[3] R. D. Mayer and R. S. Pierce, *Boolean algebras with ordered bases*, Pacific J. Math. 10 (1960), pp. 925–942.
[4] A. Mostowski and A. Tarski, *Boolesche Ringe mit geordneter Basis*, Fund. Math. 32 (1939), pp. 69–86.
[5] D. Myers, *Invarient uniformization*, Fund. Math. this volume, pp. 65–72.
[6] R. S. Pierce, *Bases of countable Boolean algebras*, J. Symb. Logic 38 (1973), pp. 212–214.

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Extension of a valuation on a lattice

by

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Abstract. In a recent paper [2], Fox and Morales give necessary and sufficient conditions in order that a strongly additive (= valuation) set function from a lattice of sets $\mathfrak{L}$ into a complete metric group be uniquely extendable to the generated (σ, δ)-lattice. It is shown in the present note that the same conditions are valid in a more general setting, i.e., when L is an arbitrary lattice and v is a valuation on L with values in a sequentially complete Hausdorff topological group. The proof is accomplished by means of the elimination of Pettis' theorem ([3], Theorem 1.2), the basic lemma in the proof of Fox and Morales.

1. Introduction. Let L be a lattice and G an Abelian topological group. A function $v\colon L\to G$ is called a *valuation* [1], [3] if

$$v(x\vee y)+v(x\wedge y) = v(x)+v(y)\,.$$

It is easy to show that ([1], p. 75) and ([4], p. 239) that if L is a relatively complemented lattice (or, in particular, a Boolean ring), then (1.1) is equivalent to

$$v(x\vee y) = v(x)+v(y) \quad \text{for} \quad x\wedge y = 0\,.$$

We do not assume, however, that a null element belongs to L. A valuation v on L is said to be (*order*) *σ-continuous* (*δ-continuous*) if, for every increasing (decreasing) sequence (x_n) such that $x_n\in L$ $(n = 1, \ldots)$ with $\sup\limits_n x_n\in L$ $(\inf\limits_n x_n\in L)$, we have $v(x_n)\to v(\sup\limits_n x_n)$ $(v(x_n)\to v(\inf\limits_n x_n))$.

v is *(σ, δ)-continuous* if it is both σ-continuous and δ-continuous.

A lattice H is said to be *σ-continuous* if it is (σ, δ)-complete (i.e. the limits of both increasing and decreasing countable sequences of elements of H are in H) and the following condition holds: $y, x_n\in H$ $(n = 1, \ldots)$, $x_n\uparrow x \Rightarrow x_n\wedge y\uparrow x\wedge y$; and dually.

All lattices occurring in the present note are supposed to be contained in a fixed σ-continuous lattice H. The only lattice operations which will be considered are restrictions of those on H. Accordingly we shall use the word lattice to mean the subset of H closed with respect to the restrictions of the lattice operations on H.

Let $v\colon L\to G$ be a (σ, δ)-continuous valuation. The aim of this note is to establish necessary and sufficient conditions for the unique extension of v to a continuous valuation on the (σ, δ)-lattice generated by L. These conditions appear to be the same as those given by G. Fox and P. Morales in Theorem 2.10 of [3] for the lattice of subsets of a fixed set T.

2. Extension theorems. Let L be a lattice contained in a σ-continuous lattice H. Further, let v be a valuation. The domain of v (always assumed to be a lattice) is denoted by $D(v)$. Let $x \in L$; then the class $\{y\colon y \in D(v), y \leqslant x\}$, if it is non-empty and directed by $\geqslant$, defines the Moore–Smith sequence $\{v(y)\}_{y\leqslant x, y\in D(v)}$. Similarly the class $\{y\colon y\in D(v), y\geqslant x\}$, if non-empty, directed by $\leqslant$, defines the Moore–Smith sequence $\{v(y)\}_{y\geqslant x, y\in D(v)}$. All lattices throughout this section are sublattices of H.

2.1. Definition. Let v, w be valuations:

(a) v is *w-lower regular* if, for every $x \in D(v)$, the set $\{y\in D(w)\colon y\leqslant x\}$ is non-empty, and $\lim\limits_{y\leqslant x, y\in D(w)} w(y) = v(x)$.

(b) v is *w-upper regular* if, for every $x\in D(v)$, the set $\{y\in D(w)\colon y\geqslant x\}$ is non-empty, and $\lim\limits_{y\geqslant x, y\in D(w)} w(y) = v(x)$.

2.2. Lemma. *We assume first that G is a complete metric group. Let v and w be valuations, and let $\varepsilon>0$ be arbitrary. If v is w-lower regular (w-upper regular) and (x_n) is a decreasing (increasing) $D(v)$-sequence, there exists a decreasing (increasing) $D(w)$-sequence (y_n) such that $y_n\leqslant x_n$ $(n = 1, \ldots)$ and $|w(y_n)-v(x_n)|<\varepsilon$. Moreover, if $z\in D(w)$ is such that $z\leqslant \inf\limits_n x_n$ $(z\geqslant \sup\limits_n x_n)$, then we may choose the sequence (y_n) so as to satisfy the additional condition*

$$z\leqslant \inf_n y_n \leqslant \inf_n x_n \quad (z\geqslant \sup_n y_n \geqslant \sup_n x_n)\,.$$

Proof. Let $\varepsilon>0$. On account of the properties of v and w, we can for every n and every $x_n\in D(v)$, find an $s_n \in D(w)$ such that $s_n\leqslant x_n$, and $|w(s_n)-v(x_n)|<\varepsilon$. Let $y_n = \bigwedge\limits_1^n s_i$, so that $y_n\in D(w)$, $y_n\leqslant x_n$, $y_n\downarrow$.

The proof that $\{y_n\}$ has the required properties is inductive. First we verify that $|v(x_2)-w(s_1\wedge s_2)|<\varepsilon$. Indeed, having assumed that $|v(x_i)-w(s_i)|<\frac{1}{3}\varepsilon$ $(i = 1, \ldots)$, we get

$$\begin{aligned}|v(x_2)-w(s_1\wedge s_2)| &= |v(x_1\wedge x_2)-w(s_1\wedge s_2)|\\ &= |v(x_1)+v(x_2)-v(x_1\vee x_2)-w(s_1)-w(s_2)+w(s_1\vee s_2)|\\ &\leqslant |v(x_1)-w(s_1)|+|v(x_2)-w(s_2)|+|v(x_1)-w(s_1\vee s_2)|\\ &< \tfrac{1}{3}\varepsilon+\tfrac{1}{3}\varepsilon+\tfrac{1}{3}\varepsilon = \varepsilon,\end{aligned}$$

since $x_1\geqslant s_1\vee s_2\geqslant s_1$.

Further, assuming that

$$|v(x_{n-1})-w(\bigwedge_1^{n-1} s_i)|<\tfrac{1}{3}\varepsilon,$$

we get

$$\begin{aligned}&|v(x_n)-w(\bigwedge_1^n s_i)|\\ &\quad= |v(\bigwedge_1^n x_i)-w(\bigwedge_1^n s_i)| = |v(\bigwedge_1^{n-1} x_i\wedge x_n)-w(\bigwedge_1^{n-1} s_i\wedge s_n)|\\ &\quad= |v(\bigwedge_1^{n-1} x_i)+v(x_n)-v((\bigwedge_1^{n-1} x_i)\vee x_n)-w(\bigwedge_1^{n-1} s_i)-w(s_n)+w((\bigwedge_1^{n-1} s_i)\vee s_n)|\\ &\quad\leqslant |v(\bigwedge_1^{n-1} x_i)-w(\bigwedge_1^{n-1} s_i)|+|v(x_n)-w(s_n)|+|v(x_{n-1})-w((\bigwedge_1^{n-1} s_i)\vee s_n)|,\end{aligned}$$

but

$$\bigwedge_1^{n-1} s_i\leqslant x_{n-1} \quad\text{and}\quad (\bigwedge_1^{n-1} s_i)\vee s_n\leqslant x_{n-1}\vee x_n = x_{n-1}.$$

Therefore

$$|v(x_n)-w(\bigwedge_1^n s_i)|<\varepsilon.$$

In the second case, choose $s_n\in D(w)$ such that $s_n\geqslant x_n$ and

$$|w(s_n)-v(x_n)|<\tfrac{1}{3}\varepsilon \quad (n = 1, \ldots)\,.$$

Let $y_n = \bigvee\limits_1^n s_i$. A similar procedure to that used before shows that y_n satisfies the condition of the lemma. If, in the respective cases, $z\in D(w)$ is such that $z\leqslant \inf\limits_n x_n$ $(z\geqslant \sup\limits_n x_n)$ then, taking $y_n' = y_n\vee z$ $(y_n' = y_n\wedge z)$, we infer, by the (σ, δ)-continuity of H, that $z\leqslant \inf\limits_n y_n' \leqslant \inf\limits_n x_n$ $(z\geqslant \sup\limits_n y_n'\geqslant \sup\limits_n x_n)$.

Henceforth v is a fixed function of domain L.

2.3. Lemma. *Let v be a σ-continuous valuation. Then the following are equivalent:*

(a) *for every $x\in L$ such that the set $\{y\in L,\ x\geqslant y\}$ is non-empty*

$$\lim_{y\leqslant x, y\in L} v(y) \quad \textit{exists},$$

(b) *for every increasing sequence $(x_n)\subset L$*

$$\lim_n v(x_n) \quad \textit{exists}.$$

Moreover, If (a) (*or* (b)) *holds, then for every increasing sequence $(x_n)\subset L$ with $\sup x_n = x$ we have*

$$\lim_{y\leqslant x, y\in L} v(y) = \lim_n v(x_n)\,.$$

Proof. (a) $\Rightarrow$ (b). Let $x = \bigvee_{1}^{\infty} x_n$; $\lim\limits_{y \leqslant x,\, y \in L} v(y) = g$ exists by (a). We shall show that $\lim\limits_n v(x_n) = g$. For a given $\varepsilon > 0$ there is a $y \in L$, $y \leqslant x$ such that $y \leqslant y' \leqslant x$, $y' \in L$ implies $|v(y)-v(y')| < \varepsilon$, $|v(y')-g| < \varepsilon$.

Then, since

$$
\begin{aligned}
|v(x_n)-v(y)| &= |v(x_n \vee y)+v(x_n \wedge y)-v(y)-v(y)| \\
&\leqslant |v(x_n \vee y)-v(y)|+|v(y)-v(x_n \wedge y)| ,
\end{aligned}
$$

we get

$$
\begin{aligned}
|v(x_n)-g| &\leqslant |v(x_n)-v(y)|+|v(y)-g| \\
&< |v(x_n \vee y)-v(y)|+|v(y)-v(x_n \wedge y)|+\varepsilon .
\end{aligned}
$$

Finally, as $y \leqslant x_n \vee y \leqslant x$, it follows that $|v(x_n \vee y)-v(y)| < \varepsilon$ and $v(y \wedge x_n) \to v(y$ by the σ-continuity of v.

(b) $\Rightarrow$ (a). If $\lim\limits_{y \leqslant x,\, y \in L} v(y)$ does not exist, then for some $\varepsilon > 0$ and every $y \in L$, there is a $y' \in L$ such that $y \leqslant y' \leqslant x$ and $|v(y)-v(y')| \geqslant \varepsilon$. (There exist $y', y'' \in L$ $y \leqslant y'$, $y'' \leqslant x$ and $|v(y'')-v(y')| \geqslant \varepsilon$. If, for instance, $|v(y)-v(y')| < \frac{1}{2}\varepsilon$, then

$$
\begin{aligned}
|v(y)-v(y'')| &= |v(y)-v(y')+v(y')-v(y'')| \\
&\geqslant \big||v(y')-v(y'')|-|v(y)-v(y')|\big| \\
&= |v(y')-v(y'')|-|v(y)-v(y')| \geqslant \varepsilon - \tfrac{1}{2}\varepsilon = \tfrac{1}{2}\varepsilon .
\end{aligned}
$$

We can construct inductively an increasing sequence (y_n) in L such that $|v(y_{n-1})-v(y_n)| \geqslant \varepsilon$, contrary to (b).

Similarly we prove the dual lemma.

2.4. Lemma. *Let v be a δ-continuous valuation. Then the following are equivalent:*

(a) *for every $x \in L$ such that the set $\{y \in L,\ y \geqslant x\}$ is non-empty*

$$\lim_{y \geqslant x,\, y \in L} v(y) \quad \textit{exists},$$

(b) *for every decreasing sequence $(x_n) \subset L$*

$$\lim_n v(x_n) \quad \textit{exists}.$$

Moreover, if (a) *(or* (b)*) holds, then for every decreasing sequence $(x_n) \subset L$ with $\inf\limits_n x_n = x$, we have $\lim\limits_{y \geqslant x, y \in L} v(y) = \lim\limits_n v(x_n)$.*

2.5. Definition. Let v be a valuation. We say that v is *monotonely convergent* if, for every monotone sequence (x_n) in L, the sequence $\{v(x_n)\}$ converges.

In what follows, up to the statement of Theorem 2.10, v is assumed to be monotonely convergent, (,)-continuous valuation.

By Lemma 2.3 v extends to the function v_σ on L_σ: $v_\sigma(x) = \lim\limits_{y \leqslant x,\, y \in L} v(y)$; and, by Lemma 2.4, v extends to the function v_δ on L_δ: $v_\delta(x) = \lim\limits_{y \geqslant x,\, y \in L} v(y)$.

2.6. Lemma. *The extension $v_\sigma(v_\delta)$ is a monotonely convergent σ-continuous (δ-continuous) valuation.*

Proof. We shall prove the lemma for v_σ; the other proof is analogous. That v is a valuation is clear. Therefore we have to show that

(a) if $x_n \uparrow x$, $x_n \in L_\sigma \Rightarrow v_\sigma(x_n) \to v_\sigma(x)$,

(b) if $x_n \downarrow$, $x_n \in L_\sigma \Rightarrow \{v_\sigma(x_n)\}$ converges.

To prove (a), for every n take $y_n \in L$, $y_n \leqslant x_n$ such that $y_n \leqslant y \leqslant x_n$ and $y \in L$ $\Rightarrow |v(y)-v_\sigma(x_n)| < n^{-1}$. For every n there exists an increasing sequence $(y_n^i)_{i=1\ldots}$ in L converging to x_n. Put $z_i = \bigvee_{k=1}^{i} (y_i^k \vee y_k)$ so that $y_i \leqslant z_i \leqslant x_i$, $z_i \in L$, $z_i \uparrow x$. Then $|v(z_m)-v_\sigma(x_m)| < m^{-1}$ and hence, by Lemma 2.3, $v(z_m) \to v_\sigma(x)$, i.e., $v_\sigma(x_m) \to v_\sigma(x)$.

To prove (b), let $\varepsilon > 0$ be arbitrary. On account of Lemma 2.2 there is a decreasing sequence (y_n) in L such that $|v(y_n)-v_\sigma(x_n)| < \frac{1}{3}\varepsilon$. Therefore $\{v_\sigma(x_n)\}$ is a Cauchy sequence.

2.7. Lemma. (a) *If v_σ is v_δ-lower regular, it is δ-continuous.*

(b) *If v_δ is v_σ-upper regular, it is σ-continuous.*

Proof. We shall demonstrate (a), (b) being similar. Let $x_n \downarrow x$, $x_n, x \in L$ and let $\varepsilon > 0$ be arbitrary. There exists a $y \in L_\delta$, $y \leqslant x$ such that $y \leqslant y' \leqslant x$ and $y' \in L_\delta$ implies that $|v_\delta(y')-v_\sigma(x)| < \frac{1}{2}\varepsilon$. By Lemma 2.2 there exists a decreasing sequence (y_n) in L_δ such that $|v_\delta(y_n)-v_\sigma(x_n)| < \frac{1}{2}\varepsilon$, and $y \leqslant \inf\limits_n y_n \leqslant x$. On account of Lemma 2.6 $\{v_\sigma(x_n)\}$ converges.

Therefore we have

$$
\begin{aligned}
|\lim_n v_\sigma(x_n)-v_\sigma(x)| &= |\lim_n v_\sigma(x_n)-\lim_n v_\delta(y_n)-v_\sigma(x)+\lim_n v_\delta(y_n)| \\
&\leqslant |\lim_n v_\delta(y_n)-v_\sigma(x)|+\tfrac{1}{2}\varepsilon < \varepsilon .
\end{aligned}
$$

2.8. Lemma. *v_σ is v_δ-lower regular iff v_δ is v_σ-upper regular.*

Proof. We shall show that if v_σ is v_δ-lower regular, then v_δ is v_σ-upper regular. Let $x \in L_\delta$. Since, on account of Lemma 2.7, v_σ is a δ-continuous monotonely convergent valuation, then $\lim\limits_{y \geqslant x,\, y \in L_\delta} v_\sigma(y) = \mu(x)$ exists (Lemma 2.4).

Take an arbitrary $\varepsilon > 0$. Then there is a $y \in L_\sigma$, $y \geqslant x$ such that $y \geqslant y' \geqslant x$ and $y' \in L_\sigma$ implies $|v_\sigma(y')-\mu(x)|$, $|v_\sigma(y')-v_\sigma(y)| < \varepsilon$. Let $x_n \downarrow x$, $x_n \in L$. We have

$$
\begin{aligned}
|v(x_n)-\mu(x)| &\leqslant |v(x_n)-v_\sigma(y)|+|v_\sigma(y)-\mu(x)| \\
&< |v_\sigma(x_n \vee y)-v_\sigma(y)|+|v_\sigma(y)-v_\sigma(x_n \wedge y)|+\varepsilon .
\end{aligned}
$$

Since $|v_\sigma(y)-v_\sigma(y \wedge x_n)| < \varepsilon$ and $v_\sigma(x_n \vee y) \to v_\sigma(y)$ (Lemma 2.7), we get $v(x_n) \to \mu(x)$. But, on the other hand, $v(x_n) \to v_\sigma(x)$ and so $\mu(x) = v_\sigma(x)$.

In the sequel we say that a valuation is v-regular if it is both v_σ-lower regular and v_δ-upper regular.

2.9. LEMMA. *Let v_σ be v_δ-lower regular (or, equivalently, let v_δ be v_σ-upper regular). Let u be a (σ,δ)-continuous monotonely convergent valuation, v-regular.*

(a) *If u is an extension of v_δ, then u_σ is v-regular and δ-continuous.*

(b) *If u is an extension of v_σ, then u_δ is v-regular and σ-continuous.*

Proof. We prove (a); part (b) is similar. Let $x\in (D(u))_\sigma$, and let $\varepsilon>0$ be arbitrary. Let $w_1(x) = \lim\limits_{y\leqslant x,\, y\in L_\sigma} v_\delta(y)$. There exists a $y\in L_\delta$, $y\leqslant x$ such that $y\leqslant y'\leqslant x$ and $y'\in L_\delta$ implies $|v_\delta(y')-w_1(x)|<\varepsilon$. By the definition of u_σ, there is a $z\in D(u)$ such that $|u(z)-u_\sigma(x)|<\varepsilon$ and, since $D(v)=L$, we have $L_\delta\subset D(u)$ by the hypotheses, and so we may suppose that $y\leqslant z$.

Since u is v_σ-lower regular, there exists a $t\in L_\delta$ such that $y\leqslant t\leqslant z$ and $|v_\sigma(t)-u(z)|<\varepsilon$. Then

$$|w_1(x)-u_\sigma(x)|\leqslant |w_1(x)-v_\sigma(t)|+|v_\sigma(t)-u(z)|+|u(z)-u_\sigma(x)|<3\varepsilon .$$

We conclude that $w_1(x)=u_\sigma(x)$; this proves that u_σ is v_δ-lower regular.

To show the v_σ-upper regularity of u_σ, note that there is an element $h\in L_\sigma$ such that $x\leqslant h$ (this follows from the fact that $x\in (D(u))_\sigma$, i.e., $x=\bigvee\limits_1^\infty h_n$, $h_n\in D(u)$, but, since u is v_σ-lower regular, every element of $D(u)$ is majorized by some element of L_σ). Therefore $\lim\limits_{y\geqslant x,\, y\in L_\sigma} v_\sigma(y) = w_2(x)$ exists. Hence there exists an element y in L_σ, $y\geqslant x$, such that $y\geqslant y'\geqslant x$ and $y'\in L_\sigma$ implies $|v_\sigma(y')-w_2(x)|<\varepsilon$. Let $x_n\uparrow x$, $x_n\in D(u)$. Then there is an increasing sequence (y_n) in L_σ such that $|v_\sigma(y_n)-u(x_n)|<\varepsilon$ and $y\geqslant \sup\limits_n y_n\geqslant x$. We then have

$$|w_2(x)-u_\sigma(x)|\leqslant |w_2(x)-v_\sigma(\sup_n y_n)|+|v_\sigma(\sup_n y_n)-u_\sigma(x)|$$
$$<\varepsilon+|\lim_n v_\sigma(y_n)-\lim_n v(x_n)|<2\varepsilon$$

and hence $w_2(x)=u_\sigma(x)$. To prove the δ-continuity of u_σ, let $x_n\downarrow x$, $x_n, x\in D(u_\sigma)$. Since u_σ is v_δ-lower regular, there exists an element $y\in L_\delta$ such that $y\leqslant x$ and $y\leqslant y'\leqslant x$ and $y'\in L_\delta$ imply $|v_\delta(y')-u_\sigma(x)|<\varepsilon$. By Lemma 2.2 there exists a decreasing sequence (y_n) in L_δ such that $|v_\delta(y_n)-u_\sigma(x_n)|<\varepsilon$ and $y\leqslant \inf\limits_n y_n\leqslant x$. We thus have

$$|\lim_n u_\sigma(x_n)-u_\sigma(x)|\leqslant |\lim_n u_\sigma(x_n)-\lim_n v_\sigma(y_n)|+|\lim_n v_\sigma(y_n)-u_\sigma(x)|$$
$$<\varepsilon+|v_\sigma(\inf_n y_n)-u_\sigma(x)|<2\varepsilon .$$

This completes the proof.

Applying Lemma 2.9 to $u=v_\delta$, we get the following

COROLLARY. *Let v_σ be v_δ-lower regular or (equivalently) let v_δ be v_σ-upper regular. Then $v_{\delta\sigma}$ is v-regular and δ-continuous.*

2.10. THEOREM. *Let v be a (σ,δ)-continuous valuation on a sublattice L of a σ-continuous lattice H, with values in a complete metric Abelian group G. Then v extends uniquely to a (σ,δ)-continuous valuation v' on a (σ,δ)-lattice L' generated by L if and only if the following conditions are satisfied:*

(a) *v is monotonely convergent,*

(b) *v_σ is v_δ-lower regular or (equivalently) v_δ is v_σ-upper regular.*

Proof of necessity. Since v' is (σ,δ)-continuous on L', it is monotonely convergent, and so is its restriction v. Let $x\in L_\sigma$ and let $\lim\limits_{y\leqslant x,\, y\in L_\delta} v_\sigma(y)=\mu(x)$. Take an arbitrary $\varepsilon>0$. Then there is a $z\in L_\delta$, $z\leqslant x$, such that $z\leqslant t\leqslant x$ and $t\in L_\delta$ imply $|v_\delta(t)-\mu(x)|$, $|v_\delta(t)-v_\sigma(z)|<\varepsilon$. Let $y_n\uparrow x$, $y_n\in L$. Then

$$|v(y_n)-\mu(x)|\leqslant |v(y_n)-v_\delta(z)|+|v_\delta(z)-\mu(x)|$$
$$<|v_\delta(y_n\vee z)-v_\delta(z)|+|v_\delta(z)-v_\delta(y_n\wedge z)|+\varepsilon .$$

But $|v_\delta(y_n\vee z)-v_\delta(z)|<\varepsilon$ and $v_\delta(z)-v_\delta(y_n\wedge z)=v'(z)-v'(y_n\wedge z)\to 0$, and therefore $v(y_n)\to\mu(x)$, so that $\mu(x)=v_\sigma(x)$.

Proof of sufficiency. Let Ω be a set of all ordered pairs (K,μ), where K is a lattice, $L_{\sigma\delta}\subset K\subset L'$ and $\mu\colon K\to G$ is an extension of $v_{\sigma\delta}$ with the following properties:

(i) μ is a (σ,δ)-continuous monotonely convergent and v-regular valuation,

(ii) μ is the only (σ,δ)-continuous valuation extending $v_{\sigma\delta}$ on K.

We partially order Ω in the usual manner:

$$(K_2,\mu_2)\geqslant (K_1,\mu_1)\ \Leftrightarrow\ K_2\supset K_1 \quad\text{and}\quad \mu_2 \text{ extends } \mu_1 .$$

By Lemma 2.6, the hypothesis and the corollary of 2.9, we have $(L_{\sigma\delta}, v_{\sigma\delta})\subset\Omega$. Let Π be any non-empty linearly ordered subset of Ω. Then $K_0=\bigcup\{K\colon (K,\mu)\in\Pi\}$ is a lattice such that $L_{\sigma\delta}\subset K_0\subset L'$. The function $\mu_0\colon K_0\to G$ is well defined if we write $\mu_0(x)=\mu(x)$, where (K,μ) is any element of Π such that $x\in K$.

We shall verify (i) and (ii) for μ_0. Clearly, μ_0 is a v-regular valuation. Let x_n be a decreasing sequence in K_0, and, let $\varepsilon>0$ be arbitrary. Because μ_0 is v_σ-lower regular, Lemma 2.2 implies the existence of a decreasing sequence (y_n) in L_δ such that $|v_\delta(y_n)-\mu_0(x_n)|<\varepsilon$, and therefore $\{\mu_0(x)\}$ is Cauchy. Further, if $x_n\downarrow x$, $x\in K_0$, then there exists a $z\in L_\delta$, $z\leqslant x$, such that $z\leqslant z'\leqslant x$ and $z'\in L_\delta$ imply $|v_\delta(z')-\mu_0(x)|<\varepsilon$. According to Lemma 2.2, we may assume that $z\leqslant\inf\limits_n y_n\leqslant x$, and therefore we conclude that $|\lim\limits_n \mu_0(x_n)-\mu_0(x)|<2\varepsilon$. From similar considerations for increasing sequences in K_0 we deduce that μ_0 is monotonely convergent and (σ,δ)-continuous. To verify (ii) for μ_0 note that every (σ,δ)-continuous valuation extending $v_{\sigma\delta}$ on K_0 coincides with the restriction of μ_0 to K, for every $(K,\mu)\in\Pi$. Application of the Zorn–Kuratowski lemma yields therefore the existence of a maximal element (K',μ') in Ω. From Lemma 2.6 we see that μ'_σ is a σ-continuous monotonely convergent valuation. By the hypotheses and Lemma 2.9 μ'_σ is v-regular and δ-continuous. By Lemma 2.3 μ'_σ is the only (σ,δ)-continuous valuation extending μ' on K'_σ. We have shown that $(K'_\sigma, \mu'_\sigma) \in \Pi$, and so, by the maximality of K', $K'_\sigma = K'$. Similarly $K'_\delta = K'$, and therefore $K' = L'$. Then $v' = \mu'$ is the required extension, and the proof is complete.

It is a well-known fact that every Hausdorff topological group is (homeomorphic with) a subset of a product of metric groups, i.e., $G \subset \Pi G_i$. Any function $u\colon L \to G$ can be regarded as a function $u\colon L \to \Pi G_i$, that is, $u = (u_i)_{i \in I}\colon L \to \Pi G_i$ where $u_i\colon L \to G_i$ and $u_i = u\,pr_i$. Since in our Theorem 2.10, the domain of each v_i is L, each v_i extends uniquely, to $v'_i\colon L' \to G_i$, provided that G_i is complete. Hence $v' = (v'_i)_{i \in I}$ is an extension of v. This proves the following

2.11. THEOREM. *Let v be a (σ, δ)-continuous valuation on a sublattice L of a σ-continuous lattice H, with values in a sequentially complete Hausdorff topological group G. Then v extends uniquely to a (σ, δ)-continuous valuation v' on a (σ, δ)-lattice L' generated by L if and only if the following conditions are satisfied*:

(a) *v is monotonely convergent*,

(b) *v_σ is v_δ-lower regular or (equivalently) v_δ is v_σ-upper regular.*

References

[1] G. Birkhoff, *Lattice Theory*, New York 1948.

[2] G. Fox and P. Morales, *Strongly additive functions on lattices*, Fund. Math. 78 (1973), pp. 99–106.

[3] B. J. Pettis, *On the extension of measures*, Ann. of Math. 54 (1951), pp. 186–197.

[4] M. Smiley, *A note on measure functions in a lattice*, Bull. Amer. Math. Soc. 46 (1940), pp. 239–241.

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On a simply connected 1-dimensional continuum without the fixed point property

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Abstract. The author answers a question of L. Tucker by giving an example of a simply connected 1-dimensional continuum X without the fixed point property such that every retract of X has the fixed point property with respect to onto maps and with respect to one-to-one maps.

Introduction. L. Tucker has asked if there exists a 1-dimensional continuum C without the fixed point property such that every retract of C has the fixed point property with respect to one-to-one maps. A similar question may be obtained by replacing "one-to-one" by "onto" in the preceding question. In [4] the author shows that an example of G. S. Young [5, p. 884] is a simply continuum satisfying the one-to-one case. An example of a planar continuum which is not arcwise connected is also given in [4] to answer the onto case. In this paper we give an example of a simply connected 1-dimensional continuum X which answers both questions simultaneously. Our example is obtained by adding a countable number of "$\sin(1/x)$ arcs" to Young's example [5, p. 884).

1. Construction of the continuum X. Let C_1 be a continuum in the right half xy-plane joining the point $(0,3)$ to the interval $I_1 = [-3, -1]$ of the y-axis, C_1 being homeomorphic to the closure of the graph of $y = \sin(1/x)$, $0 < x \leqslant 1/\pi$, with I_1 corresponding to the limiting interval of the graph. Let $C_2(I_2)$ be the image of $C_1(I_1)$ under the rotation of the xy-plane about the origin 0 through an angle of π. Let $T = T_1 \cup T_2 \cup T_3$ be a triod consisting of the subintervals T_1, T_2 on the y-axis joining the origin 0 to $(0, -1)$, respectively $(0, 1)$, and an arc T_3 which joins 0 to $a = (0, 4)$ and whose interior lies below the xy-plane. Let A be a set lying in the xy-plane homeomorphic to a half-open interval such that A (1) has only its endpoint $a = (0, 4)$ in common with $C_1 \cup C_2 \cup T$ and (2) "converges" to $C_1 \cup C_2$ in such a way that (a) there is a sequence of arcs $S_1, S_2, S_3, \ldots$ filling up A such that $S_i \cap S_j = \emptyset$ for $j \neq i-1, i+1$, and is an endpoint of each for $j = i-1, i+1$, and (b) $C_1 = \lim S_{2j-1}$, $C_2 = \lim S_{2j}$. It may be assumed that C_1

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