{n+1, n+2, \dots, 2n+1\}$.

Now we describe a homotopy of another type in categories (Δ_a^*, M) which correspond to "prismatic models". A construction of a required functor $\Delta_a\to(\Delta_a^*, \mathrm{Set})$ is in fact a construction of some special triangulation of prisms $\Delta_n\times I$, appropriately related to maps induced by maps in Δ_a. We describe one such triangulation, called τ-triangulation. It is not as good as the standard triangulation, which corresponds to the functor P, because maps $|\bar{\eta}^i|\times 1\colon \Delta_n\times I\to\Delta_{n-1}\times I$ map τ-simplexes onto τ-simplexes, but are not affine in general (compare formula $(19)_2$).

We can obtain another useful functor $\Delta_a\to(\Delta_a^*, \mathrm{Set})$ by constructing another triangulation of prisms. Over each simplex σ of a barycentric subdivision of Δ_n (with natural ordering of vertices induced by an inclusion of faces) we build the standard triangulation of a prism $\sigma\times I$. The sum of all such triangulations determines the triangulation of a prism $\Delta_n\times I$ appropriately related to maps in Δ_a. Imbeddings of Δ_n into lower and upper faces of $\Delta_n\times I$ are simplicial if we consider Δ_n with a barycentric triangulation. Consequently, we have to replace objects X in (Δ_a^*, M) by their simplicial subdivisions, generalizing the Kan construction in (Δ^*, Set). We shall not discuss this subject here.

For fixed n ($n = 0, 1, \dots$) we consider sequences $(i_0, i_1, \dots, i_{m-1}; \delta)$ such that $0\leqslant m\leqslant n$, $\delta = 0, 1$, $i_0, i_1, \dots, i_{m-1}$ are different integers and $0\leqslant i_k\leqslant n$ for $k = 0, 1, \dots, m-1$. For each such sequence we denote by $i_m, \dots, i_n$ such integers that $\{i_m, \dots, i_n\} = \{0, \dots, n\}\setminus\{i_0, \dots, i_{m-1}\}$ and $i_m < \dots < i_n$. We can identify a sequence $(i_0, \dots, i_{m-1})$ with a monomorphic map $i\colon [m-1]\to[n]$ in Δ_a. We denote by $\tau_{n+1}(i_0, \dots, i_{m-1}; \delta)\colon \Delta_{n+1}\to\Delta_n\times I$ such an affine map that

$$\tau_{n+1}(i_0, \dots, i_{m-1}; \delta)(j) = \begin{cases}\left(b(i_j, i_{j+1}, \dots, i_n), \tfrac{1}{2}\right) & \text{for } 0\leqslant j\leqslant m,\\ (i_{j-1}, \delta) & \text{for } m<j\leqslant n+1,\end{cases}$$

where $b(i_j, \dots, i_n)$ denotes a barycenter of a face of Δ_n determined by the vertices $i_j, \dots, i_n$. If $m = 0$, then we have

$$\tau_{n+1}(\ ;\delta)(0) = \left(b(0, \dots, n), \tfrac{1}{2}\right) \quad\text{and}\quad \tau_{n+1}(\ ;\delta)(j) = (j-1, \delta)$$

for $0<j\leqslant n+1$. It is easy to see that all maps $\tau_{n+1}(i_0, \dots, i_{m-1}; \delta)$ determine a triangulation of a prism $\Delta_n\times\Delta_1$ and we call it the τ-triangulation.

PROPOSITION 6. *Maps $\tau_{n+1}(i_0, \dots, i_{m-1}; \delta)$ satisfy the following relations*:

(13) $$\tau_{n+1}(\ ;\delta)\circ|\varepsilon^0| = i_\delta, \qquad \delta = 0, 1.$$

(14) *If $m>0$, i, i' are monomorphic maps and the diagram*

$$\begin{array}{ccc}[m-2] & \xrightarrow{\varepsilon^0} & [m-1]\\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i}\\ [n-1] & \xrightarrow{\varepsilon^{i_0}} & [n]\end{array}$$

is commutative, then $\tau_{n+1}(i_0, \dots, i_{m-1}; \delta)\circ|\varepsilon^0| = (|\varepsilon^{i_0}|\times 1)\circ\tau_n(i'_0, \dots, i'_{m-2}; \delta)$.

(15) *If $0<j<m$, then*

$$\tau_{n+1}(i_0, \dots, i_{m-1};\delta)\circ|\varepsilon^j| = \tau_{n+1}(i_0, \dots, i_j, i_{j-1}, \dots, i_{m-1};\delta)\circ|\varepsilon^j|.$$

(16) *If $m<j\leqslant n+1$, $m<n$, then*

$$\tau_{n+1}(i_0, \dots, i_{m-1};\delta)\circ|\varepsilon^j| = \tau_{n+1}(i_0, \dots, i_{m-1}, i_{j-1};\delta)\circ|\varepsilon^{m+1}|.$$

(17) $$\tau_{n+1}(i_0, \ldots, i_{n-1}; \delta) \circ |\varepsilon^{n+1}| = \tau_{n+1}(i_0, \ldots, i_{n-1};\ 1-\delta) \circ |\varepsilon^{n+1}|\,.$$

(18) *Let* $\pi\colon [n]\to[n]$ *be an automorphism in* Δ_a *and let* $i'_m < \ldots < i'_n$ *be such integers that* $\{i'_m, \ldots, i'_n\} = \pi\{i_m, \ldots, i_n\}$; *then we define an automorphism* $\varrho\colon [n+1]\to[n+1]$ *as follows:* $\varrho(p) = p$ *for* $0\leqslant p\leqslant m$ *and* $\varrho(p) = q$ *if* $m<p \leqslant n+1$ *and* $\pi(i_{p-1}) = i'_{q-1}$. *Thus we have*

$$(|\pi|\times 1) \circ \tau_{n+1}(i_0, \ldots, i_{m-1};\delta) = \tau_{n+1}(\pi(i_0), \ldots, \pi(i_{m-1});\delta) \circ |\varrho|\,.$$

(19) *Consider a fixed map* $\tau_{n+1}(i_0, \ldots, i_{m-1};\delta)$ *and a fixed integer* j *such that* $0\leqslant j\leqslant n$. *Let* k, l *be such integers that* $\{j, j+1\} = \{i_k, i_l\}$ *and* $k<l$.

(19)$_1$ *If* $k\leqslant m-1$, *then*

$$(\bar{\eta}^j\times 1) \circ \tau_{n+1}(i_0, \ldots, i_{m-1};\delta) = \tau_n(\eta^j(i_0), \ldots, \widehat{\eta^j(i_k)}, \ldots, \eta^j(i_{m-1});\ \delta) \circ |\eta^k|.$$

(19)$_2$ *If* $k>m-1$, *then*

$$(\bar{\eta}^j\times 1) \circ \tau_{n+1}(i_0, \ldots, i_{m-1};\delta) = \tau_n(\eta^j(i_0), \ldots, \eta^j(i_{m-1});\ \delta) \circ \bar{\bar{\eta}}{}^{k+1}_m$$

where $\bar{\bar{\eta}}{}^{k+1}_m\colon \Delta_{n+1}\to\Delta_n$ *is defined by the formula*

$$\bar{\bar{\eta}}{}^{k+1}_m(tx+(1-t)y) = tx+(1-t)\bar{\eta}^{k+1}(y)$$

for all points x, y *lying on faces of* Δ_{n+1} *determined by the sets of vertices* $\{0, 1, \ldots, m\}$, $\{m+1, \ldots, n+1\}$ *and* $0\leqslant t\leqslant 1$ (¹).

The following definition of homotopy in (Δ_a^*, M) is related to the τ-triangulation of prisms in a similar way as homotopy in (Δ^*, M) is related to the standard triangulation of prisms. We preserve the notation of Proposition 6.

DEFINITION 7. A τ-homotopy of maps of an object X in Y in a category (Δ_a^*, M) is a *family of maps* $\{h_{i_0,\ldots,i_{m-1};\delta,n}\}$, $h_{i_0,\ldots,i_{m-1};\delta,n}\colon X_n\to Y_{n+1}$ which satisfy the relations

(14′) $$d_0 h_{i_0,\ldots,i_{m-1};\delta,n} = h_{i'_0,\ldots,i'_{m-2};\delta,n-1} d_{i_0}\,,$$

(15′) $$d_j h_{i_0,\ldots,i_{m-1};\delta,n} = d_j h_{i_0,\ldots,i_j,i_{j-1},\ldots,i_{m-1};\delta,n}\,,$$

(16′) $$d_j h_{i_0,\ldots,i_{m-1};\delta,n} = d_{m+1} h_{i_0,\ldots,i_{m-1},i_{j-1};\delta,n}\,,$$

(17′) $$d_{n+1} h_{i_0,\ldots,i_{m-1};\delta,n} = d_{n+1} h_{i_0,\ldots,i_{m-1};1-\delta,n}\,,$$

(18′) $$h_{i_0,\ldots,i_{m-1};\delta,n}\tilde{\pi} = \tilde{\rho} h_{\pi(i_0),\ldots,\pi(i_{m-1});\delta,n-1}\,,$$

(19′)$_1$ $$h_{i_0,\ldots,i_{m-1};\delta,n} s_j = s_k h_{\eta^j(i_0),\ldots,\widehat{\eta^j(i_k)},\ldots,\eta^j(i_{m-1});\delta,n-1}\,,$$

(19′)$_2$ $$h_{i_0,\ldots,i_{m-1};\delta,n} s_j = s_{k+1} h_{\eta^j(i_0),\ldots,\eta^j(i_{m-1});\delta,n-1}\,.$$

The homotopy $\{h_{i_0,\ldots,i_{m-1};\delta,n}\}$ joins maps $f_0, f_1\colon X\to Y$ where $(f_0)_n = d_0 h_{;0,n}$, $(f_1)_n = d_0 h_{;1,n}$.

(¹) Les us remark that $|\eta^{k+1}|(tx+(1-t)y) = tx+(1-t)|\eta^{k+1}|(y)$.

§ 3. Preservation of homotopy in categories of simplicial objects

Let Top denote the category of topological spaces and continuous maps. By Proposition 1.3 of Chapter II of [3] it follows that there exist pairs of adjoint functors $(\Delta_s^*, \mathrm{Set}) \underset{\mathrm{Sing}_s}{\overset{|\ |_s}{\rightleftarrows}} \mathrm{Top}$, $(\Delta_a^*, \mathrm{Set}) \underset{\mathrm{Sing}_a}{\overset{|\ |_a}{\rightleftarrows}} \mathrm{Top}$, where $\mathrm{Sing}_s = z_1 \circ \mathrm{Sing}$, Sing_a are functors "simplicial set of singular simplexes", $|\ |_s$, $|\ |_a$ and $|\ |\colon (\Delta^*, \mathrm{Set})\to\mathrm{Top}$ are geometric realization functors and we have $|\ |_s = |\ | \circ {}^*z_1 = |\ |_a \circ {}^*z$, $|\ | = |\ |_a \circ {}^*z_2$.

It is well known (see [3]) that $|\Delta[n]\times\Delta[m]| \approx |\Delta[n]|\times|\Delta[m]|$ and this implies that $|X\times\Delta[1]| \approx |X|\times|\Delta[1]|$; thus the functor $|\ |$ preserves homotopy. The definitions immediately imply

COROLLARY 1. *The forgetful functor* $z_1\colon (\Delta^*, M)\to(\Delta_s^*, M)$ *preserves homotopy.*

COROLLARY 2. *The forgetful functors* $z_2\colon (\Delta_a^*, M)\to(\Delta^*, M)$, $z\colon (\Delta_a^*, M) \to(\Delta_s^*, M)$ *preserve* G*-homotopy.*

To prove that the functor ${}^*z_1$ preserves homotopy we need the following

PROPOSITION 3. *There exists such an equivalence* φ *of functors that the diagrams*

$$\begin{array}{ccc} {}^*z_1(X\times\Delta_s[1]) & \xrightarrow{\varphi(X)} & {}^*z_1(X)\times\Delta[1] \\ & {\scriptstyle {}^*z_1(i_\delta(X))}\nwarrow \quad \nearrow{\scriptstyle j_\delta({}^*z_1(X))} & \\ & {}^*z_1(X) & \end{array}$$

$\delta = 0, 1$ *are commutative for all objects* X *in* (Δ_s^*, M); i_δ, j_δ *denote the appropriate imbeddings.*

Proof. Let $w''_{k,q}\colon {}^*z_1(X)_q\to{}^*z_1(X)_q\times(\Delta[1])_q$, $k = -1, 0, \ldots, q$, denote imbeddings; then we define a natural transformation φ and its inverse ψ by the formulas

$$\varphi_n(X)\, w_\eta w_{k,q} = \tilde{\eta} w''_{k,q} w_{\eta^k}\,, \qquad k = 0, 1, \ldots, q-1\,,$$
$$\varphi_n(X)\, w_\eta w'_{k,q} = \tilde{\eta} w''_{k,q} w_{1_{[q]}}\,, \qquad k = -1, 0, \ldots, q$$

for all epimorphic maps $\eta\colon [n]\to[q]$ in Δ, and if $\eta = \eta^{j_1}\ldots\eta^{j_t}$ with $0\leqslant j_1 < \ldots < j_t < n$ then

$$\psi_n(X)\, w''_{k,n} w_\eta = \begin{cases} w_\eta w'_{k-m,q} & \text{for} \quad j_m<k<j_{m+1}\,, \\ w_{\eta^{j_1}\ldots\eta^{j_m}\eta^{j_{m+2}}\ldots\eta^{j_t}} w_{k-m,q+1} & \text{for} \quad k = j_{m+1}\,. \end{cases}$$

A standard and lengthy computation shows that φ and ψ are in fact natural transformations and that ψ is inverse to φ.

THEOREM 4. *The functor* ${}^*z_1 = (\Delta_s^*, M)\to(\Delta^*, M)$ *preserves homotopy.*

Proof. Let a homotopy $\{h_{j,n}\}$ join maps $f_0, f_1\colon X\to Y$ in (Δ_s^*, M). By Theorem 4 of part 2 to this homotopy corresponds a map $h\colon X\times\Delta_s[1]\to Y$ such that $h\circ j_0 = f_0$,

$h \circ j_1 = f_1$ and $h_{j,n} = h_{n+1} w_{j,n+1}$. The map $\bar{h} = {}^*z_1(h) \circ \psi(X)$: ${}^*z_1(X) \times \varDelta[1] \to {}^*z_1(Y)$ determines a homotopy which joins ${}^*z_1(f_0)$ and ${}^*z_1(f_1)$. The components $\bar{h}_{j,n}$ of a homotopy corresponding to $\bar{h}$ are given by the formula

$$\bar{h}_{j,n} w_\eta = w_{\eta^{j_1} \ldots \eta^{j_m} \eta^{j_{m+1}+1} \ldots \eta^{j_t+1}} h_{j-m,q},$$

where $\eta = \eta^{j_1} \ldots \eta^{j_t}$ and $j_m < j \leqslant j_{m+1}$.

In [2] it is proved that the functors k, N, K preserve homotopy.

THEOREM 5. *Let A be an Abelian category.*

(i) *The functor k_s: $(\varDelta_s^*, A) \to (\mathrm{Ch}, A)$ preserves homotopy.*

(ii) *The functors ${}^*z_1 \circ z_1$ and $1_{(\varDelta^*, A)}$ are homotopically equivalent.*

Proof. (i) Let a homotopy $\{h_{j,n}\}$ join maps f_0, f_1: $X \to Y$ in $(\varDelta_s^*, A)$; then the maps $h_n = \sum_{j=0}^{n} (-1)^j h_{j,n}$ determine homotopy which joins the maps $k_s(f_0)$ and $k_s(f_1)$ in (CH, A).

(ii) Since $k = k_s \circ z_1$, by Theorem 3(i) of part 1 it follows that $N \circ {}^*z_1 \circ z_1 \approx k_s \circ z_1 = k$; thus by Theorem 3(v) of part 1 (Kan-Dold Theorem) we get ${}^*z_1 \circ z_1 \approx K \circ k$. Since K preserves homotopy and the functors k and N are homotopically equivalent, $K \circ k$ is homotopically equivalent to the identity functor.

THEOREM 6. *The functors k_a and N_a' preserve τ-homotopy.*

Proof. Let a τ-homotopy $\{h_{i_0, \ldots, i_{m-1}; \delta, n}\}$ join maps f_0, f_1; $X \to Y$. We define the maps h_n: $(k_a X)_n = X_n \to (k_a Y)_{n+1} = Y_{n+1}$ as follows:

$$h_n = \sum (-1)^a h_{i_0, \ldots, i_{m-1}; \delta, n},$$

where the sum is taken over all admissible indices of h's, $a = \delta + I(i_0, \ldots, i_{m-1}) + \sum_{k=0}^{m-1} (i_k + k + 1)$ and $I(i_0, \ldots, i_{m-1})$ denotes the number of such pairs (k, k') that $0 \leqslant k < k' < m$ and $i_k > i_{k'}$. Using relations (14')–(19'), one can compute that $\partial_{n+1} h_n + h_{n-1} \partial_n = (k_a(f_0 - f_1))_n$; thus the maps $k_a(f_0)$ and $k_a(f_1)$ are homotopic.

To prove that the maps $N_a'(f_0)$ and $N_a'(f_1)$ are homotopic it is sufficient to show that $h_n(D_n(X)) \subset D_{n+1}(Y)$. This easily follows from the formula

$$I(i_0, \ldots, i_{m-1}) + I(\pi(i_m), \ldots, \pi(i_n)) + \sum_{k=0}^{m-1} i_k$$
$$\equiv I(\pi(0), \ldots, \pi(n)) + I(\pi(i_0), \ldots, \pi(i_{m-1})) + \sum_{k=0}^{m-1} \pi(i_k) \pmod 2,$$

which holds for each automorphism π: $[n] \to [n]$ in $\varDelta_a$.

Theorem 4 and the formula $|\ |_s = |\ | \circ {}^*z_1$ imply

COROLLARY 7. *The functor $|\ |_s$: $(\varDelta_s^*, \mathrm{Set}) \to \mathrm{Top}$ preserves homotopy.*

To prove a similar statement for $|\ |_a$ we need a lemma.

LEMMA 8. *There exists such a natural transformation σ of functors that the diagrams*

$$\begin{array}{ccc} |X|_a \times |\varDelta_a[1]|_a & \xrightarrow{\sigma(X)} & |X \times \varDelta_a[1]|_a \\ & {\scriptstyle j_\delta(|X|_a)} \nwarrow \quad \nearrow {\scriptstyle |i_\delta(X)|_a} & \\ & |X|_a & \end{array}$$

$\delta = 0, 1$ are commutative, where i_δ, j_δ denote the appropriate imbeddings.

Proof. At first we consider the case $X = \varDelta_a[n]$. Then it is easy to see that there exists an isomorphism $\varDelta_a[2n+1] \approx \varDelta_a[n] \times \varDelta_a[1]$ which maps $1_{[2n+1]}$ onto $g = (g_1, g_2)$, where g_1: $[2n+1] \to [n]$, g_2: $[2n+1] \to [1]$ satisfy $g_1(i) + (n+1) g_2(i) = i$ for all $i \in [2n+1]$. We denote $b_i = |\tilde{g}|_a(i)$ for all $i \in [2n+1]$ and define

$$\sigma(\varDelta_a[n])(t_0 a_0 + \ldots + t_n a_n, t) = (1-t)(t_0 b_0 + \ldots + t_n b_n) + t(t_0 b_{n+1} + \ldots + t_n b_{2n+1}),$$

where $a_0, \ldots, a_n$ denote vertices of $|\varDelta_a[n]|_a \approx \varDelta_n$, $t, t_0, \ldots, t_n$ belong to the unit interval and $t_0 + \ldots + t_n = 1$. For an arbitrary object X in $(\varDelta_s^*, \mathrm{Set})$ we extend the definition of σ as follows. Let $x_n \in X_n$; then $\sigma(X)$ is the only map such that

$$\sigma(X)(|\tilde{x}_n|_a \times 1) = |\tilde{x}_n \times 1|_a \circ \sigma(\varDelta_a[n]).$$

THEOREM 9. *The functor $|\ |_a$: $(\varDelta_a^*, \mathrm{Set}) \to \mathrm{Top}$ preserves G-homotopy.*

Proof. Let h: $X \times \varDelta_a[1] \to Y$ be a map which joins f_0 and f_1; then $|f_\delta|_a = |h \circ i_\delta|_a = |h|_a \circ |i_\delta|_a = |h|_a \circ \sigma(X) \circ j_\delta$ for $\delta = 0, 1$; consequently $|h|_a \circ \sigma(X)$ joins $|f_0|_a$ and $|f_1|_a$.

THEOREM 10. *Let f_0, f_1: $X \to Y$ be maps in the category $(\varDelta_s^*, \mathrm{Set})$ and suppose that the maps ${}^*z(f_0)$, ${}^*z(f_1)$ are τ-homotopic. Then the maps $|{}^*z(f_0)|_a$, $|{}^*z(f_1)|_a$ are homotopic.*

Proof. We know that $|{}^*z(X)|_a \approx |X|_s$; thus we can represent $|{}^*z(X)|_a$ as a cokernel of a pair of maps $\bigsqcup_{\varepsilon} \bigsqcup_{x_n} \varDelta_{m,x_n} \underset{\mu}{\overset{\lambda}{\rightrightarrows}} \bigsqcup_{x_n} \varDelta_{n,x_n}$, where ε: $[m] \to [n]$ runs over maps in $\varDelta_s$, x_n runs over X_n, $\varDelta_{m,x_n} = \varDelta_m$, $\varDelta_{n,x_n} = \varDelta_n$ and $\lambda \circ w_{(\varepsilon, x_n)} = w_{(n,x_n)} \circ |\varepsilon|$, $\mu \circ w_{(\varepsilon, x_n)} = w_{(m, \tilde{\varepsilon} x_n)}$. Let ν: $\bigsqcup \varDelta_{n,x_n} \to |{}^*z(X)|_a$ be a natural map.

Relations (15')–(17') correspond to all pairs of adjacent $(n+1)$-dimensional simplexes of τ-triangulation of a prism $\varDelta_n \times \varDelta_1$. Using those relations, we show that for each $x_n \in X_n$ there exists a unique map H_{x_n}: $\varDelta_n \times \varDelta_1 \to |{}^*z Y|_a$ such that $H_{x_n} \circ \tau_{n+1}(i_0, \ldots, i_{m-1}; \delta) = |\tilde{h}_{i_0, \ldots, i_{m-1}; \delta, n}(x_n)|_a$. In fact, using for example (16'), we get (abbreviation: $h = h_{i_0, \ldots, i_{m-1}; \delta, n}$, $h' = h_{i_0, \ldots, i_{m-1}, i_j; \delta, n}$)

$$|\tilde{h}(x_n)|_a \circ |\varepsilon^j|_a = |\tilde{h}(x_n) \circ \varDelta[\varepsilon^j]|_a = |\widetilde{d_j h}(x_n)|_a \doteq |\widetilde{d_{m+1} h'}(x_n)|_a = |\tilde{h}'(x_n)|_a \circ |\varepsilon^{m+1}|_a$$

and similarly for (15′), (17′). Thus the maps $|\tilde{h}(x_n)|_a$ induce H_{x_n}. By (14′) it follows that the maps H_{x_n} induce such a map $H\colon |X|_s\times\Delta_1\to|{}^*z(Y)|_a$ that $H\circ(\nu\times 1)\circ(w_{(n,x_n)}\times 1)=H_{x_n}$. In fact, we have (abbreviation: $h''=h_{i'_0,\ldots,i'_{m-2};\delta,n}$).

$$\begin{aligned} H_{x_n}\circ(|\varepsilon^{i_0}|\times 1)\circ\tau_n(i'_0,\ldots,i'_{m-2};\delta) &= H_{x_n}\circ\tau_{n+1}(i_0,\ldots;i_{m-1};\delta)\circ|\varepsilon^0| = |\tilde{h}(x_n)|_a\circ|\varepsilon^0|_a\\ &= |\widetilde{d_0h}(x_n)|_a = |\tilde{h}''(d_{i_0}x_n)|_a\\ &= H_{d_{i_0}(x_n)}\circ\tau_n(i'_0,\ldots,i'_{m-2};\delta);\end{aligned}$$

thus for any ε holds $H_{x_n}\circ(|\varepsilon|\times 1)=H_{\tilde{\varepsilon}(x_n)}$. If we put $H'=\bigsqcup H_{x_n}\colon \bigsqcup \Delta_{n,x_n}\times\Delta_1\to|{}^*z(Y)|_a$, then

$$\begin{aligned} H'\circ(\lambda\times 1)(w_{(\varepsilon,x_n)}\times 1) &= H'\circ(w_{(n,x_n)}\times 1)\circ(|\varepsilon|\times 1) = H_{x_n}\circ(|\varepsilon|\times 1) = H_{\tilde{\varepsilon}(x_n)}\\ &= H'\circ(w_{(m,\tilde{\varepsilon}x_n)}\times 1) = H'\circ(\mu\times 1)(w_{(\varepsilon,x_n)}\times 1);\end{aligned}$$

thus H' induces H. It is easy to see that H joins $|{}^*z(f_0)|_a$ and $|{}^*z(f_1)|_a$.

References

[1] S. Balcerzyk, P. H. Chan and R. Kiełpiński, *On three types of simplicial objects*, preprint
[2] A. Dold and D. Puppe, *Homologie nicht-additiver Funktoren. Anwendungen*, Ann. Inst Fourier 11 (1961), pp. 201–312.
[3] P. Gabriel and M. Zisman, *Calculus of fractions and homotopy theory*, Berlin 1967.
[4] R. Godement, *Topologie algébrique et théorie des faisceaux*, Paris 1958.
[5] A. Hanna, *On the homology theory of categories*, J. Reine und Angew. Math. 248 (1971), pp. 133–146, 252 (1972), pp. 112–127.
[6] S. Mac Lane, *Categories for the Working Mathematician*, Berlin 1971.
[7] C. P. Rourke and B. J. Sanderson, *Δ-sets. Homotopy theory*, Quart. J. Math. 22 (1971), pp. 321–338.

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