A constructive modification of Vietoris homology

by

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Abstract. An Abelian category is developed to serve as the range for a homology theory, on the category of compact metric spaces, that is constructive in the sense of Bishop. As an application it is shown, constructively, that a compact subset of the plane has trivial homology if and only if its complement is pathwise connected. Of classical interest is the fact that, in this theory, the homology functors are continuous and exact.

In this paper we introduce an Abelian category $\mathfrak{C}$ to serve as the range for a homology theory on the category of compact metric spaces. The category $\mathfrak{C}$ contains the category $\mathfrak{A}$ of Abelian groups as a full exact subcategory, and the homology functors take values in $\mathfrak{A}$ on finite polyhedra. Moreover there is a retraction of $\mathfrak{C}$ upon $\mathfrak{A}$ that transforms the homology functors into Vietoris homology functors. The theory satisfies all the Eilenberg–Steenrod axioms [3] and, in addition, is continuous in the sense that the homology functors commute with countable inverse limits. This contrasts with the Vietoris (or Čech) theory for which the exactness axiom fails.

The category $\mathfrak{C}$ is introduced to provide homology objects containing the numerical information needed to develop a useful theory that is constructive in the sense of Bishop [2]. As an example of the application of this theory we show that a compact subspace K of the plane has trivial one-dimensional homology if and only if given any two points a and b in the (metric) complement of K, we can construct a path joining a to b that is bounded away from K. The Vietoris theory cannot be used to obtain this result in a constructive way. To see why not, let K be the result of removing a small open arc of unknown size from the unit circle. Then we know that the one-dimensional Vietoris group is trivial since there can be no 1-cycles. But this knowledge provides no information concerning the size of the gap, so we have no way of constructing a path joining the origin to a point outside the circle that is bounded away from K, since we have no way to construct the bound.

The Vietoris groups can be considered as inverse limits of ε-homology groups, where the ε-homology group of a space X is the homology group of the simplicial complex whose vertices are the points of X and whose simplices are those finite subsets of X of diameter less than ε. By passing to the limit we lose valuable information. In the example above what we need, from the constructive point of view, is that for every $\varepsilon>0$ there be a $\delta>0$ so that the image of the δ-homology group in the ε-homology group is trivial. Thus we are led to consider the entire system of ε-homology groups, or at least a cofinal subsequence, as a homology object. By suitably defining maps, and equality of maps, the objects become homeomorphism invariants, and are trivial exactly under the circumstances described above. The resulting homology theory carries more information than the Vietoris theory even from a classical point of view, since it distinguishes between a p-adic solenoid and a single point.

1. A category. We define a category $\mathfrak{C}$ as follows. An *object* A of $\mathfrak{C}$ is a family of Abelian groups A_r indexed by the set Z^+ of positive integers, together with homomorphisms $A_s^r\colon A_r\to A_s$, defined for $r\geqslant s$, such that $A_t^sA_s^r = A_t^r$ if $r\geqslant s\geqslant t$. If A and B are objects of C, then a *map* $f\colon A\to B$ is a family of homomorphisms $f_s^r\colon A_r\to B_s$ indexed by a subset $D(f)$ of $Z^+\times Z^+$ such that

1) If $s\in Z^+$, then $(r, s)\in D(f)$ for some r in Z^+.

2) If $(r, s)\in D(f)$ and $u\geqslant r$ and $v\leqslant s$, then $(u, v)\in D(f)$ and $f_v^u = B_v^sf_s^rA_r^u$.

We define *equality* of two maps f and g by setting $f = g$ if for each s in Z^+ there is an r in Z^+ such that $f_s^r = g_s^r$.

If $g\colon C\to A$ and $f\colon A\to B$, then we define $fg\colon C\to B$ as follows. Let $D(fg) = D(g)\circ D(f) = \{(r, t)\colon (r, s)\in D(g)$ and $(s, t)\in D(f)$ for some s in $Z^+\}$. If $(r, t)\in D(fg)$ set $(fg)_t^r = f_t^sg_s^r$. To see that this does not depend on s, suppose $u\geqslant v$ are possible choices for s. Then $f_t^ug_u^r = f_t^vA_v^ug_u^r = f_t^vg_v^r$. To show that this composition respects equality, suppose $g = h$. Then for each t there is an s so that $(s, t)\in D(f)$, and an r such that $g_s^r = h_s^r$. Hence $(fg)_t^r = f_t^sg_s^r = f_t^sh_s^r = (fh)_t^r$. So for each t there is an r such that $(fg)_t^r = (fh)_t^r$, which means that $fg = fh$. Similarly, suppose $f = k$. Then for each t there is an s so that $f_t^s = k_t^s$, and an r such that $(r, s)\in D(g)$. Hence $(fg)_t^r = f_t^sg_s^r = k_t^sg_s^r = (kg)_t^r$. Thus $fg = kg$.

If f and g are two maps from A to B we let $D(f+g) = D(f)\cap D(g)$ and define $(f+g)_s^r = f_s^r+g_s^r$. This operation clearly respects equality and turns the set of maps from A to B into an Abelian group. It is also clear that composition distributes over this addition on both sides. Thus $\mathfrak{C}$ is a pre-additive category [4]. If A and B are objects, define $(A\oplus B)_r = A_r\oplus B_r$ and $(A\oplus B)_s^r = A_s^r\oplus B_s^r$. The obvious injection and projection maps make $A\oplus B$ a direct sum of A and B. Hence $\mathfrak{C}$ is an additive category. We proceed to the construction of kernels and cokernels.

THEOREM 1. *Let $f\colon A\to B$. Define K by $K_s \bigcap\limits_{t\leqslant s} \ker f_t^s$, where the intersection is understood to be limited to those t such that $(s, t)\in D(f)$, and let K_s^r be the restriction of A_s^r to K_r. Let $k\colon K\to A$ be the natural injection. Then k is a kernel of f, and we write $K = \ker f$.*

Proof. Clearly $fk = 0$. Suppose $g\colon C\to A$ and $fg = 0$. We must show that there is a unique map $i\colon C\to K$ such that $ki = g$. Let $D(i) = \{(r, s)\in D(g)\colon$

$\operatorname{im} g_s^r\subseteq K_s\}$, and let i_s^r be the map induced by g_s^r from C_r to K_s. We must show that for each s there is an r such that $(r, s)\in D(i)$. Since $fg = 0$, for each s we can find r such that $(fg)_s^r = 0$ and $(r, s)\in D(g)$. Then if $t\leqslant s$ and $(s, t)\in D(f)$ we have $(fg)_t^r = B_t^s(fg)_s^r = 0$, so $f_t^sg_s^r = (fg)_t^r = 0$. Thus $(r, s)\in D(i)$. It is clear that $g = ki$. Now suppose $j\colon C\to K$ and $g = kj$. Then for each s there is an r such that $g_s^r = k_s^sj_s^r$. Hence $(r, s)\in D(i)$, so $g_s^r = k_s^si_s^r$, which implies that $j_s^r = i_s^r$. Thus $j = i$.

THEOREM 2. *Let $g\colon C\to A$. Define K by $K_s = A_s/(\bigcup\limits_{r\geqslant s} \operatorname{im} g_s^r)$, where the union is understood to be restricted to those r such that $(r, s)\in D(g)$, and let K_s^r be the map from K_r to K_s induced by A_s^r. Let $k\colon A\to K$ be the natural projection. Then k is a cokernel of g, and we write $K = \operatorname{coker} g$.*

Proof. Clearly $kg = 0$. Suppose $f\colon A\to B$ and $fg = 0$. We must show that there is a unique map $i\colon K\to B$ so that $f = ik$. Let

$$D(i) = \{(s, t)\in D(f)\colon \bigcup_{r\geqslant s} \operatorname{im} g_s^r\subseteq \ker f_t^s\},$$

and let i_t^s be the map induced by f_t^s from K_s to B_t. We must show that for each t there is an s such that $(s, t)\in D(i)$. Since $fg = 0$, for each t we can find an s such that $(fg)_t^s = 0$ and $(s, t)\in D(f)$. Then if $r\geqslant s$ and $(r, s)\in D(g)$ we have $(fg)_t^r = (fg)_t^sC_s^r = 0$, so $f_t^sg_s^r = (fg)_t^r = 0$. Thus $(s, t)\in D(i)$. It is clear that $f = ik$. Now suppose $j\colon K\to B$ and $f = jk$. Then for each t there is an s such that $f_t^s = j_t^sk_s^s$. Hence $(s, t)\in D(i)$ so $f_t^s = i_t^sk_s^s$ which implies that $j_t^s = i_t^s$. Thus $j = i$.

To show that $\mathfrak{C}$ is Abelian we must verify that the kernels and cokernels fit together right.

THEOREM 3. *Let $f\colon A\to B$. Then the natural map g from $\operatorname{coker}\ker f$ to $\ker\operatorname{coker} f$ is an isomorphism.*

Proof. We have

$$(\operatorname{coker}\ker f)_r = A_r/(\ker f)_r = A_r/\bigcap_{s\leqslant r} \ker f_s^r,$$

and

$$(\ker\operatorname{coker} f)_s = \bigcup_{r\geqslant s} \operatorname{im} f_s^r.$$

Also $D(g) = D(f)$ and g_s^r is induced by f_s^r. Define $h\colon \ker\operatorname{coker} f\to\operatorname{coker}\ker f$ as follows. Let $D(h) = \{(s, t)\colon s\geqslant t\}$. If $y\in(\ker\operatorname{coker} f)_s$, then $y = f_s^r(x)$ for some $r\geqslant s$ and x in A_r. Let $h_s^s(y) = A_s^r(x)+(\ker f)_s\in A_s/(\ker f)_s = (\operatorname{coker}\ker f)_s$. The homomorphism h_s^s is well defined, for if $y = 0$ then $f_s^r(x) = 0$ so

$$A_s^r(x)\in\bigcap_{t\leqslant s} \ker f_t^s = (\ker f)_s.$$

To show that h is the inverse of g note that if $\bar{x} = x+(\ker f)_r\in(\operatorname{coker}\ker f)_r$, then

$$(hg)_s^r(\bar{x}) = h_s^sg_s^r(\bar{x}) = h_s^sf_s^r(x) = A_s^r(x)+(\ker f)_s.$$

Hence hg is equal to the identity map on $\operatorname{coker}\ker f$. Similarly, if $y = f_s(x) \in (\ker\operatorname{coker} f)_s$, then

$$(gh)_t^s(y) = g_t^s h_s^s(y) = g_t^s(A_s^r(x)+(\ker f)_s) = f_t^s A_s^r(x) = f_t^r(x) = B_t^s(y)\,.$$

Hence gh is equal to the identity map on $\ker\operatorname{coker} f$.

We now show how to construct inverse limits of countable systems in $\mathfrak{C}$.

THEOREM 4. *Let $A(j)$ be an object of $\mathfrak{C}$ for $j = 0, 1, 2, \ldots$, and f_j^i: $A(i)\to A(j)$ a family of maps, defined when $i\geqslant j$, such that f_j^j is the identity and $f_k^i = f_k^j f_j^i$ if $i\geqslant j\geqslant k$. Then we can construct an inverse limit $A(\infty)$, with maps f_j^∞: $A(\infty)\to A(j)$, for this system.*

Proof. Choose an ascending sequence of positive integers $p(i)$ so that $p(i)\geqslant i$ and $(p(i), p(i-1)) \in D(f_j^i)$ if $i>j$. Define $A(\infty)$ by setting $A_r(\infty) = A_{p(r)}(r)$. Let the map $A_s^r(\infty)$: $A_r(\infty)\to A_s(\infty)$ be defined for $r\geqslant s$ by $A_s^r(\infty) = (f_s^r)_{p(s)}^{p(r)}$. This works because $(p(r), p(s)) \in D(f_s^r)$ if $r>s$. Define f_j^∞: $A(\infty)\to A(j)$ as follows. Let

$$D(f_j^\infty) = \{(r, s)\colon j\leqslant r \text{ and } (p(r), s) \in D(f_j^r)\}$$

and define $(f_j^\infty)_s^r$: $A_r(\infty)\to A_s(j)$ by $(f_j^\infty)_s^r = (f_j^r)_s^{p(r)}$.

Suppose B is an object of $\mathfrak{C}$ and g_j: $B\to A(j)$ are maps satisfying $f_j^i g_i = g_j$ for $i\geqslant j$. We must exhibit a map g_∞: $B\to A(\infty)$ such that $f_j^\infty g_\infty = g_j$ for $j = 0, 1, 2, \ldots$ First we show that g_∞ is unique by proving that for each j there is an r such that $(g_\infty)_j^r = (g_j)_{p(j)}^r$. If $f_j^\infty g_\infty = g_j$ then for some r and s we have $(f_j^\infty)_{p(j)}^s (g_\infty)_s^r = (g_j)_{p(j)}^r$. But $(f_j^\infty)_{p(j)}^s = (f_j^s)_{p(j)}^{p(s)} = A_j^s(\infty)$ so $(g_j)_{p(j)}^r = A_j^s(\infty)(g_\infty)_s^r = (g_\infty)_j^r$. To define g_∞ we let

$$D(g_\infty) = \{(r, s)\colon (r, p(s)) \in D(g_s)\}$$

and let

$$(g_\infty)_s^r = (g_s)_{p(s)}^r\colon B_r\to A_{p(s)}(s) = A_s(\infty)\,.$$

2. A homology theory. The category $\mathfrak{K}$ of compact metric pairs is defined as follows. An *object* of $\mathfrak{K}$ is a pair (X, A) where X is a compact metric space and A is a compact subspace of X. A *map* f: $(X, A)\to(Y, B)$ is a (uniformly) continuous function f: $X\to Y$ such that $f(A)\subseteq B$.

To each compact metric pair (X, A) and positive number ε we associate a simplicial complex X_ε and a subcomplex A_ε of X_ε. The vertices of the complex X_ε are the points of X, and the simplices are those finite families of vertices whose diameters are less than ε, while A_ε is the subcomplex of X_ε whose simplices lie in A. Let $\varepsilon(r)$ be a sequence of positive numbers decreasing to 0. If n is a nonnegative integer, then we denote the nth homology group of the pair $(X_{\varepsilon(r)}, A_{\varepsilon(r)})$ by $H_n(X, A)_r$. If $r\geqslant s$ we have a natural map from $H_n(X, A)_r$ to $H_n(X, A)_s$. Thus, for every pair (X, A) in $\mathfrak{K}$, we have associated an object $H_n(X, A)$ in $\mathfrak{C}$. It is easy to see that the object $H_n(X, A)$ is, up to isomorphism, independent of the sequence ε.

If f maps (X, A) into (Y, B), and ω is a modulus of continuity for f, then f induces homomorphisms $H_n(X, A)_r\to H_n(Y, B)_s$ whenever $\varepsilon(r)<\omega(\varepsilon(s))$. It is easily verified that this gives a map f_{*n} from $H_n(X, A)$ to $H_n(Y, B)$ in the category $\mathfrak{C}$, and that in fact we have constructed a functor from $\mathfrak{K}$ to $\mathfrak{C}$ for each n. Moreover, the boundary operators ∂_r: $H_n(X, A)_r\to H_{n-1}(A)_r$ constitute a boundary operator ∂: $H_n(X, A)\to H_{n-1}(A)$.

THEOREM 5. *The boundary operator ∂: $H_n(X, A)\to H_{n-1}(A)$ is natural; that is, if f: $(X, A)\to(Y, B)$, then $\partial f_{*n} = f_{*n-1}\partial$: $H_n(X, A)\to H_{n-1}(B)$. Moreover, the sequence $H_{n+1}(X, A)\xrightarrow{\partial} H_n(A)\to H_n(X)\to H_n(X, A)\xrightarrow{\partial} H_{n-1}(A)$ is exact for each n.*

Proof. To see that ∂ is natural, let $\langle x_0, \ldots, x_n\rangle$ be an n-simplex of diameter less than $\varepsilon(r)$ in X. If $(r, s) \in D(f_{*n})$, then

$$\begin{aligned}(\partial f_{*n})_s^r\langle x_0, \ldots, x_n\rangle &= \partial_s(f_{*n})_s^r\langle x_0, \ldots, x_n\rangle = \partial_s\langle f(x_0), \ldots, f(x_n)\rangle\\ &= \textstyle\sum(-1)^i\langle f(x_0), \ldots, \widehat{f(x_i)}, \ldots, f(x_n)\rangle\\ &= (f_{*n-1})_s^r\textstyle\sum(-1)^i\langle x_0, \ldots, \hat{x}_i, \ldots, x_n\rangle\\ &= (f_{*n-1})_s^r\partial_r\langle x_0, \ldots, x_n\rangle = (f_{*n-1}\partial)_s^r\langle x_0, \ldots, x_n\rangle.\end{aligned}$$

Exactness of the long sequence follows easily from the exactness of the corresponding sequences with subscripts r, and the characterization of kernels and cokernels in the category $\mathfrak{C}$.

We now show that the excision axiom is satisfied in this theory.

THEOREM 6. *Let (X, A) be a compact metric pair. Let Y be a compact subset of X such that $Y\cap A$ is compact (this latter condition is superfluous classically). Suppose that there is a $\delta>0$ such that for each x in X either $x \in Y$ or every point within δ of x is in A. Then the homomorphism $H_n(Y, Y\cap A)\to H_n(X, A)$ induced by the inclusion $(Y, Y\cap A)\subseteq(X, A)$ is an isomorphism for each n.*

Proof. We shall construct the inverse map g. Let $D(g) = \{(r, s)\colon \varepsilon(r)<\delta$ and $s\leqslant r\}$. Define g_s^r: $H_n(X, A)_r\to H_n(Y, Y\cap A)_s$ as follows. Each n-simplex of diameter less than $\varepsilon(r)$ is either contained in A or contained in Y. Let

$$g_s^r\Big(\sum_J k_j\sigma_j\Big) = \sum_I k_j\sigma_j$$

where $J = I\cup I'$ so that if $j \in I$, then $\sigma_j\subseteq Y$, while if $j \in I'$, then $\sigma_j\subseteq A$. This is independent of the choice of I since the ambiguous simplices are in $Y\cap A$. It is clear that g is the inverse map.

Next we show that the homotopy axiom is satisfied.

THEOREM 7. *If two maps between compact metric pairs are homotopic, then they induce equal maps on homology.*

Proof. It suffices to show that the two maps from X to $X\times I$ defined by $f(x) = (x, 0)$ and $g(x) = (x, 1)$ induce equal maps from $H_n(X, A)$ to $H_n(X\times I, A\times I)$. So it suffices to show that f and g induce equal homomorphisms from $H_n(X, A)_r$ to $H_n(X\times I, A\times I)_r$ for each positive integer r. Choose a positive integer k so that $1/k<\varepsilon(r)$ and define f_j: $X\to X\times I$ by $f_j(x) = (x, j/k)$ for $0\leqslant j\leqslant k$. It suffices to show that f_j and f_{j+1} induce equal homomorphisms on homology for $0 \leqslant j \leqslant k-1$. But this follows from the fact that the induced simplicial maps are contiguous.

Finally we show that homology commutes with countable inverse limits.

THEOREM 8. *Let (X_i, A_i) be a compact metric pair for $i = 0, 1, 2, \ldots$, and $f_j^i\colon (X_i, A_i) \to (X_j, A_j)$ a family of maps, defined when $i \geqslant j$, such that f_i^i is the identity and $f_k^i = f_k^j f_j^i$ if $i \geqslant j \geqslant k$. Let (X_∞, A_∞) be the inverse limit of this system and $f_i^\infty\colon (X_\infty, A_\infty) \to (X_i, A_i)$ the associated maps. If for each j and $\delta > 0$ there is an i so that every point of $f_j^i(X_i)$ is within δ of some point of $f_j^\infty(X_\infty)$, and every point of $f_j^i(A_i)$ is within δ of some point of $f_j^\infty(A_\infty)$, then $H_n(X_\infty, A_\infty)$ is isomorphic to $\underleftarrow{\lim}\, H_n(X_i, A_i)$.*

Proof. Let $L = \underleftarrow{\lim}\, H_n(X_i, A_i)$ as constructed in the proof of Theorem 4. The maps f_i^∞ induce a unique map $\varphi\colon H_n(X_\infty, A_\infty) \to L$. We shall show that φ is an isomorphism by constructing an inverse map ψ. Let $p(r)$ be the ascending sequence of positive integers associated with the inverse limit L. Let $D(\psi) = \{(r, s)$: for some $s \leqslant j \leqslant r$, if $y, z \in X_\infty$ and $d(f_j^\infty(y), f_j^\infty(z)) < 3\varepsilon(p(r-1))$, then $d(y, z) < \frac{1}{3}\varepsilon(s)$, and for each x in X_r or A_r there is a y in X_∞ or A_∞ respectively such that $d(f_t^\infty(y), f_t^r(x)) < \varepsilon(p(r-1))$ if $t \leqslant j\}$. That $D(\psi)$ is a suitable domain follows from the fact that the metric on X_∞ comes from the product metric on ΠX_i, and the hypotheses. Note that since $(p(r), p(r-1)) \in D(f_{j*}^r)$ we have $\varepsilon(p(r)) < \omega_j^r(\varepsilon(p(r-1)))$ where ω_j^r is the modulus of continuity for f_j^r. To define ψ_s^r let $g\colon (X_r, A_r) \to (X_\infty, A_\infty)$ be a function (an *operation* in the sense of Bishop) such that $d(f_t^\infty(g(x)), f_t^r(x)) < (p(r-1))$ if $t \leqslant j$. Then $\psi_s^r\colon L_r = H_n(X_r, A_r)_{p(r)} \to H_n(X_\infty, A_\infty)_s$ is defined by setting $\psi_s^r(\langle x_0, \ldots, x_n\rangle) = \langle g(x_0), \ldots, g(x_n)\rangle$. If $d(x, y) < \varepsilon(p(r))$ for $x, y \in X_r$, then $d(f_j^r(x), f_j^r(y)) < \varepsilon(p(r-1))$ so $d(f_j^\infty(g(x)), f_j^\infty(g(y))) < 3\varepsilon(p(r-1))$, whereupon $d(g(x), g(y)) < \frac{1}{3}\varepsilon(s) < \varepsilon(s)$, so ψ_s^r does indeed map into $H_n(X_\infty)_s$. This observation also shows that ψ_s^r is independent of the choice of g (that is, the operation ψ_s^r is a function) since the elements $\langle g(x_0), \ldots, g(x_n)\rangle$ are homologous in $H_n(X_\infty, A_\infty)_s$ for different choices of g.

To see that $\varphi\psi$ is equal to the identity, for given t choose s and r so that $(s, t) \in D(\varphi)$ and $(r, s) \in D(\psi)$. Then

$$\varphi_t^s \psi_s^r(\langle x_0, \ldots, x_n\rangle) = \varphi_t^s(\langle g(x_0), \ldots, g(x_n)\rangle) = \langle f_t^\infty g(x_0), \ldots, f_t^\infty g(x_n)\rangle$$

which is homologous in L_t to $\langle f_t^r(x_0), \ldots, f_t^r(x_n)\rangle = L_t^r(\langle x_0, \ldots, x_n\rangle)$. To see that $\psi\varphi$ is equal to the identity, for given s choose r and q so that $(r, s) \in D(\psi)$ and $(q, r) \in D(\varphi)$. Then

$$\psi_s^r \varphi_r^q(\langle z_0, \ldots, z_n\rangle) = \psi_s^r(\langle f_r^\infty(z_0), \ldots, f_r^\infty(z_n)\rangle) = \langle g f_r^\infty(z_0), \ldots, g f_o^\infty(z_n)\rangle.$$

But $d(f_j^\infty g f_r^\infty(z_i), f_j^\infty(z_i)) = d(f_j^\infty g f_r^\infty(z_i), f_j^r f_r^\infty(z_i)) < \varepsilon(p(r-1))$ so $d(g f_r^\infty(z_i), z_i) < \varepsilon(s)/3$. Hence $\langle g f_r^\infty(z_0), \ldots, g f_r^\infty(z_n)\rangle$ is homologous to $\langle z_0, \ldots, z_n\rangle$ in $H_n(X_\infty, A_\infty)_s$, so $\psi\varphi$ is equal to the identity on $H_n(X_\infty, A_\infty)$.

Note that the last hypothesis of Theorem 8 is superfluous from the classical point of view. For if this condition failed then, for $Y_i = A_i$ or $Y_i = X_i$, the sets

$B_k = \{x \in \Pi Y_i\colon d(f_j^k(x_k), f_j^\infty(Y_\infty)) \geqslant \delta$ and $x_i = f_i^k(x_k)$ for all $i \leqslant k\}$ would be nonempty and compact for all $k \geqslant j$. But $B_j \supseteq B_{j+1} \supseteq \ldots$ so the intersection would be nonempty. However any point in $\bigcap B_k$ is in Y_∞, a contradiction. In the most important case, when $f_j^i(X_i) = X_j$ and $f_j^i(A_i) = A_j$, the condition is clearly met, even constructively.

3. Some observations. If G is an Abelian group, then we can identify G with the object $F(G)$ in $\mathfrak{C}$ such that $F(G)_r = G$ for all r and $F(G)_s^r$ is the identity homomorphism for all $s \leqslant r$. It is clear that under this identification the category $\mathfrak{A}$ of Abelian groups is imbedded as a full exact subcategory of $\mathfrak{C}$. On the other hand, if A is an object of $\mathfrak{C}$, then we may construct the Abelian group $L(A) = \underleftarrow{\lim}\, A_r$. It is easy to verify that L is a functor from $\mathfrak{C}$ to $\mathfrak{A}$ and that LF is the identity functor on $\mathfrak{A}$. If we follow the homology functors of the preceding section by the functor L we obtain the Victoris homology functors. Thus, not surprisingly, the former functors carry at least as much information as the latter.

To see that this homology distinguishes spaces that Vietoris homology does not, consider the unit circle C in the complex plane and the map $f\colon C \to C$ defined by $f(z) = z^2$. Let K be the inverse limit of the system $\ldots \xrightarrow{f} C \xrightarrow{f} C \xrightarrow{f} C$. Then, by Theorem 8, we have $H_1(K) \cong \underleftarrow{\lim}\, H_1(C)$ so $H_1(K)_r$ is an infinite cyclic group and $H_1(K)_s^r$ is multiplication by a positive power of 2 for $r > s$. Thus $H_1(K)$ is nontrivial, but the one-dimensional Victoris homology group is $\underleftarrow{\lim}\, H_1(K)_r$ (in the category $\mathfrak{A}$) which is trivial.

The groups $H_n(X, A)_r$ are presented as very large sets with lots of elements identified. In general these groups will not be discrete, that is, given two elements x and y, we may not be able to decide if $x = y$. However we can find an object V in $\mathfrak{C}$ so that V_r is a finitely generated discrete group for each r, and $H_n(X, A) \cong V$. The groups V_r are the homology groups of pairs of finite abstract simplicial complexes which approximate (X, A). Following Bishop we say that (Y, B) is a *δ-approximation to a compact metric pair* (X, A) if Y and B are finite sets, $(Y, B) \subseteq (X, A)$ and for each x in X or A there is a y in Y or B respectively such that $d(x, y) < \delta$. A compact metric pair has a δ-approximation for each $\delta > 0$.

THEOREM 9. *Let (X, A) be a compact metric pair. Then we can find a sequence $\{\varepsilon(r)\}$ of positive numbers that is strictly decreasing to zero, and an $\frac{1}{2}(\varepsilon(r) - \varepsilon(r+1))$-approximation (Y_r, B_r) to (X, A) for each r, so that if $x, y \in Y_r$ then $d(x, y) \neq \varepsilon(r)$. Let $V_r = H_n(Y_r, B_r)_r$ and define V_s^r for $r > s$ by $V_s^r(\langle x_0, \ldots, x_n\rangle) = \langle z_0, \ldots, z_n\rangle$ where $d(x_i, z_i) < \frac{1}{2}(\varepsilon(s) - \varepsilon(s+1))$, and $z_i \in B_s$ if $x_i \in B_r$. Then V_r is a finitely generated discrete group for each r and $H_n(X, A) \cong V$.*

Proof. Let $\varepsilon(1) = 1$. Suppose we have constructed $\varepsilon(1), \ldots, \varepsilon(r)$ and $(Y_1, B_1), \ldots, (Y_{r-1}, B_{r-1})$. Choose $\varepsilon(r+1) < \frac{1}{2}\varepsilon(r)$ and let (Y_r, B_r) be a δ-approximation to (X, A) for some $\delta < \frac{1}{2}(\varepsilon(r) - \varepsilon(r+1))$. Choose $\varepsilon'(r)$ between $\varepsilon(r+1) + 2\delta$ and $\varepsilon(r)$ so that $\varepsilon'(r) \neq d(x, y)$ if $x, y \in Y_r$. Redefine $\varepsilon(r)$ to be $\varepsilon'(r)$. Note that (Y_r, B_r) is still an $\frac{1}{2}(\varepsilon(r) - \varepsilon(r+1))$-approximation to (X, A). This procedure gives the desired sequence $\{\varepsilon(r)\}$.

Define a homomorphism $\psi_s^r\colon H_n(X, A)_r \to V_s$ for $r > s$ by $\psi_s^r(\langle x_0, \ldots, x_n\rangle) = \langle z_0, \ldots, z_n\rangle$ where $z_i \in Y_s$ and $d(x_i, z_i) < \frac{1}{2}(\varepsilon(s) - \varepsilon(s+1))$, and $z_i \in B_s$ if $x_i \in B_r$. The homomorphism ψ_s^r is well defined since $d(z_i, z_j) \leqslant d(z_i, x_i) + d(z_j, x_j) + d(x_i, x_j) < \varepsilon(s) - \varepsilon(s+1) + \varepsilon(r) \leqslant \varepsilon(s)$, which shows both that $\operatorname{diam}(z_0, \ldots, z_n) < \varepsilon(s)$ and that different choices of the z_j give homologous simplices. The inclusion $(Y_s, B_s) \subseteq (X, A)$ gives a natural homomorphism $\varphi_t^s\colon V_s \to H_n(X, A)_t$ for $s \geqslant t$. Clearly $V_t^r = \varphi_t^{r+1}\psi_{r+1}^r$ for $r > s$. It is now readily verified that V is an object of $\mathfrak{C}$, and that the maps φ and ψ are inverses of each other, showing that $H_n(X) \cong V$. Since V_r is the nth homology group of a finite simplicial complex relative to a finite subcomplex, V_r is finitely generated and discrete (see [3; p. 135 ff]).

4. An application. We shall use the preceding homology theory to prove constructively a special case of Alexander duality in the plane, namely that the homology of a compact set X in the plane E^2 is trivial if and only if the complement of X is connected. First we observe that the latter condition entails a certain uniformity.

LEMMA. *Let X be a compact subset of E^2 such that if $d(a, X) > 0$ and $d(b, X) > 0$, then a and b can be joined by a path that is bounded away from X. Then for each $\varepsilon > 0$ there is a $\delta > 0$ such that if $d(a, X) > \varepsilon$ and $d(b, X) > \varepsilon$ then a and b can be joined by a path that is bounded away from X by δ.*

Proof. Let C be a circle containing X and bounded away from X by 2ε. Let Y be an $\frac{1}{3}\varepsilon$-approximation to the inside of C. Let $Y = A \cup B$ where $d(y, X) > \frac{1}{3}\varepsilon$ if $y \in A$ and $d(y, X) < \frac{1}{2}\varepsilon$ if $y \in B$. Choose $\delta > 0$ so that all points in A may be joined by paths that are bounded away from X by δ. Clearly a and b can be joined by straight line segments that are bounded by δ away from X to points in A. Hence a and b can be joined by a path that is bounded away from X by δ.

By $\sim X$ we mean the *metric complement* of X in E^2, that is, $\sim X = \{a \in E^2\colon d(a, X) > 0\}$. Classically $\sim X$ is just the complement of X; constructively, if we know that $a \in \sim X$ then we have a lower bound on $d(a, X)$.

THEOREM 10. *Let X be a compact subset of E^2 such that $\sim X$ is pathwise connected. Then $H_1(X) = 0$.*

Proof. We must show that for each s there is an r such that the homomorphism $H_1(X)_r \to H_1(X)_s$ is trivial. Let $\varepsilon = \frac{1}{4}\varepsilon(s)$ and let δ be as in the lemma. Choose r so that $\varepsilon(r) < \delta$. It suffices to show that if $x_0, \ldots, x_n$ are distinct points of X such that $d(x_{i-1}, x_i) < \varepsilon(r)$ for $i = 1, \ldots, n$ and $d(x_n, x_0) < \varepsilon(r)$, then $\sigma = \langle x_n, x_0\rangle + \sum_{i=1}^{n} \langle x_{i-1}, x_i\rangle$ is homologous to zero in $H_1(X)_s$. Let $\alpha < \varepsilon(r)$ be the supremum of $d(x_n, x_0)$ and the $d(x_{i-1}, x_i)$ for $i = 1, \ldots, n$. Then we may assume that if x_i and x_j are not adjacent, then $d(x_i, x_j) > \alpha$ since otherwise $d(x_i, x_j) < \varepsilon(r)$ so σ is homologous to the sum of two smaller cycles in $H_1(X)_r$. Thus $x_0, \ldots, x_n$ describes a simple closed polygon P in E^2.

To show that σ is homologous to zero in $H_1(X)_s$ we triangulate the inside of the polygon P and construct a two-chain τ in E^2 whose simplices have diameters less than $\varepsilon(r)$ such that $\partial\tau = \sigma$. If v_i is a vertex of τ inside P, then there is a point w_i in X such that $d(w_i, v_i) < \frac{1}{3}\varepsilon(s)$, for otherwise there would be a point a inside P such that $d(a, X) > \frac{1}{4}\varepsilon(s)$. But such an a could not be joined to any point b outside of P by a path that was bounded by δ away from X, since the distance between successive vertices of P is less than $\varepsilon(r) < \delta$.

Let τ' be the two-chain constructed from τ by replacing the v's by the w's. Then the simplices of τ' have diameter less than $\varepsilon(s)$ and $\partial\tau' = \sigma$. Hence σ is homologous to zero in $H_1(X)_s$.

Our proof of the converse of Theorem 10 uses a construction from [1].

THEOREM 11. *Let X be a compact subset of E^2 such that $H_1(X) = 0$. Then $\sim X$ is pathwise connected.*

Proof. Suppose a is a point of E^2 such that $d(a, X) > 0$. Choose s such that $d(a, X) > \varepsilon(s)$ and choose r such that the homomorphism $H_1(X)_r \to H_1(X)_s$ is trivial. We shall show that a can be connected to "infinity" by a path that is bounded away from X.

Tessellate the inside of a large circle containing X with regular hexagons of diameter $h < \frac{1}{6}\varepsilon(r)$ so that a lies at the center of a hexagon H_a. Choose h so that $\frac{5}{2}h < \varepsilon(s) - d(a, X)$. Let L be the set of line segments joining centers of adjacent hexagons. Partition L into two subsets L_0 and L_1 such that if $\lambda \in L_0$ then $d(\lambda, X) < 2h$, and if $\lambda \in L_1$ then $d(\lambda, X) > h$. Let A be the set of hexagons whose centers are connected to a by a sequence of segments in L_1, and let B be the set of hexagons that are not in A. We shall show that if H_b is a hexagon on the rim of the tessellation, then $H_b \in A$, so a can be joined to the rim by a polygonal path that is bounded away from X.

Let b be the center of H_b and suppose by way of contradiction that $H_b \in B$. Connect b to a by a sequence of segments from L and let H_v be the first hexagon in A that this sequence enters. Let H_w be the hexagon from which H_v was first entered. Then the edge E between H_v and H_w separates a hexagon in A from one in B, and it is readily seen that E lies on a unique simple closed polygon P comprised of such edges. Since the sides of P are edges of hexagons in A we have $d(P, X) > \frac{1}{2}h$. Since b is outside P, we must have a inside P.

If p_i is a vertex of P we can find a point x_i in X such that $d(p_i, x_i) < \frac{5}{2}h$ since p_i is on an edge that separates a hexagon in A from a hexagon in B and hence is crossed by a segment in L_0. Let Y be the (closure of the) space consisting of the hexagons in B. Then $X \subseteq Y$. Now P determines a cycle σ in $H_1(Y)_r$ that is homologous in $H_1(Y)_r$ to the cycle τ in $H_1(X)_r$ obtained from σ by replacing each p_i by x_i. By hypothesis τ is homologous to zero in $H_1(X)_s$ so to get our contradiction we need only show that σ is not homologous to zero in $H_1(Y)_s$. But a is inside P and $d(a, P) > d(a, X) - \frac{5}{2}h > \varepsilon(s)$. Thus the disc of radius $\varepsilon(s)$ about a is bounded away from Y and inside P so σ cannot be homologous to zero in $H_1(Y)_s$.

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Accepté par la Rédaction le 27. 6. 1974

An annulus theorem for suspension spheres

by

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Abstract. A space X is called a *suspension* $(n-1)$-*sphere* if $S(X)$, the suspension of X, is homeomorphic to S^n. Kirby has shown [3] that any orientation preserving self homeomorphism of S^n is stable for $n\geqslant 5$. The author shows that Kirby's result implies the following. If $n\geqslant 5$ and X is a suspension $(n-1)$-sphere, then for any two embeddings $f_i\colon X\to S^n$, $i=1,2$, so that $f_1(X)$ and $f_2(X)$ are disjoint and bicollared in S^n then M, the closed region in S^n bounded by $f_1(X)$ and $f_2(X)$, is homeomorphic to $X\times I$.

1. Introduction. If X_1 and X_2 are compact metric generalized manifolds then we shall say that X_1 and X_2 are *h-cobordant* if there exists a compact metric space M so that (i) M is a generalized manifold with boundary (as in [5]); (ii) there is a homeomorphism f from the disjoint union X_1+X_2 onto ∂M, the boundary of M; (iii) the restrictions $f_i=f|X_i$ induce isomorphisms between the homotopy groups of X_i and those of M, $i=1,2$. In addition, for the objects we shall consider it will be necessary to impose two further conditions: (iv) $\mathrm{Int}M=M-\partial M$ is a manifold and (v) ∂M is collared in M, that is there is a homeomorphism h from $\partial M\times[0,1)$ onto an open set in M so that for each $x\in\partial M$, $f(x,0)=x$.

When conditions (i)-(v) are satisfied we call M an *h-cobordism* between X_1 and X_2. This terminology was suggested to the author by L. C. Siebenmann.

It should be noted that conditions (iv) and (v) require $X_i\times R$ to be a manifold. Thus X_1, X_2 and M are generalized manifolds with respect to homology and cohomology over any coefficient domain. If $X\times R$ is a manifold then $X\times I$ is an h-cobordism between $X\times\{0\}$ and $X\times\{1\}$.

For any space X, $S(X)=S^0*X$ will be the suspension of X, $C(X)=I^0*X$ will be the cone over X and $OC(X)=C(X)-X$ will be the open cone over X. If $S(X)\approx S^n$, that is $S(X)$ and S^n are homeomorphic, we call X a *suspension* $(n-1)$-*sphere*.

Consider the proposition

(HCB_n): *If X_1 and X_2 are suspension $(n-1)$-spheres then up to homeomorphism there is exactly one h-cobordism M between X_1 and X_2.*

The purpose of this note is to show that (HCB_n) is true for $n\geqslant 5$. A fairly elementary proof is given for $n\geqslant 6$; the case for $n=5$ was originally proved by us