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Metacompactness and the class MOBI

by

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Abstract. We construct examples of open compact mappings which are defined on metacompact complete Moore spaces. The examples show that the range of such a mapping can be either a Moore space which is not metacompact or a regular nondevelopable space. This solves some problems connected with the class MOBI.

Let MOBI_i denote the minimal class of T_i spaces containing all metric spaces and closed under open compact mappings (see [1, Definition 5.4], and [10]).

It is known that MOBI_2 contains hereditarily paracompact nonmetrizable spaces [13, Example 2] (a similar example is constructed in [2]) and nondevelopable nonmetacompact spaces [13, Example 3] (a similar example is constructed in [3]).

On the other hand, it is shown in [13, Theorem 2] implicitly (and independly in [10]) that the paracompact members of MOBI_3 are metrizable.

The purpose of this note is to construct a space Y in $\mathrm{MOBI}_{3\frac{1}{2}}$ which is neither metacompact nor developable.

More exactly, we shall construct an example of an open compact mapping of a completely regular metacompact developable Čech complete space X onto a completely regular space Y which is not a p-space and contains a closed subset which is not a G_δ-subset; moreover, Y has not a G_δ-diagonal (Examples 2.2 and 2.4).

From the results of the generalized base of countable order theory of H. H. Wicke and J. M. Worrell, Jr., it follows that Y is not θ-refinable (see [6] for simpler proofs and definitions); hence Y is neither metacompact nor subparacompact.

The example gives an answer to Problems 7.1,2,3,5,6 and, partially, to 12 (1) from [10] (see also Question 2 from [2]), and some questions from [3].

In the first section we present a general method of constructing open compact mappings. This method is used in the second section to construct various spaces in $\mathrm{MOBI}_{3\frac{1}{2}}$.

We shall use the terminology and notation from [7].

(1) It is easy to see that Problem 7.1 is equivalent to the negation of Problem 7.5.

1. General constructions. To explain the idea of our construction we shall first demonstrate a method of constructing open compact mappings which do not preserve metacompactness. This method was used in Example 3 of [13].

EXAMPLE 1.1. Let Y be a T_1 space such that the set A of the accumulation points of Y is discrete. Let ωA denote the Alexandroff compactification of A. Consider the set

$$X = A \cup [(Y \setminus A) \times \omega A]$$

with the topology generated by the open subsets of the space $(Y \setminus A) \times \omega A$ and by the sets of the form

$$\{a\} \cup [(U \setminus A) \times \{a\}],$$

where $a \in A$ and U is a neighbourhood of a in Y.

It is easy to see that X is a completely regular metacompact space and a function f which is the identity on A and the projection on $(Y \setminus A) \times \omega A$ is an open compact mapping from X onto Y.

To get an example of an open compact mapping which does not preserve metacompactness it suffices to take as the space Y the space obtained from the Niemycki plane by isolating each point of the upper half-plane.

Let us notice that if Y is regular and the set A is countable, then Y is paracompact. On the other hand, if A is uncountable, then X is not first countable (cf. Example 3 of [13]). It follows that in order to obtain nonmetacompact spaces in MOBI_3 we have to modify the above construction.

THEOREM 1.2. *If Y is a T_1 space with a point countable base* (²) *and the set A of the accumulation points of Y is discrete, then there exists a metacompact completely regular complete Moore space and an open compact mapping f from X onto Y.*

Proof. Let $\mathfrak{B}$ be a point countable base of Y such that each element of $\mathfrak{B}$ contains one point from A at most. For each isolated point y of Y let $A(y) = A \cap \mathrm{St}(y, \mathfrak{B})$.

Each set $A(y)$ is countable and discrete. Let $\omega A(y)$ denote the Alexandroff compactification of $A(y)$ if $A(y)$ is infinite, and let $\omega A(y) = A(y)$ if $A(y)$ is finite. Consider the set

$$X = A \cup \bigoplus_{y \in Y \setminus A} (\{y\} \times \omega A(y))$$

with the topology generated by the open subsets of the space $\bigoplus_{y \in Y \setminus A} \{y\} \times \omega A(y)$ and by the sets of the form

$$\{a\} \cup \bigcup_{y \in U \setminus A} \{(y, a)\},$$

where $a \in A$ and $V \in \mathfrak{B}$ is a neighborhood of a.

(²) It is easy to see that this condition is necessary.

One can easily check that the space X and a function f which is the identity on A and maps $\omega A(y)$ onto y have the desired properties.

Let us notice that the space X in the above constructions is normal if and only if Y is metacompact.

In the next section we shall construct various examples of completely regular spaces which satisfy the assumptions of Theorem 1.2, and therefore belong to the class $\mathrm{MOBI}_{3\frac{1}{2}}$.

The following remark will be our main tool:

Remark 1.3. Let Z be a completely regular first countable space such that the set A of the accumulation points of Z is discrete. Let Z^* denote the set of all pairs (z, φ), where $z \in Z \setminus A$ and φ is a countable subset of A (³).

Consider the set $Y = A \cup Z^*$ with the topology generated by all subsets of Z^* and by the sets of the form

$$\{a\} \cup \{(z, \varphi) \in Z^* \colon a \in \varphi \text{ and } z \in U\},$$

where $a \in A$ and U is a neighbourhood of a in Z.

The space Y is completely regular and satisfies the assumptions of Theorem 1.2. Moreover,

(a) Y is metacompact iff Z is metacompact,

(b) A is G_δ in Y iff A is G_δ in Z,

(c) if A is G_δ in Z, then Y is Čech complete iff Z is Čech complete,

(d) if A is not G_δ in Z, then Y is not a p-space and does not have a G_δ-diagonal.

The proofs of these facts are easy and therefore omitted (⁴).

2. Examples. In this section we shall construct various completely regular first countable spaces with discrete sets of accumulation points. Next, we shall use the modification described in Remark 1.3 in order to obtain spaces in $\mathrm{MOBI}_{3\frac{1}{2}}$.

EXAMPLE 2.1. A complete Moore space in $\mathrm{MOBI}_{3\frac{1}{2}}$ which is not metacompact.

Let Z_1 be the space obtained from the Niemycki plane by isolating each point of the upper half-plane.

Consider the space Y_1 obtained from Z_1 by the modification described in Remark 1.3. From Remark 1.3 and Theorem 1.2 it follows that Y_1 has the desired properties.

EXAMPLE 2.2. A space in $\mathrm{MOBI}_{3\frac{1}{2}}$ which is not a p-space does not have a G_δ-diagonal, and contains a closed subset which is not a G_δ-set.

(³) This idea is taken from [12]. It is easy to observe that the constructions in 1.1 and 1.2 are based on the same method.

(⁴) The proof of (d) is based on characterizations from [11] and [5]. For the proofs of (b) and (d) in a special case see the exposition of Example 2.4.

Let V be the set of countable ordinals, and let A be a maximal family of monotonically increasing functions from the set N of natural numbers into V such that if $a, a' \in A$, then the set $a(N) \cap a'(N)$ is finite.

Consider the set $Z_2 = A \cup V$ with the topology generated by all subsets of V and by the sets of the form

$$\{a\} \cup a(\{n \in N: n \geqslant k\}),$$

where $a \in A$ and $k \in N$.

The space Z_2 is completely regular first countable and the set A of the accumulation points of Z_2 is not a G_δ-subset of Z_2 (see [6, Example 2.9]).

Again, using Remark 1.3 and Theorem 1.2, we can construct a space Y_2 with the desired properties.

Let us recall that such a space cannot be θ-refinable [6, Theorems 2.8 and 3.2].

EXAMPLE 2.3. A Moore space in $\mathrm{MOBI}_{3\frac{1}{2}}$ which is neither metacompact nor complete ([5]).

We shall use the notation introduced in Example 2.2.

Let $Z_3 = A \cup (V \times N)$ be a topological space such that $V \times N$ is an open and discrete subset of Z_3 and the neighbourhoods of $a \in A$ are of the form

$$\{a\} \cup [a(\{n \in N: n \geqslant k\}) \times \{n \in N: n \geqslant k\}],$$

where $k \in N$.

The space Z_3 is completely regular first countable and the set A is G_δ in Z_3. Hence Z_3 is a Moore space. Using the fact that A is not G_δ in Z_2, one can easily prove (see [8, Theorem 9]) that Z_3 is not Čech complete.

From the maximality of A it follows that Z_3 is not metacompact.

To construct a space Y_3 with the desired properties it suffices to use Remark 1.3 and Theorem 1.2.

Finally we shall construct an example of a space in $\mathrm{MOBI}_{3\frac{1}{2}}$ which has the same properties as the space described in Example 2.2 and is screenable ([6]). Since countably θ-refinable screenable spaces are θ-refinable, it follows that this space is neither countably metacompact nor countably subparacompact.

EXAMPLE 2.4. Let Q denote the set of rational numbers and P the set of irrational numbers.

([5]) The example is not very surprising. We present it here to complete the list of modifications. The idea of this modification is taken from [8, Theorem 9].

([6]) This example shows that in Theorem 2.8 of [6] θ-refinability cannot be replaced by screenability.

For each $q \in Q$ let $A(q)$ be a maximal family of one-to-one functions from the set N of natural numbers into P such that

(1) $a \in A(q)$ implies $|a(n) - q| < 1/n$ for $n \in N$,

(2) $a, a' \in A(q)$ implies $a(N) \cap a'(N)$ is finite.

Put $A = \bigcup \{A(q): q \in Q\}$ and consider the set $Z_4 = A \cup P$ with the topology generated by all subsets of P and by the sets of the form

$$U(a, k) = \{a\} \cup a(\{n \in N: n \geqslant k\}),$$

where $a \in A$ and $k \in N$.

The space Z_4 is completely regular first countable and each $A(q)$ is a G_δ-set in Z_4. We shall show that A is not a G_δ-set in Z_4 ([7]).

Suppose that A is a G_δ-set in Z_4. It follows that $P = \bigcup_{k=1}^{\infty} B_k$, where each B_k is closed in Z_4. Since P is not an F_σ-subset of the reals, there exist a k and a rational number q such that q is an accumulation point of B_k in the topology of the real line. Hence there exists a function a from N into B_k satisfying (1). From the maximality of $A(q)$ it follows that the closure of B_k in Z_4 contains a point from $A(q)$. The contradiction shows that A is not a G_δ-set in Z_4 ([8]).

We shall modify the space Z_4 in order to obtain a space Y_4 with the desired properties.

Let P^* be the set of all ordered pairs (p, φ) such that $p \in P$, $\varphi \subseteq A$ and, for each $q \in Q$, φ contains at most one element of $A(q)$.

Let $Y_4 = A \cup P^*$ be a topological space such that P^* is open and discrete in Y_4 and the neighbourhoods of a point $a \in A$ are of the form

$$U^*(a, k) = \{a\} \cup \{(p, \varphi) \in P^*: a \in \varphi \text{ and } p \in U(a, k)\}.$$

The space Y_4 is completely regular, satisfies the assumptions of Theorem 1.2, and has a σ-disjoint base.

To show that A is not a G_δ-set in Y_4 assume that $A = \bigcap \{G_l: l \in N\}$ and

$$G_l = \bigcup \{U^*(a, k(a, l)): a \in A\},$$

where $k(a, l) \geqslant l$ and $k(a, l+1) \geqslant k(a, l)$ for each $a \in A$ and $l \in N$.

([7]) The space Z_4 contains a discrete countable family of closed G_δ-sets such that the union of this family is not a G_δ-set. Hence Z_4 is not normal. Moreover, it is easy to see (cf. [9, Problem 5.I]) that Z_2 and the subspaces $A(q) \cup P$ of Z_4 are not normal.

([8]) The space Z_4 is a locally compact nondevelopable space with a G_δ-diagonal (the last fact is an easy consequence of condition (1)). A similar space is constructed in [4]. Another example of a space having the same properties as Z_4 can be obtained by means of a construction from Theorem 9 of [8].

From the fact that A is not a G_δ-subset of Z_4 we infer that there exist a p_0 and elements a_l of A such that

$$p_0 \in \bigcap \{U(a_l, k(a_l, l)) \colon l \in N\}.$$

Let $\varphi'_0 = \{a_l \colon l \in N\}$. Since

$$A(q) = \bigcap_{l=1}^{\infty} \bigcup \{U(a, l) \colon a \in A(q)\}$$

and $l \leqslant k(a, l)$, $A(q) \cap \varphi'_0$ is finite for $q \in Q$. Therefore, we can define the set φ_0 of all $a_l \in \varphi'_0$ such that, for a certain $q \in Q$, $\{a_m \colon m \geqslant l\} \cap A(q) = \{a_l\}$.

It is easy to see that $(p_0, \varphi_0) \in P^*$ and $(p_0, \varphi_0) \in \bigcap \{G_l \colon l \in N\}$. Hence we have proved that A is not a G_δ-set in Y_4.

Moreover, if $U_l^* = U^*(a_l, k(a_l, l))$, $(p_0, \varphi_0) \in P^*$, $(p_0, \varphi_0) \in \bigcap \{U_l^* \colon l \in N\}$ and there exists a $q \in Q$ such that $\varphi_0 \cap A(q)$ is empty, then the uncountable closed discrete subset $\{(p_0, \varphi) \in P^* \colon \varphi_0 \subseteq \varphi\}$ is contained in the intersection of the family $\{U_l^* \colon l \in N\}$. This, together with the fact that A is not a G_δ-subset of Y_4, implies that Y_4 neither is a p-space nor has a G_δ-diagonal (9).

The remark following the proof of Theorem 1.2 suggests the following problem:

PROBLEM. Is each space in MOBI_4 metacompact?

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(9) The space Y_4 contains a subspace Y_5 which is not developable and has a G_δ-diagonal. Namely, $Y_5 = A \cup \{(p, \varphi) \in P^* \colon p \in P \text{ and } \varphi \in \mathfrak{A}\}$, where $\mathfrak{A}$ is a maximal subfamily of $\{\varphi \subseteq A \colon \varphi$ is countable and φ contains at most one element of $A(q)$ for $q \in Q\}$ such that $\varphi, \varphi' \in \mathfrak{A}$ implies $\varphi \cap \varphi'$ is finite.

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