have shown that $(K'_\sigma, \mu'_\sigma) \in \Pi$, and so, by the maximality of K', $K'_\sigma = K'$. Similarly $K'_\delta = K'$, and therefore $K' = L'$. Then $v' = \mu'$ is the required extension, and the proof is complete.

It is a well-known fact that every Hausdorff topological group is (homeomorphic with) a subset of a product of metric groups, i.e., $G \subset \Pi G_i$. Any function $u\colon L \to G$ can be regarded as a function $u\colon L \to \Pi G_i$, that is, $u = (u_i)_{i \in I}\colon L \to \Pi G_i$ where $u_i\colon L \to G_i$ and $u_i = upr_i$. Since in our Theorem 2.10, the domain of each v_i is L, each v_i extends uniquely, to $v'_i\colon L' \to G_i$, provided that G_i is complete. Hence $v' = (v'_i)_{i \in I}$ is an extension of v. This proves the following

2.11. THEOREM. *Let v be a (σ, δ)-continuous valuation on a sublattice L of a σ-continuous lattice H, with values in a sequentially complete Hausdorff topological group G. Then v extends uniquely to a (σ, δ)-continuous valuation v' on a (σ, δ)-lattice L' generated by L if and only if the following conditions are satisfied*:

(a) *v is monotonely convergent*,

(b) *v_σ is v_δ-lower regular or (equivalently) v_δ is v_σ-upper regular.*

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On a simply connected 1-dimensional continuum without the fixed point property

by

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Abstract. The author answers a question of L. Tucker by giving an example of a simply connected 1-dimensional continuum X without the fixed point property such that every retract of X has the fixed point property with respect to onto maps and with respect to one-to-one maps.

Introduction. L. Tucker has asked if there exists a 1-dimensional continuum C without the fixed point property such that every retract of C has the fixed point property with respect to one-to-one maps. A similar question may be obtained by replacing "one-to-one" by "onto" in the preceding question. In [4] the author shows that an example of G. S. Young [5, p. 884] is a simply continuum satisfying the one-to-one case. An example of a planar continuum which is not arcwise connected is also given in [4] to answer the onto case. In this paper we give an example of a simply connected 1-dimensional continuum X which answers both questions simultaneously. Our example is obtained by adding a countable number of "$\sin(1/x)$ arcs" to Young's example [5, p. 884).

1. Construction of the continuum X. Let C_1 be a continuum in the right half xy-plane joining the point $(0, 3)$ to the interval $I_1 = [-3, -1]$ of the y-axis, C_1 being homeomorphic to the closure of the graph of $y = \sin(1/x)$, $0 < x \leqslant 1/\pi$, with I_1 corresponding to the limiting interval of the graph. Let $C_2(I_2)$ be the image of $C_1(I_1)$ under the rotation of the xy-plane about the origin 0 through an angle of π. Let $T = T_1 \cup T_2 \cup T_3$ be a triod consisting of the subintervals T_1, T_2 on the y-axis joining the origin 0 to $(0, -1)$, respectively $(0, 1)$, and an arc T_3 which joins 0 to $a = (0, 4)$ and whose interior lies below the xy-plane. Let A be a set lying in the xy-plane homeomorphic to a half-open interval such that A (1) has only its endpoint $a = (0, 4)$ in common with $C_1 \cup C_2 \cup T$ and (2) "converges" to $C_1 \cup C_2$ in such a way that (a) there is a sequence of arcs $S_1, S_2, S_3, \ldots$ filling up A such that $S_i \cap S_j = \emptyset$ for $j \neq i-1, i+1$, and is an endpoint of each for $j = i-1, i+1$, and (b) $C_1 = \lim S_{2j-1}$, $C_2 = \lim S_{2j}$. It may be assumed that C_1

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and A have been constructed so that C_1 passes through $(2,0)$ and the intersection of each S_i with the set $\{(x,y) \mid -1 \leqslant y \leqslant 1\}$ is a vertical line segment with length 2 such that S_{2j-1} passes through $p_{2j-1} = (2+1/j, 0)$ and S_{2j} passes through $P_{2j} = (-2-1/j, 0)$. Let A_{2j-1} denote the curve whose equation is

$$y = \frac{1}{2j-1}\sin\left(\frac{\pi}{j(j+1)(x-2)-j}\right) \quad \text{for} \quad 2+\frac{1}{j+1} < x \leqslant 2+\frac{1}{j},$$

i.e. A_{2j-1} is a $\sin(1/x)$ curve with amplitude $1/(2j-1)$ and whose closure joins the point p_{2j-1} to the limiting interval in S_{2j+1} whose length is $2/(2j-1)$ and whose midpoint is p_{2j+1}. Let A_{2j} be the image of A_{2j-1} under the rotation of the xy-plane about the origin 0 through an angle of π. Define $X = C_1 \cup C_2 \cup T \cup A \cup (\bigcup_{i=1}^{\infty} A_i)$. Then clearly X is a simply connected 1-dimensional continuum.

We define a fixed point free map $f\colon X \to X$ which is a composition of two discontinuous functions f_1 and f_2. Let $f_1\colon X \to X$ be such that on $C_1 \cup C_2 \cup T_1 \cup T_2$ it is a rotation in the xy-plane about 0 through an angle of π, and is the identity otherwise. Let $f_2\colon X \to X$ be a function that is a homeomorphism on A such that for each i, S_i is mapped on S_{i+1}, that maps each A_i homeomorphically onto A_{i+1}, that is the identity on $C_1 \cup C_2$, that maps T_j, $j = 1, 2$, homeomorphically onto $T_j \cup T_3$, and maps T_3 homeomorphically onto S_1. Then $f = f_2 f_1$ is continuous and fixed point free.

2. Proof that every retract of X has the fixed point property with respect to one-to-one maps and with respect to onto maps. To facilitate notation we write $A = [a, \infty)$ and if b, c are points in A with $b < c$, then $[b, c]$ shall denote the unique subarc of A with endpoints b and c, and $[b, \infty)$ shall denote the unique infinite subarc of A with initial point b.

(i) First we show that X has the fixed point property with respect to one-to-one maps. Let $h\colon X \to X$ be a one-to-one map. Then h must preserve triods and hence triple points. Thus $h(0) = p_i$ for some i, or $h(0) = 0$. But if $h(0) = p_i$, then h must map $T_3 \cup A$ into one of (a) A_i, (b) $C_1 \cup C_2 \cup T \cup [a, p_i]$, or (c) $[p_i, \infty)$. However cases (a) and (b) cannot occur since infinitely many of the triple points p_j would be mapped by h onto points which are not triple points. Also case (c) cannot occur since $h(C_i)$ would not be connected if h were a one-to-one function preserving triods. Hence $h(0) = 0$ and thus h has a fixed point.

(ii) Next we show that X has the fixed point property with respect to onto maps. Suppose $h\colon X \to X$ is an onto map with no fixed points. Let $q_1 = (2, 0)$ and $q_2 = (-2, 0)$, i.e. $q_1 = \lim p_{2i-1}$ and $q_2 = \lim p_{2i}$. Let U_1 be a neighborhood of $h(q_1)$ disjoint from q_1 such that U_1 intersects at most two sets of the form A_{2i-1}. Since X is locally connected at q_1 there exists a connected neighborhood V_1 of q_1 such that $V_1 \cap U_1 = \emptyset$ and $h(V_1)$ lies in a single component K_1 of U_1. Hence for all but finitely many i, $h(A_{2i-1}) \subset K_1$. A similar argument holds for $h(q_2)$, q_2 and sets of the form A_{2i}. Since h is onto it follows that $h(q_1) = q_2$ and $h(q_2) = q_1$ for

otherwise infinitely many A_i would not be covered. Then for infinitely many i, $h(p_{2i-1}) \neq h(p_{2i+1})$, $\lim h(p_{2i-1}) = q_2$ and $h([p_{2i-1}, p_{2i+1}])$ contains the unique arc in X with endpoints $h(p_{2i-1})$ and $h(p_{2i+1})$. Thus it follows that $h(C_1 \cup C_2) = C_1 \cup C_2$.

Let $S = h(T_3 \cup A) \cap (T_3 \cup A \cup (\bigcup_{i=1}^{\infty} A_i))$. Let e be a homeomorphism from $T_3 \cup A$ onto the non-negative real numbers. Since S can contain no A_i there exists a retraction $r\colon S \to h(T_3 \cup A) \cap (T_3 \cup A)$ defined by

$$r(h(x)) = \begin{cases} h(x) & \text{if} \quad h(x) \in T_3 \cup A, \\ p_i & \text{if} \quad h(x) \in A_i. \end{cases}$$

Define a map H from S into the real numbers $\boldsymbol{R}$ by $H(x) = erh(x) - e(x)$ for all x in S. Since h does not have the fixed point property we claim that either $H(x) > 0$ for all x in S or $H(x) < 0$ for all x in S. For otherwise, since S is arcwise connected there is a point c in S such that $H(c) = 0$ and hence $rh(c) = c$. If $h(c) \in T_3 \cup A$, then $h(c) = c$. Hence $h(c) \in \operatorname{Int} A_i$ for some i and thus $c = p_i$. Since $\operatorname{Cl} A_i$ has the fixed point property, h cannot map $\operatorname{Cl} A_i$ into itself. Regard A_i as a directed arc with initial point p_i, and let a_i be the first point in A_i such that $h(a_i) = p_i$. It then follows that there is a point in A_i between p_i and a_i which is fixed under h. Since h is onto and has no fixed points an infinite subarc of A_1 must lie in $h(A_i)$ for some $i > 1$. Hence $h(p_{i+2}) \in \operatorname{Cl} A_i$. Thus $H(p_{i+2}) \in \operatorname{Cl} A_i$. Thus $H(p_{i+2}) < 0$ and therefore $H(x) < 0$ for all x in S. If h does not map $C_1 \cup C_2 \cup T_1 \cup T_2$ into itself, since $H(x) < 0$ for all x in S and $h(C_1 \cup C_2) = C_1 \cup C_2$ it follows that there is a point in $\operatorname{Int}(T_1 \cup T_2)$ which is a fixed point under h. Therefore h must map $C_1 \cup C_2 \cup T_1 \cup T_2$ into itself. But it easy to show that $C_1 \cup C_2 \cup T_1 \cup T_2$ has the fixed point property and therefore h has a fixed point which is a contradiction.

(iii) Now we show that every proper retract Y of X has the fixed point property with respect to one-to-one maps and with respect to onto maps. First we consider the case for a proper retract Y which contains A. Since X is an arcwise connected continuum every retract of X must be an arcwise connected continuum. We note that Y must contain all but finitely many of the A_i. For otherwise, suppose $r\colon X \to Y$ is a retraction such that for infinitely many i, A_i is not a subset of Y. But then for infinitely many i we have $r(A_i) \supset [p_i, p_{i+2}]$ and therefore r could not be continuous at the points $q_1 = (2, 0)$ and $q_2 = (-2, 0)$. The remaining argument for the case of proper retracts containing A is completely analogous to that used for X itself in (i) and (ii).

The second case we consider is that of a retract Y of X lying in $C_1 \cup C_2 \cup T$. Now Y cannot contain a neighborhood N of I_1 (or I_2). For otherwise, infinitely many subarcs of A would be retracted onto the non-locally connected space N which is impossible. Consequently, Y must be a point, arc, or triod and hence has the fixed point property. Of course any dendrite D in X is an absolute retract for the class of compact metric spaces and hence has the fixed point property [2, p. 138].

Evidently, an arcwise continuum Y consisting of a compact subarc of A and finitely many A_i is a retract of X (i.e. Y consists of finitely many "$\sin(1/x)$ circles" each intersecting a common subarc of A in an arc which contains their respective limit intervals). The $\sin(1/x)$ circle has the fixed point property [1, p. 123] and a tedious but elementary argument can be used to show that Y has the fixed point property. If D is a dendrite in X such that $D \cap Y$ consists of a single point, then $D \cup Y$ must have the fixed point property [1, p. 121]. Hence retracts of X which are obtained from Y in this manner also have the fixed point property and this completes (iii).

Problem 1. The following question posed in [4] still remains open. Namely, can a planar example be found?

Problem 2. In [3] J. M. Łysko gives an example of a contractible continuum of dimension 3 which does not have the fixed point property for homeomorphisms. Does there exist a simply connected 1-dimensional continuum which does not have the fixed point property with respect to homeomorphisms?

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Rings in which every proper right ideal is maximal

by

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Abstract. We study the structure of rings in which every proper right ideal is maximal. We generalize some results of Perticani for non-commutative rings.

Recently, Perticani [2] has studied the structure of commutative rings with a unit element in which every proper ideal is maximal. In this paper we shall follow his line to discuss some generalizations for non-commutative rings.

A right ideal (or an ideal) of a ring R is said to be *proper* if it is different from (0) and R. Throughout this paper R will denote a ring (not necessarily commutative) with $R^2 = R \neq (0)$ in which every proper right ideal is maximal. We shall prove that R must be one and only one of the following types:

(1) R is a division ring;
(2) R is isomorphic to a 2×2 matrix ring over a division ring;
(3) R is isomorphic to the direct sum of two division rings;
(4) R is a left pseudo field over a division ring in the sense of Thierrin [3];
(5) R is a right pseudo field over a Galois field $GF(p)$ in the sense of Thierrin;
(6) R is a local ring (i.e., with unit and unique maximal ideal I) such that R/I is a division ring and $I^2 = (0)$.

Finally we shall show that in a ring A with $A^2 = A \neq (0)$, every proper right ideal is almost maximal if and only if every proper right ideal is maximal. Thus, this paper also provides a further classification for rings in which every proper right ideal is almost maximal given by Koh [1, Prop. 5.28].

We begin with

Lemma 1. *R has at most two proper ideals.*

Proof. Suppose that I, J, K are distinct proper ideals in R. Then I, J, K are maximal right ideals and $I+J = R$. If $K \cap I \neq (0)$, then $K \cap I$ would be a maximal right ideal contained properly in K, a contradiction. Hence $K \cap I = (0)$ and $KI = (0)$. Similarly, $KJ = (0)$. It follows that $KR = K(I+J) \subseteq KI+KJ = (0)$ and $K \subseteq R^l$, the left annihilator of R. Since $R^l \neq R$ is an ideal containing K, $R^l = K$. Using a similar argument, we can show that $R^l = I$. This contradicts the fact that $I \neq K$.