where $\{{}^*F_i\}$ forms a decreasing sequence of *closed sets, is said to be **resolvable with respect to P.*

${}^*F_{2n-1} - {}^*F_{2n}$ $(n = 1, 2, \ldots)$ have the Baire property by Theorem 6, then a *resolvable set E with respect to the first category defined by (7) also have the same property, since the class of sets having the Baire property is a σ-algebra. Hence we have

THEOREM 7. *X has the Baire property if and only if X is *resolvable with respect to the first category.*

§ 6. The measurable set. We shall take the property to be of Lebesgue measure zero as P. We denote by an ${}^*F_\sigma$ set the union of a countable family of *closed sets, and by a ${}^*G_\delta$ set the intersection of a countable family of *open sets. Then, by Theorem 1, an ${}^*F_\sigma$ set is of the form F_σ set plus a nullset, and a ${}^*G_\delta$ set is of the form G_δ set minus a nullset. Evidently the inverse of each of these holds true. Hence we obtain the following:

THEOREM 8. *X is measurable if and only if X is the set both ${}^*F_\sigma$ and ${}^*G_\delta$ with respect to Lebesgue measure zero.*

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References

[1] H. Hashimoto, *On the resemblance of point sets*, Mathematica Japonicae 4 (1954), pp. 53–56.
[2] C. Kuratowski, *Topologie* I, Warszawa 1933.
[3] K. Kuratowski, *Topology*, Vol. I, New York–London–Warszawa 1966.
[4] J. C. Oxtoby, *Measure and Category*, Graduate Texts in Math. 2 (1971).

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Topological completeness of first countable Hausdorff spaces II *

by

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Abstract. This article continues the study of basic completeness, a concept introduced in part I. It analyzes basic completeness into more primitive components: pararegularity (a generalization of regularity), monotonic completeness (a natural form of topological completeness), and base closurewise of countable order (a form of uniform first countability). The analysis finds expression in these theorems: 1. A space is basically complete if and only if it is T_1, locally monotonically complete, and has a base closurewise of countable order. 2. A space is basically complete if and only if it is a pararegular monotonically complete T_0-space having a base of countable order.

In addition there are results concerning pararegularity and monotonic completeness. It is shown that a pararegular space which is pseudo-m-complete (a modification of Oxtoby's pseudo-completeness) satisfies the Baire category theorem. The technique of primitive sequences exposited in I is further elaborated and applications are made. A number of examples are given.

This paper analyzes the concept of basic completeness, introduced in [22], into more primitive components: pararegularity, monotonic completeness, and base closurewise of countable order. These isolate, respectively, features of regularity, of completeness, and of uniform first countability. Each of them is discussed in a separate section where examples are given and relations to other concepts are established. A Baire category theorem is proved for pararegular spaces satisfying a weak completeness condition. The final section presents some characterizations of basic completeness in terms of these components.

The first section continues the development of the technique of primitive sequences initiated in I. Here some results are established in general form which are used in the subsequent proofs and which are useful in other contexts as well. This section may be regarded as a complement to Section 2 of I. These two sections begin a systematic presentation of a powerful technique for dealing with monotonically contracting sequences. In particular, they have application to spaces and concepts whose definitions involve monotonically contracting sequences of open coverings such as the spaces which are the subject of this investigation.

* This paper is a continuation of [22] which will be referred to herein as I. We use the notation, definitions, and results of I throughout: references such as Lemma I.2.1 are to Lemma 2.1 of I.

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1. Primitive sequences. A method, or technique, which we propose calling the *method of primitive sequences*, has evolved from our work on base of countable order theory. Some of the fundamental definitions and lemmas pertaining to the application of the technique are given in I, where the use of the method is well illustrated. For other applications see, e.g., [20, 21, 23]. The present section develops the method further. Its contents are applied in this paper, but the results are phrased in suffcient generality so as to be applicable in other situations as well. The reader may wish to postpone reading the section until reference is made to its contents.

DEFINITION 1.1. If $\mathscr{W}$ is a set well-ordered by $\preccurlyeq$ and $W \in \mathscr{W}$, then $\pi(W, \mathscr{W}, \preccurlyeq)$ denotes $\{x \in W\colon$ If $x \in V \in \mathscr{W}$, then $W \preccurlyeq V\}$. When the well-ordering $\preccurlyeq$ involved is obvious from the context, or implicit, the notation $\pi(W, \mathscr{W})$ will be used.

LEMMA 1.1. *Suppose M is a set and $\mathscr{W}_1, \ldots, \mathscr{W}_n$ are well-ordered coverings of M. Let $\mathscr{V}$ denote the collection of all sets $W_1 \cap \ldots \cap W_n$ such that $W_i \in \mathscr{W}_i$ for $1 \leqslant i \leqslant n$ and $\bigcap \{\pi(W_i, \mathscr{W}_i)\colon 1 \leqslant i \leqslant n\} \cap M \neq \emptyset$. Let $\preccurlyeq$ denote the collection of all pairs $(V, V') \in \mathscr{V} \times \mathscr{V}$ such that either* (a) $V = V'$ *or* (b) $V \neq V'$ *and $V = W_1 \cap \ldots \cap W_n$, $V' = W_1' \cap \ldots \cap W_n'$ and for the first $i \leqslant n$ such that $W_i \neq W_i'$, W_i precedes W_i' in $\mathscr{W}_i$.*

Then $\preccurlyeq$ is a well-ordering with field $\mathscr{V}$, $\mathscr{V}$ covers M, and $M \cap \pi(V, \mathscr{V}, \preccurlyeq) \neq \emptyset$ for each $V \in \mathscr{V}$.

Proof. If $V \in \mathscr{V}$ and $V = \bigcap \{W_i\colon 1 \leqslant i \leqslant n\} = \bigcap \{W_i'\colon 1 \leqslant i \leqslant n\}$, then $W_i = W_i'$ for all $i \leqslant n$. For there exist

$$x \in \bigcap \{\pi(W_i, \mathscr{W}_i)\colon 1 \leqslant i \leqslant n\} \cap M \quad \text{and} \quad y \in \bigcap \{\pi(W_i', \mathscr{W}_i)\colon 1 \leqslant i \leqslant n\} \cap M.$$

Because $x, y \in V$, it follows that for each $i \leqslant n$, W_i does not precede W_i' and *vice versa*. Therefore $W_i = W_i'$. From this it readily follows that $\preccurlyeq$ is a well ordering with field $\mathscr{V}$. Because each $x \in M$ belongs to exactly one $\pi(W_i, \mathscr{W}_i)$ for each $i \leqslant n$, it follows that $x \in V = \bigcap \{W_i\colon 1 \leqslant i \leqslant n\} \in \mathscr{V}$. Thus $\mathscr{V}$ covers M. It is easily seen that $x \in \pi(V, \mathscr{V}, \preccurlyeq)$.

DEFINITION 1.2. A collection $\mathscr{V}$ obtained from well-ordered sets $\mathscr{W}_1, \ldots, \mathscr{W}_n$ as in the statement of Lemma 1.1 and well-ordered by $\preccurlyeq$ will be said to be *lexicographically derived* from $\mathscr{W}_1, \ldots, \mathscr{W}_n$.

LEMMA 1.2. *Suppose that $(\mathscr{H}_i^1)_{i \in N}, \ldots, (\mathscr{H}_i^n)_{i \in N}$ are primitive sequences of M in S. For each $i \in N$ let $\mathscr{V}_i$ be lexicographically derived from $\mathscr{H}_i^1, \ldots, \mathscr{H}_i^n$. Then $(\mathscr{V}_i)_{i \in N}$ is a primitive sequence of M in S. If $(V_i)_{i \in N}$ is a decreasing representative of $(\mathscr{V}_i)_{i \in N}$, then for each j $(1 \leqslant j \leqslant n)$, there exists a decreasing representative $(H_i^j)_{i \in N}$ of $(\mathscr{H}_i^j)_{i \in N}$ such that for each $k \in N$, H_k^j is the first element of $\mathscr{H}_k^j$ that includes a term of $(V_i)_{i \in N}$.*

Proof. By Lemma 1.1 each $\mathscr{V}_i$ is a well-ordered collection of subsets of S which covers M and $M \cap \pi(V, \mathscr{V}_i) \neq \emptyset$ for all $V \in \mathscr{V}_i$. Thus the first two conditions of being a primitive sequence are satisfied. Suppose $x \in M$, $m < k$, and V and V' are the first elements of $\mathscr{V}_m$ and $\mathscr{V}_k$, respectively, that contain x. Then if $V = \bigcap \{W_j\colon 1 \leqslant j \leqslant n\}$, $V' = \bigcap \{W_j'\colon 1 \leqslant j \leqslant n\}$, where each $W_j \in \mathscr{H}_m^j$ and each $W_j' \in \mathscr{H}_k^j$, then $x \in \pi(W_j, \mathscr{H}_m^j) \cap \pi(W_j', \mathscr{H}_k^j)$. Because each $(\mathscr{H}_i^j)_{i \in N}$ is a primitive sequence, $W_j' \subset W_j$. Therefore $V' \subset V$. It follows that $(\mathscr{V}_i)_{i \in N}$ is a primitive sequence of M in S. Suppose $(V_i)_{i \in N}$ is a decreasing representative of $(\mathscr{V}_i)_{i \in N}$ and $1 \leqslant j \leqslant n$. For each $i \in N$ there exists $x \in V_i \cap M$ such that the first element of $\mathscr{H}_i^j$ that contains x includes V_i. For

$$V_i = \bigcap \{W_k\colon 1 \leqslant k \leqslant n\} \text{ where } W_k \in \mathscr{H}_i^k \text{ and } \bigcap \{\pi(W_k, \mathscr{H}_i^k)\colon 1 \leqslant k \leqslant n\} \cap M \neq \emptyset.$$

By Lemma I.2.2, the remainder of the conclusion follows.

DEFINITION 1.3. If $(\mathscr{V}_i)_{i \in N}$ is a primitive sequence obtained from primitive sequences $(\mathscr{H}_i^1)_{i \in N}, \ldots, (\mathscr{H}_i^n)_{i \in N}$ as in the statement of Lemma 1.2, we shall say that $(\mathscr{V}_i)_{i \in N}$ is the *lexicographic refinement* of $(\mathscr{H}_i^1)_{i \in N}, \ldots, (\mathscr{H}_i^n)_{i \in N}$.

Remark 1.1. Use of the definite article may be justified by checking that the construction described results in a unique sequence.

The second part of the following simple lemma follows from Lemma 1.2 and serves as an example of its use.

LEMMA 1.3. *Suppose $(\mathscr{H}_n)_{n \in N}$ is a primitive sequence of M in S and $L \subset S$. Then $(\mathscr{H}_n)_{n \in N}$ is a primitive sequence of $L \cap M$ in S. If $\mathscr{V}_n = \{L \cap H \cap M\colon H \in \mathscr{H}_n$ and $M \cap L \cap \pi(H, \mathscr{H}_n) \neq \emptyset\}$ for each n, then $(\mathscr{V}_n)_{n \in N}$ is a primitive sequence of $L \cap M$ in itself with the property that for any decreasing representative $(V_n)_{n \in N}$ of $(\mathscr{V}_n)_{n \in N}$ there is a decreasing representative $(H_n)_{n \in N}$ of $(\mathscr{H}_n)_{n \in N}$ such that for each $n \in N$, the first element of $\mathscr{H}_n$ that includes a term of $(V_n)_{n \in N}$ is H_n.*

Proof. The first conclusion follows directly from Definition I.2.1. The second follows from Lemma 1.2 on considering the primitive sequence $(\mathscr{H}_n^2)_{n \in N}$ of $L \cap M$ in S defined by $\mathscr{H}_n^2 = \{L \cap M\}$ for all $n \in N$ and taking $\mathscr{H}_n^1 = \mathscr{H}_n$ for all $n \in N$.

Suppose A is a well-ordered set and $(\mathscr{W}_\alpha)_{\alpha \in A}$ is a family of well-ordered sets. Define a relation $\preccurlyeq$ with field $\mathscr{W} = \bigcup \{\mathscr{W}_\alpha\colon \alpha \in A\}$ by $W \preccurlyeq W'$ if and only if either a) $W = W'$ or b) $W \neq W'$ and if α and α' are the first elements of A such that $W \in \mathscr{W}_\alpha$ and $W' \in \mathscr{W}_{\alpha'}$ then either (i) α precedes α' or (ii) $\alpha = \alpha'$ and W precedes W' in $\mathscr{W}_\alpha$. It is well known [8], p. 102, that this procedure defines a well-ordering with field $\mathscr{W}$. We will call this the *natural well-ordering* on $\mathscr{W}$.

LEMMA 1.4. *Suppose $\mathscr{W}$ is a well-ordered collection of subsets of a set S such that* 1) *$\mathscr{W}$ covers $M \subseteq S$, and* 2) *if $W \in \mathscr{W}$, then $\pi(W, \mathscr{W}) \neq \emptyset$.*

Suppose that for each $W \in \mathscr{W}$ there exists a primitive sequence $(\mathscr{H}_n^W)_{n \in N}$ of W in itself. For each $n \in N$, let $\mathscr{H}_n$ denote the collection of all sets H such that for some $W \in \mathscr{W}$, $H \in \mathscr{H}_n^W$ and $\pi(H, \mathscr{H}_n^W) \cap M \cap \pi(W, \mathscr{W}) \neq \emptyset$. Consider $\mathscr{H}_n$ under the well-ordering induced by the natural well-ordering on $\bigcup \{\mathscr{H}_n^W\colon W \in \mathscr{W}\}$.

Then $(\mathscr{H}_n)_{n \in N}$ is a primitive sequence of M in S such that each $\mathscr{H}_n \subset \bigcup \{\mathscr{H}_n^W\colon W \in \mathscr{W}\}$. If $(H_n)_{n \in N}$ is a decreasing representative of $(\mathscr{H}_n)_{n \in N}$, then there exists $W \in \mathscr{W}$ and $j \in N$ such that $H_n \in \mathscr{H}_n^W$ for all $n \geqslant j$.

Proof. Each $\mathscr{H}_n$ is well-ordered and if $x \in M$ there exists a first $W \in \mathscr{W}$ such that $x \in W$. There exists a first $H \in \mathscr{H}_n^W$ such that $x \in H$. Therefore $H \in \mathscr{H}_n$ and $(\mathrm{Pl})_n$ of Definition I.2.2 is satisfied for all $n \in N$. Suppose $H \in \mathscr{H}_n$ and W is the first element of $\mathscr{W}$ such that $H \in \mathscr{H}_n^W$ and $\pi(H, \mathscr{H}_n^W) \cap M \cap \pi(W, \mathscr{W}) \neq \emptyset$. Let $x \in \pi(H, \mathscr{H}_n^W) \cap M \cap \pi(W, \mathscr{W})$. If $x \in H' \in \mathscr{H}_n$ there exists a first $W' \in \mathscr{W}$ such that $H' \in \mathscr{H}_n^{W'}$ and $\pi(H', \mathscr{H}_n^W) \cap M \cap \pi(W', \mathscr{W}) \neq \emptyset$. Then $x \in W'$ since $H' \subset W'$. Therefore W' does not precede W. By the definition of natural ordering, H' does not precede H and thus $x \in \pi(H, \mathscr{H}_n)$. Therefore $(P2)_n$ of Definition I.2.2 is satisfied for all $n \in N$.

Suppose $x \in M$, $j<n$, and H and H' are the first elements of $\mathscr{H}_j$ and $\mathscr{H}_n$ respectively that contain x. Let W and W' be the first elements of $\mathscr{W}$ such that $H \in \mathscr{H}_j^W$ and $H' \in \mathscr{H}_n^{W'}$ and $\pi(H, \mathscr{H}_j^W) \cap M \cap \pi(W, \mathscr{W}) \neq \emptyset \neq \pi(H', \mathscr{H}_n^{W'}) \cap \cap M \cap \pi(W', \mathscr{W})$. It is easy to see that $x \in \pi(W, \mathscr{W}) \cap \pi(W', \mathscr{W})$ and therefore $W = W'$. Therefore $H' \subset H$, because $(\mathscr{H}_i^W)_{i \in N}$ is primitive. Therefore $(P3)_n$ of Definition I.2.2 is satisfied for all $n \in N$. Suppose $(H_i)_{i \in N}$ is a decreasing representative of $(\mathscr{H}_i)_{i \in N}$. For each $i \in N$, there exists a first $W_i \in \mathscr{W}$ such that $H_i \in \mathscr{H}_i^{W_i}$. Let W denote the first element of $\{W_i: i \in N\}$. There exists j such that $W = W_j$. Suppose $n \geqslant j$. Then $H_n \subset H_j \subset W$. Because $H_n \cap \pi(W_n, \mathscr{W}) \neq \emptyset$, it follows that W does not precede W_n. Therefore $W = W_n$.

LEMMA 1.5. *Suppose $(\mathscr{B}_n)_{n \in N}$ is a sequence of well-ordered bases for a topological space X and for each $B \in \bigcup \{\mathscr{B}_n: n \in N\}$ there exists a primitive sequence $P(B)$ of B in itself such that the members of each $P(B)_i$ are open.*

Then there exists a primitive sequence $(\mathscr{H}_i)_{i \in N}$ of X such that for each $n \in N$:

$(1)_n$ $\mathscr{H}_n \subset \mathscr{B}_n$.

$(2)_n$ *If $j<n$, $x \in X$, and H and H' are the first elements of $\mathscr{H}_j$ and $\mathscr{H}_n$, respectively, that contain x, then if x is in a proper open subset of H, then H' is a proper subset of H.*

(3) *If $(H_i)_{i \in N}$ is a decreasing representative of $(\mathscr{H}_i)_{i \in N}$ such that $\pi(H_j, \mathscr{H}_j) \cap \cap \pi(H_{j+1}, \mathscr{H}_{j+1}) \neq \emptyset$ for all $j \in N$, then for each $j \in N$ there exists a decreasing representative $(G_{jm})_{m \in N}$ of $P(H_j)$ such that for each $k \in N$, the set G_{jk} is the first element of $P(H_j)_k$ that includes a term of $(H_i)_{i \in N}$.*

Proof. Let $\mathscr{H}_1 = \{H \in \mathscr{B}_1: \pi(H, \mathscr{B}_1) \neq \emptyset\}$. Suppose collections $\mathscr{H}_1, \ldots, \mathscr{H}_k$ exist satisfying $(P1)_n$, $(P2)_n$, and $(P3)_n$ of Definition I.2.2 and conditions $(1)_n$ and $(2)_n$ above for $1 \leqslant n \leqslant k$, and $(*)_k$: if $x \in X$ and H_j is the first element of $\mathscr{H}_j$ that contains x for $1 \leqslant j \leqslant k$ and H_{jm} is the first element of $P(H_j)_m$ that contains x for $1 \leqslant m \leqslant k-j$ then $H_n \subset H_{j,n-j} \subset \ldots \subset H_{j1} \subset H_j$ for $1 \leqslant j<n \leqslant k$. We suppose that X is well ordered by a relation $\leqslant_0$. Given a well ordering $\leqslant_{k-1}$ on X, define a well-ordering $\leqslant_k$ on X as in the proof of Lemma I.2.1 (taking $M = X$) and let X_k denote the resulting well-ordered set. Suppose $x \in X_k$ and t is a function on $s(x)$ (the initial segment of X_k determined by x) into $\mathscr{B}_{k+1}$. If there exists a first $x' <_k x$ such that $x \in t(x')$ let $f(t)$ denote $t(x')$. Suppose no such x' exists. For each $j = 1, \ldots, k$ let H_j be the first element of $\mathscr{H}_j$ that contains x. Then sequences as described above in $(*)_k$ exist. Since $x \in H_k$, for each j, $1 \leqslant j \leqslant k$, there exists a first element $H_{j,k+1-j} \in P(H_j)_{k+1-j}$ containing x. If x is in a proper open subset of $Y = \bigcap \{H_{j,k+1-j}: 1 \leqslant j \leqslant k\}$, then let $f(t)$ denote the first element of $\mathscr{B}_{k+1}$ that contains x and is a proper

subset of Y. Otherwise $Y \in \mathscr{B}_{k+1}$ and we let $f(t) = Y$. By the transfinite recursion theorem [7], p. 70 there exists a function $U_k: X_k \to \mathscr{B}_{k+1}$ such that $U_k(x) = f(U_k|s(x))$ for all $x \in X_k$.

As in the proof of Lemma I.2.1 it may be shown that a well-ordered collection $\mathscr{H}_{k+1}$ may be obtained from the range of U_k satisfying conditions $(P1)_{k+1}$, $(P2)_{k+1}$, $(P3)_{k+1}$ of Definition I.2.2 and, conditions $(1)_{k+1}$ and $(2)_{k+1}$. We will show that $(*)_{k+1}$ is satisfied. Suppose that $x \in X$, that H_j is the first element of $\mathscr{H}_j$ that contains x for $1 \leqslant j \leqslant k+1$, and that H_{jm} is the first element of $P(H_j)_m$ that contains x for $1 \leqslant m \leqslant k+1-j$. If $1 \leqslant j<n \leqslant k$, then $H_n \subset H_{j,n-j} \subset \ldots \subset H_{j1} \subset H_j$ by the assumption $(*)_k$. Let y denote the first element of X_k such that $U_k(y) = H_{k+1}$. Then $y \in \pi(H_{k+1}, \mathscr{H}_{k+1})$, by the definition of order on $\mathscr{H}_{k+1}$ given in the proof of Lemma I.2.1, and H_j is the first element of $\mathscr{H}_j$ that contains y for $1 \leqslant j \leqslant k$. Therefore $H_{k+1} = f(U_k|s(y))$ is either the first element of $\mathscr{B}_{k+1}$ which is a proper subset of $Y = \bigcap \{H_{j,k+1-j}: 1 \leqslant j \leqslant k\}$ and which contains x or else it is Y. It follows that $(*)_{k+1}$ is valid. Since $\mathscr{H}_1$ obviously satisfies the conditions $(P1)_1$, $(P2)_1$, $(P3)_1$, $(1)_1$, $(2)_1$, and $(*)_1$ it follows by induction that a primitive sequence $(\mathscr{H}_n)_{n \in N}$ of X exists satisfying $(1)_n$ and $(2)_n$ for each $n \in N$. We shall show that condition (3) of the lemma is satisfied. Suppose that $(H_n)_{n \in N}$ is a decreasing representative of $(\mathscr{H}_n)_{n \in N}$ such that $\pi(H_n, \mathscr{H}_n) \cap \pi(H_{n+1}, \mathscr{H}_{n+1}) \neq \emptyset$ for all $n \in N$. It is easy to see from this condition and $H_{n+1} \subset H_n$ that H_n is the first element of $\mathscr{H}_n$ that includes H_{n+1}. Furthermore, if $j<n$ and H is the first element of $\mathscr{H}_j$ that includes H_n and $x \in \pi(H_n, \mathscr{H}_n)$ then $x \in \pi(H, \mathscr{H}_j)$. Therefore for all $k \in N$, condition $(*)_k$ will apply to the sets $H_1, \ldots, H_k$. It follows that if $j, m \in N$, then there exists a first $H_{jm} \in P(H_j)_m$ that includes H_{j+m}. Let G_n denote H_{j+n} for all $n \in N$. If x is the first element of X_{j+n-1} such that $U_{j+n-1}(x) = G_n$, then H_{jn} is the first element of $P(H_j)_n$ that contains x and H_{jn} includes G_n. Therefore Lemma I.2.2 applies to $(G_n)_{n \in N}$ and $(P(H_j)_n)_{n \in N}$. It follows that there exists a decreasing representative $(G_{jm})_{m \in N}$ of $P(H_j)$ with the property stated.

2. **Pararegularity.** We have emphasized in part I that our main results are obtained without the use of regularity. This emphasis is justified, in our opinion, by the extent of the class of nonregular basically complete spaces and the nature of the results concerning them. There is however, as is shown in § 3, a residue of regularity present in basically complete spaces. We call the concept involved in this residue *pararegularity* and devote this section to its explication. The idea permits deeper analysis of basic completeness and has interest in itself as a natural weakening of regularity. Evidence for the latter claim is presented here; another result proved later is a Baire category theorem for pararegular countably monotonically complete spaces (Theorem 5.1). The concept also plays a significant role in some characterization theorems of § 7.

The definition is very much in the spirit of this work; it involves monotonically contracting sequences of open coverings and their decreasing representatives. The techniques being exposited in these articles have been developed to handle just such situations. We proceed to define the concept, present some examples and counterexamples, and prove some basic results concerning it.

In connection with the following definition, the phrase "a monotonically contracting sequence of open coverings of U" refers to the case of Definition I.2.1 where $U = M = S$.

DEFINITION 2.1. A space X is called *pararegular* if and only if for every open $U \subset X$ there exists a monotonically contracting sequence $(\mathcal{G}_n)_{n \in N}$ of open coverings of U such that if $(G_n)_{n \in N}$ is a decreasing representative of $(\mathcal{G}_n)_{n \in N}$, then $\bigcap \{\overline{G}_n \colon n \in N\} \subset U$.

Every such sequence $(\mathcal{G}_n)_{n \in N}$ will be called a *pararegularizing sequence* (abbreviated by *p-sequence*) *of U*. The reference to U will be omitted when contextually clear.

Several of the theorems of this section and Section 3 either provide examples of pararegular spaces or give methods of constructing them. We give some examples below which, among other matters, compare pararegular spaces to regular spaces, to semiregular spaces in the sense of M. H. Stone [17], and to completely Hausdorff [16] spaces (due to Uryson [19]).

EXAMPLE 2.1. The well-known example of Aleksandrov and Uryson [2], p. 5 in which the underlying set is $[0, 1]$ and the topology is generated by the union of the usual topology $\mathcal{I}$ and the family of all sets $U \setminus \{1/n \colon n \in N\}$ where $U \in \mathcal{I}$, is a basically complete nonregular space which is pararegular and completely Hausdorff but is not semiregular.

EXAMPLE 2.2. Example 4 of I is another space with the properties listed above in 2.1.

EXAMPLE 2.3. The unnumbered example following Example 4 of I is a T_1 nonpararegular space which has λ-bases locally.

EXAMPLE 2.4. The space of Example 88 of [16] is a completely Hausdorff semiregular space which is not pararegular.

EXAMPLE 2.5. The space of Example 100 of [16] is a basically complete (and therefore pararegular) semiregular Hausdorff space which is not completely Hausdorff.

Additional examples are given in §§ 3, 4, and 5.

We list the following theorems concerning pararegularity and give the proofs at the end of the section. The theorems show that pararegularity has many of the properties of regularity. A relation to basic completeness is given in § 3.

THEOREM 2.1. *Every regular space is pararegular.*

THEOREM 2.2. *If a space is pararegular, then all of its subspaces are pararegular.*

THEOREM 2.3. *If X is a space having a base each element of which has a pararegularizing sequence, then X is pararegular.*

THEOREM 2.4. *Every pararegular space is essentially T_1.*

THEOREM 2.5. *Every pararegular T_0-space is Hausdorff.*

THEOREM 2.6. *The Cartesian product space of any family of pararegular spaces is pararegular.*

THEOREM 2.7. *The topological sum of any disjoint family of pararegular spaces is pararegular.*

LEMMA 2.1. *If X is a space, then an open $U \subset X$ has a pararegularizing sequence in X if and only if it has a primitive sequence $(\mathcal{H}_i)_{i \in N}$ in itself of collections of open sets such that if $(H_i)_{i \in N}$ is a decreasing representative of $(\mathcal{H}_i)_{i \in N}$, then $\bigcap \{\overline{H}_i \colon i \in N\} \subset U$.*

Proof. A proof may easily be obtained with the use of the definitions and Lemmas I.2.1, I.2.2, and I.2.3.

DEFINITION 2.2. A primitive sequence of U in itself with the property described in Lemma 2.1 will be called a *primitive pararegularizing sequence* (abbreviated *primitive p-sequence*) (*of U*).

Proof of 2.1. Suppose X is regular and U is open in X. For each $i \in N$ let $\mathcal{G}_i = \{V \subset U \colon V$ is open and $\overline{V} \subset U\}$. If $(G_i)_{i \in N}$ is a decreasing representative of $(\mathcal{G}_i)_{i \in N}$, then $\bigcap \{\overline{G}_i \colon i \in N\} \subset U$.

Proof of 2.2. Suppose X is pararegular and $Y \subset X$. If U is open in Y, then there exists $V \subset X$ such that $U = Y \cap V$. The set V has a primitive p-sequence $(\mathcal{H}_i)_{i \in N}$. Form a primitive sequence $(\mathcal{V}_n)_{n \in N}$ of U in itself as in Lemma 1.3, taking $Y = L$ and $V = M$. If $(V_i)_{i \in N}$ is a decreasing representative of $(\mathcal{V}_n)_{n \in N}$, then there exists a decreasing representative $(H_i)_{i \in N}$ of $(\mathcal{H}_i)_{i \in N}$ such that for each $n \in N$ the first element of $\mathcal{H}_i$ that includes a term of (V_i) is H_i. Because $\overline{V}_j^Y = \overline{V}_j^X \cap Y$ and $\bigcap \{\overline{V}_j^X \colon j \in N\} \subset \bigcap \{\overline{H}_j^X \colon j \in N\} \subset V$, it follows that $\bigcap \{\overline{V}_j^Y \colon j \in N\} \subset U$. Lemma 2.1 implies that U has a p-sequence in Y.

Proof of 2.3. Suppose $\mathcal{B}$ is a base for a space X such that each element of $\mathcal{B}$ has a p-sequence. Suppose $U \subset X$ is open. Let $\mathcal{W}' = \{B \in \mathcal{B} \colon B \subset U\}$ be well-ordered and let $\mathcal{W}$ denote $\{W \in \mathcal{W}' \colon \pi(W, \mathcal{W}') \neq \emptyset\}$ with the well-ordering induced by that of $\mathcal{W}'$. Then $\mathcal{W}$ is a collection of subsets of U covering U and each $W \in \mathcal{W}$ has a primitive p-sequence $(\mathcal{H}_i^W)_{i \in N}$ by Lemma 2.1. By Lemma 1.4, there exists a primitive sequence $(\mathcal{H}_n)_{n \in N}$ of U in itself satisfying the conclusion of the lemma. If $(H_n)_{n \in N}$ is a decreasing representative of $(\mathcal{H}_n)_{n \in N}$ there exists $j \in N$ and $W \in \mathcal{W}$ such that $H_n \in \mathcal{H}_n^W$ for all $n \geqslant j$. Therefore $\bigcap \{\overline{H}_n \colon n \in N\} \subset W$. It follows from Lemma 2.1 that U has a p-sequence.

Proof of 2.4. Suppose X is a pararegular space and $x, y \in X$. Suppose $z \in \overline{\{x\}} \cap \overline{\{y\}}$ and $w \in \overline{\{x\}}$. The set $X \setminus \overline{\{y\}}$ has a p-sequence $(\mathcal{G}_n)_{n \in N}$. If $w \notin \overline{\{y\}}$ there exists a decreasing representative $(G_n)_{n \in N}$ of $(\mathcal{G}_n)_{n \in N}$ such that $w \in \bigcap \{G_n \colon n \in N\}$. Therefore $x \in \bigcap \{G_n \colon n \in N\}$ and it follows that $z \in \bigcap \{\overline{G}_n \colon n \in N\} \subset X \setminus \overline{\{y\}}$. Therefore $w \in \overline{\{y\}}$. Similarly, $\overline{\{y\}} \subset \overline{\{x\}}$.

Proof of 2.5. Suppose X is T_0 and pararegular. Suppose $x, y \in X$ and there exists an open set U containing x but not y. The set U has a p-sequence $(\mathcal{G}_n)_{n \in N}$ which has a decreasing representative $(G_n)_{n \in N}$ with $x \in \bigcap \{G_n \colon n \in N\}$. Because $y \notin U$, there exists m such that $y \notin \bar{G}_m$. Therefore G_m and $X \setminus \bar{G}_m$ are disjoint open sets containing x and y respectively.

Proof of 2.6. Suppose $(X_\alpha)_{\alpha \in A}$ is a family of pararegular spaces. Let X denote the product with the topology introduced by Tychonoff [5], p. 73. Consider a basic open set $\langle U_{\alpha_1}, \ldots, U_{\alpha_k}\rangle = \bigcap \{p_{\alpha_i}^{-1}(U_{\alpha_i}) \colon 1 \leqslant i \leqslant k\}$ (where p_α is the projection map of X onto X_α). Each U_{α_i} is open in X_{α_i} and therefore has a p-sequence $(\mathscr{G}_n^i)_{n \in N}$. For each $n \in N$, let $\mathscr{G}_n$ denote the collection $\{\langle G_{\alpha_1}, \ldots, G_{\alpha_k}\rangle \colon G_{\alpha_i} \in \mathscr{G}_n^i \text{ for } 1 \leqslant i \leqslant k\}$. Suppose $(V_n)_{n \in N}$ is a decreasing representative of $(\mathscr{G}_i)_{i \in N}$. Each $V_n = \langle G_{\alpha_1}^n, \ldots, G_{\alpha_k}^n\rangle$. Because $V_{n+1} \subset V_n$, it follows that $G_{\alpha_j}^{n+1} \subset G_{\alpha_j}^n$ for all $n \in N$ and $1 \leqslant j \leqslant k$. Therefore $\bigcap \{\overline{G_{\alpha_j}^n} \colon n \in N\} \subset U_{\alpha_j}$ for $1 \leqslant j \leqslant k$. But $\bar{V}_n = \langle \overline{G_{\alpha_1}^n}, \ldots, \overline{G_{\alpha_k}^n}\rangle$ and

$$\bigcap \{\bar{V}_n \colon n \in N\} = \langle \bigcap \{\overline{G_{\alpha_1}^n} \colon n \in N\}, \ldots, \bigcap \{\overline{G_{\alpha_k}^n} \colon n \in N\}\rangle \subset \langle U_{\alpha_1}, \ldots, U_{\alpha_k}\rangle .$$

Therefore the last-named set has a p-sequence. The proof may be completed by applying Theorem 2.3.

Proof of 2.7. Suppose that $(X_\alpha)_{\alpha \in A}$ is a family of disjoint pararegular spaces. Let X denote their topological sum [5], p. 70. If U is open in X, then each $U \cap X_\alpha$ is open in X_α. Therefore each $U \cap X_\alpha$ has a p-sequence $(\mathscr{G}_i^\alpha)_{i \in N}$. For each $i \in N$, let $\mathscr{G}_i = \{G \colon G \in \mathscr{G}_i^\alpha \text{ and } \alpha \in A\}$. If $(G_i)_{i \in N}$ is a decreasing representative of $(\mathscr{G}_i)_{i \in N}$, then there exists $\alpha \in A$ such that $G_i \in \mathscr{G}_i^\alpha$ for all $i \in N$. Therefore $\bigcap \{\bar{G}_i^{X_\alpha} \colon i \in N\} \subset U \cap X_\alpha$. If $G \subset X_\alpha$, then $\bar{G}^{X_\alpha} = \bar{G}^X$. It follows that $(\mathscr{G}_i)_{i \in N}$ is a p-sequence for U.

3. Pararegularity and basic completeness. This section contains theorems relating the concepts of the title and also a characterization of basic completeness.

THEOREM 3.1. *If X is basically complete, then X is pararegular.*

COROLLARY 3.1. *Every T_2 first countable scattered space is pararegular.*

Remark. Example 2.3 shows that a T_1-space having λ-bases locally is not necessarily pararegular.

THEOREM 3.2. *If X is pararegular and has a λ-base, then X has λ-bases locally.*

THEOREM 3.3. *If X is a T_2-space having a λ-base, then X is basically complete if and only if X is pararegular.*

THEOREM 3.4. *If X is a T_0-space then X is basically complete if and only if every open $U \subset X$ has a monotonically contracting sequence $(\mathscr{G}_n)_{n \in N}$ of open coverings such that every decreasing representative $(G_n)_{n \in N}$ of $(\mathscr{G}_n \setminus \{\emptyset\})_{n \in N}$ converges to some $x \in U$ and $\{x\} = \bigcap \{\bar{G}_n \colon n \in N\}$.*

Proof of 3.1. Suppose X is basically complete and $U \subset X$ is open. Then there exists a sequence $(\mathscr{G}_i)_{i \in N}$ of bases for U satisfying (Λ) of Theorem I.3.2 with X replaced by U. Each member of each $\mathscr{G}_i$ is open in X. If $(G_i)_{i \in N}$ is a decreasing representative of $(\mathscr{G}_i \setminus \{\emptyset\})_{i \in N}$ there exists $x \in U$ such that $\{G_i \colon i \in N\}$ converges to x. If $y \neq x$ there exist disjoint open sets V and W such that $x \in V$ and $y \in W$. Therefore $G_n \subset V$ for some $n \in N$ and thus $y \notin \bar{G}_n$. Thus $\bigcap \{\bar{G}_n \colon n \in N\} = \{x\} \subset U$; therefore U has a p-sequence.

Corollary 3.1 is immediate from Theorem 3.1 and Theorem I.3.8.

Proof of 3.2. Suppose X is pararegular and has a λ-base. By Theorem I.3.2 there is a sequence $(\mathscr{G}_i)_{i \in N}$ of bases for X satisfying (Λ). Suppose U is open in X. Then U has a primitive p-sequence $(\mathscr{W}_i)_{i \in N}$ by Lemma 2.1. For each i, let $\mathscr{V}_i' = \{G \in \mathscr{G}_i \colon G \subset U\}$. Then $(\mathscr{V}_i')_{i \in N}$ is monotonically contracting. By Lemma I.2.1 a primitive sequence $(\mathscr{V}_i)_{i \in N}$ may be derived from $(\mathscr{V}_i')_{i \in N}$ such that each $\mathscr{V}_i \subset \mathscr{V}_i'$. By Lemma 1.2 the lexicographic refinement $(\mathscr{U}_i)_{i \in N}$ of $(\mathscr{W}_i)_{i \in N}$ and $(\mathscr{V}_i)_{i \in N}$ exists. If $(U_i)_{i \in N}$ is a decreasing representative of $(\mathscr{U}_i \setminus \{\emptyset\})_{i \in N}$, then there exist decreasing representatives $(V_i)_{i \in N}$ and $(W_i)_{i \in N}$ such that for each $i \in N$ the first elements of $\mathscr{V}_i$ and $\mathscr{W}_i$ that include a term of $(U_i)_{i \in N}$ are V_i and W_i respectively. There exists $x \in \bigcap \{\bar{V}_i \colon i \in N\}$. Because $x \in \bar{W}_i$ for all $i \in N$, it follows that $x \in U$. Clearly, $x \in \bigcap \{\bar{U}_i \colon i \in N\}$. If $y \in \bigcap \{\bar{U}_i \colon i \in N\}$ and W is open and $y \in W$, then there exists V_j such that $V_j \subset W$. It follows that there exists $U_n \subset V_j \subset W$. Thus $\{U_i \colon i \in N\}$ converges to y. Therefore, with the use of Lemma I.2.3, a λ-base for U may be obtained from $(\mathscr{U}_i)_{i \in N}$. (Theorem I.3.3 shows that the construction results in a λ-base.) Thus X has λ-bases locally.

Proof of 3.3. This is immediate from the definition of *basically complete* and the two preceding theorems.

Proof of 3.4. The necessity follows readily from Theorem I.3.2 and the property of being a Hausdorff space. If X is a space satisfying the given condition, then X is pararegular and therefore T_2. Since the condition obviously implies that each open subset of X has a λ-base (by Theorem I.3.3), it follows that X has λ-bases locally.

4. Monotonic completeness. The purpose of this section is to discuss a simple and natural idea of topological completeness which is used later to analyze basic completeness. The underlying idea is called *monotonic completeness*; some modifications are also discussed. A countable version for metric spaces was used in [1, 18] and forerunners of it are to be found in Cantor's work [3]. Metrizable countably monotonically complete spaces are metrically topologically complete [18].

DEFINITION 4.1. A base $\mathscr{B}$ for a space is said to be (*countably*) *monotonically complete* [20] if and only if the closures of the elements of every (countable) monotonic subcollection of $\mathscr{B}$ have a point in common. If a space has such a base it is called a (*countably*) *monotonically complete space*. If a space has a base $\mathscr{B}$ such that each element of $\mathscr{B}$ is (countably) monotonically complete the space is called a *locally* (*countably*) *monotonically complete space*.

We use the terms *locally* (*countably*) *compact space* to mean a space each point of which belongs to an open set with a (countably) compact closure. *Countable compactness* and *compactness* are used as in [16]; we do not assume T_2.

The proofs of the following four theorems are left to the reader.

THEOREM 4.1. *A locally* (*countably*) *compact space is* (*countably*) *monotonically complete.*

THEOREM 4.2. *A space having a λ-base* (*λ-bases locally*) *is monotonically complete* (*locally monotonically complete*). *Therefore every basically complete space is locally monotonically complete.*

THEOREM 4.3. *A regular* (*countably*) *monotonically complete space is locally* (*countably*) *monotonically complete.*

De Groot introduced the following concept in two different forms ([4], remark on p. 763). The second form is used here because the spaces considered in this paper are not assumed to be regular.

DEFINITION 4.2 [4]. A space is said to be (*countably*) *subcompact* if it has a base $\mathscr{B}$ such that the closures of the elements of any (countable) filter base included in $\mathscr{B}$ have a point in common.

In [6] the following concept, stronger than subcompactness, is defined.

DEFINITION 4.3 [6]. A space is said to be *basis compact* if it has a base $\mathscr{B}$ such that the closures of the elements of every subcollection of $\mathscr{B}$ with the finite intersection property have a point in common.

The example of Theorem 9 of [15] shows that there are subcompact Moore spaces which are not basis compact. Theorem 7.3 below shows that subcompactness and countable monotonic completeness are equivalent for pararegular spaces having bases of countable order.

THEOREM 4.4. *Every* (*countably*) *subcompact space is* (*countably*) *monotonically complete*; *therefore all basis compact spaces are monotonically complete.*

EXAMPLE 4.1. The domain of Example 2 of [20] is a monotonically complete metacompact developable completely Hausdorff space which is not pararegular and not locally monotonically complete and consequently not basically complete. It is basis compact.

EXAMPLE 4.2. The so-called Michael line [11] in which the reals are given a topology generated by the union of the usual topology with $\{\{x\}\colon x \text{ is irrational}\}$ is locally monotonically complete; in fact, it is basis compact, but it is not basically complete. The subset of the rational numbers is closed but it is not countably monotonically complete and thus *a fortiori* not basis compact. Thus (countable) monotonic completeness is not hereditary with respect to closed subspaces.

EXAMPLE 4.3. The example of Theorem 8 of [15] is a Moore space which is not a subspace of any regular monotonically complete space having a base of countable order.

EXAMPLE 4.4. The example of Theorem 9 of [15] is a monotonically complete Moore space (therefore a subcompact Moore space) which is not a complete Moore space. This contrasts interestingly with the situation in metrizable spaces, when one considers that a metrizable space complete in Moore's sense is metrically topologically complete. (Note that basis compact Moore spaces are complete Moore spaces.)

A way in which the concept of monotonic completeness generalizes the concept of compactness is brought out by comparison with the concept of perfect compactness [13]. A space X is called *perfectly compact* if for every monotonic collection $\mathscr{M}$ of nonempty subsets of X the closures of the elements of $\mathscr{M}$ have a point in common. The equivalence of this concept with compactness is well known: [2, 9, 12]. It is clear that (countable) monotonic completeness is a weakening of (countable) perfect compactness.

R. L. Moore proved that closed subspaces of Moore spaces are compact if and only if they are countably compact [13]. If "Moore spaces" is replaced by "spaces having bases of countable order" in the phrase following "that" in the preceding sentence, the resulting statement is not a theorem as the example of the countable ordinals with the order topology shows. However, if monotonic completeness is considered in place of compactness, the following analogue is obtained.

THEOREM 4.5. *A space having a base of countable order is monotonically complete if and only if it is countably monotonically complete.*

Proof. Suppose X is countably monotonically complete and has a base of countable order. By Theorem I.3.4 it has a countably monotonically complete base $\mathscr{B}$ of countable order. We may assume $\emptyset \notin \mathscr{B}$. Suppose $\mathscr{M}$ is a monotonic subcollection of $\mathscr{B}$. If some member of $\mathscr{M}$ is a subset of all elements of $\mathscr{M}$ then the closures of the elements of $\mathscr{M}$ have a point in common. If no such member of $\mathscr{M}$ exists, then $\mathscr{M}$ is perfectly decreasing. By induction there exists a sequence $(M_i)_{i\in N}$ such that each $M_i \in \mathscr{M}$ and M_{i+1} is a proper subset of M_i. Suppose $A \in \mathscr{M}$. If no $M_i \subset A$, then $A \subset M_i$ for all $i \in M$. Since $A \neq \emptyset$, and $\mathscr{B}$ is a base of countable order, it follows that some $M_j \subset A$. But then $M_j \subset M_{j+1}$ which is a contradiction. Therefore for each $A \in \mathscr{M}$ there exists $M_i \subset A$. By countable monotonic completeness, there exists $x \in \bigcap\{\overline{M}_i\colon i \in N\}$. Therefore $x \in \overline{A}$ for all $A \in \mathscr{M}$ and X is thus monotonically complete.

THEOREM 4.6. *The Cartesian product space of any family of* (*locally*) (*countably*) *monotonically complete spaces is* (*locally*) (*countably*) *monotonically complete.*

THEOREM 4.7. *The topological sum of any disjoint family of* (*locally*) (*countably*) *monotonically complete spaces is* (*locally*) (*countably*) *monotonically complete.*

The proofs of the preceding two theorems are straightforward.

5. A Baire category theorem. J. C. Oxtoby has made a penetrating contribution [14] to the theory of Baire category. Theorem 5.1 of [14] gives quite general sufficient conditions that a space be a Baire space (i.e., that the intersection of a countable family of dense open sets is dense). It follows easily from this theorem that a regular countably monotonically complete space is a Baire space. We repeat Oxtoby's definitions here, give some examples, and show that, with a suitable modification of his completeness concept, a Baire category theorem holds for pararegular spaces.

DEFINITION 5.1 [14]. *A pseudo-base* for a space is a collection of nonempty open sets such that any nonempty open set includes some member of the collection. A space is *quasi-regular* if every nonempty open set includes the closure of some nonempty open set. A space is *pseudo-complete* if it is quasi-regular and has a sequence $(\mathscr{B}_n)_{n\in N}$ of pseudo-bases such that if $(B_n)_{n\in N}$ is a decreasing representative of $(\mathscr{B}_n)_{n\in N}$ such that $\overline{B}_{n+1}\subset B_n$ for all $n\in N$, then $\bigcap\{B_n\colon n\in N\}\neq\emptyset$.

In [14], among other interesting results, Oxtoby proves that *every pseudo-complete space is a Baire space.*

In order to deal with pararegular spaces which are not quasi-regular the following modification of Oxtoby's completeness condition seems appropriate.

DEFINITION 5.2. A space is said to be *pseudo-m-complete* if it has a sequence $(\mathscr{B}_n)_{n\in N}$ of pseudo-bases such that if $(B_n)_{n\in N}$ is a decreasing representative of $(\mathscr{B}_n)_{n\in N}$, then $\bigcap\{\overline{B}_n\colon n\in N\}\neq\emptyset$.

Clearly a pseudo-m-complete space which is quasi-regular is pseudo-complete. Every countably monotonically complete space is pseudo-m-complete as is every locally countably compact space.

EXAMPLE 5.1. *A locally monotonically complete Hausdorff pararegular space which is not quasi-regular.* Let S denote the simple extension [10] of the real numbers R by the irrationals; i.e., let $S = R$ and give S the topology σ generated by the union of the usual topology $\mathscr{T}$ and the set whose only element is the set P of irrational numbers. Then X is not quasi-regular because P is open but P does not include the closure of any nonempty open set. If U is open in S then $\overline{U}^S = \overline{U}^R$. From this it is easy to see that S is monotonically complete. It may also be seen that S is locally monotonically complete. The space S is pararegular. If B is an open interval in the usual topology, let $\mathscr{G}_n = \{U\in\sigma\colon \overline{U}^R\subset B\}$ for each $n\in N$. Then $(\mathscr{G}_n)_{n\in N}$ is a p-sequence. Consider an open set $B\cap P$ where B is an open interval in $\mathscr{T}$. Let $(r_i)_{i\in N}$ be a one-one sequence whose range is Q. For each n, let $\mathscr{G}_n = \{U\in\sigma\colon U\subset B\cap P$ and $\{r_1,\dots,r_n\}\cap\overline{U}^S = \emptyset\}$. Then each $\mathscr{G}_n$ covers $B\cap P$ and the sequence $(\mathscr{G}_n)_{n\in N}$ is a p-sequence for U. This space does not have a base of countable order because of Theorem 7.2 below and the fact that the rationals are a closed subspace of S.

EXAMPLE 5.2. *A pararegular pseudo-m-complete space which is not quasi-regular and not monotonically complete.* Let R denote the real numbers and Q the rationals. For each $n\in N$ let $S_n = S\times\{n\}$, and let $L = \bigcup\{S_n\colon n\in N\}\cup Q$, where S is as in Example 5.1. Generate a topology by giving each S_n the product topology and for $x\in Q$ take as neighborhood base the collection of all sets

$$V(x,n) = \bigcup\{B(x,1/n)\times\{m\}\colon n\leqslant m \text{ and } m\in N\}\cup(B(x,1/n)\cap \mathrm{Q}),$$

where $B(x,r) = \{y\in R\colon |x-y|<r\}$. That this space is not monotonically complete may be seen by an argument based on one which shows that Q is not monotonically complete.

The next example shows that monotonic completeness cannot be replaced by pseudo-m-completeness in the characterization theorems of § 7.

EXAMPLE 5.3. *A metrizable pseudo-m-complete space which is not metrically topologically complete.* Such an example may be obtained as a subspace of Example 1.2 of [5], p. 107 (a modification of an example in [2]). Let $D_i = \{(x,y)\colon y = i$ and $x\in Q\cap[0,1]\}$, for $i = 0,1$. Let $Z = D_0\cup D_1$. For $z\in D_1$ let a neighborhood base at z be $\{\{z\}\}$ and for $z\in D_0$ let such a base consist of all sets

$$\{(x,y)\in Z\colon 0<|x-z^1|<1/n\}\cup\{z\},$$

where $z = (z^1, 0)$. Then Z is metrizable, but is not metrically topologically complete because a homeomorph of Q is a closed subspace of Z. It is pseudo-m-complete since the set D_1 is a dense subspace each of whose points is isolated; therefore $\{\{z\}\colon z\in D_1\}$ is a monotonically complete pseudo-base.

The following theorem shows that the spaces of Example 5.1-5.2 are Baire spaces although they do not satisfy Oxtoby's conditions.

THEOREM 5.1. *A pararegular pseudo-m-complete space is a Baire space.*

Proof. Suppose X is a nonempty pararegular space and $(\mathscr{B}_n)_{n\in N}$ is a sequence of pseudo-bases satisfying the condition of Definition 5.2 such that $\emptyset\notin\mathscr{B}_n$ for all $n\in N$. Suppose $(E_i)_{i\in N}$ is a sequence of dense open subsets of X and $U\subset X$ is open and nonempty. Let $(\mathscr{G}_n)_{n\in N}$ and $(\mathscr{G}_n^i)_{n\in N}$ denote p-sequences for U and E_i, respectively. Because $E_1\cap U\neq\emptyset$ there exists $G_1\in\mathscr{G}_1$ and $G_{11}\in\mathscr{G}_1^1$ such that $G_1\cap G_{11}\neq\emptyset$. Since $\mathscr{B}_1$ is a pseudo-base, there exists $B_1\in\mathscr{B}_1$ such that $B_1\subset G_1\cap G_{11}$. Let $G_{i0} = X = B_0 = G_0$ for all $i\in N$. Suppose sequences $(B_i)_{1\leqslant i\leqslant k}$, $(G_i)_{1\leqslant i\leqslant k}$, and $(G_{nj})_{1\leqslant j\leqslant k-n+1}$, where $1\leqslant n\leqslant k$, have been defined satisfying for each $i\leqslant k$ these conditions:

$(1)_i$ $B_i\in\mathscr{B}_i$, $G_i\in\mathscr{G}_i$, and $G_{nj}\in\mathscr{G}_j^n$ for $1\leqslant n\leqslant i$ and $1\leqslant j\leqslant i-n+1$.

$(2)_i$ $G_i\subset G_{i-1}$.

$(3)_i$ $G_{nj}\subset G_{n,j-1}$, $1\leqslant n\leqslant i$ and $1\leqslant j\leqslant i-n+1$.

$(4)_i$ $B_i\subset B_{i-1}\cap G_i\cap(\bigcap\{G_{rs}\colon 1\leqslant r\leqslant i,\ r+s = i+1\})$.

There exists $x_k\in B_k\cap E_{k+1}$. There exist $G_{k+1}\in\mathscr{G}_{k+1}$ and $G_{rs}\in\mathscr{G}_s^r$ for $1\leqslant r\leqslant k+1$ and $r+s = k+2$ such that $x_k\in B_k\cap G_{k+1}\cap\bigcap\{G_{rs}\colon 1\leqslant r\leqslant k+1,\ r+s = k+2\}$ and $G_{rs}\subset G_{r,s-1}$ for $1\leqslant r\leqslant k$, $r+s = k+2$. There exists $B_{k+1}\in\mathscr{B}_{k+1}$ such that $(4)_{k+1}$ is satisfied. Thus conditions $(1)_{k+1}$-$(4)_{k+1}$ are satisfied. By induction, sequences $(G_k)_{k\in N}$ and $(G_{rs})_{s\in N}$, for all $r\in N$, exist satisfying $(1)_k$-$(4)_k$ for all $k\in N$. By the pseudo-m-completeness of X, there exists $x\in\bigcap\{\overline{B}_k\colon k\in N\}$. By the conditions $(1)_k$-$(4)_k$, $x\in\bigcap\{\overline{G}_k\colon k\in N\}\subset U$ and $x\in\bigcap\{\overline{G}_{rs}\colon s\in N\}\subset E_r$ for each $r\in N$. This concludes the proof.

If in the definition of *pararegularity* the monotonically contracting sequence $(\mathscr{G}_n)_{n\in N}$ is replaced by a sequence of pseudo-bases then a generalization of quasi-regularity is obtained; i.e., for each open $U\subset X$ there is a sequence $(\mathscr{B}_n)_{n\in N}$ of pseudo-bases for U such that if $(B_i)_{i\in N}$ is a decreasing representative, then $\bigcap\{\overline{B}_i\colon i\in N\}\subset U$. It is straightforward to verify that a pseudo-m-complete space with this property is a Baire space.

Theorem 5.1 above and Oxtoby's Theorem 5.1 of [14] each apply to situations not covered by the other as is shown by Example 5.2 and the following example.

EXAMPLE 5.4. *A T_1 quasi-regular space having λ-bases locally which is not* *pararegular*. Take as a base for a topology on N the collection of all sets which are of the form $\{2n\}$ for $n \in N$ or of the form $\{2n+1\} \cup \{m \in N\colon m \geqslant p \geqslant 2n+1\}$ for $n \in N$ and $p \in N$. This space is not pararegular since it is T_1 but not T_2. Every nonempty open set includes a set of the form $\{2n\}$ so it is quasi-regular. The space has λ-bases locally and is therefore pseudo-complete.

6. Bases closurewise of countable order. The concept introduced here is a strengthening of that of a base of countable order. It incorporates a regularity like condition in its definition. In this connection see Theorem 6.4.

DEFINITION 6.1. A collection $\mathscr{B}$ of subsets of a space X is called a *base closurewise of countable order* if and only if it is a base for X and if $x \in X$ and $\mathscr{K}$ is a perfectly decreasing subcollection of $\mathscr{B}$ such that the closure of each element of $\mathscr{K}$ contains x, then any open set containing x includes a member of $\mathscr{K}$.

Examples 2.3 and 5.4 are T_1-spaces which have λ-bases locally (and therefore have bases of countable order) but do not have bases closurewise of countable order. This follows from the fact that a T_1-space having a base closurewise of countable order is T_2. Note also that a T_2-space having a λ-base has a base closurewise of countable order. We list here three theorems which follow readily by means of techniques used in I, [23], and above.

THEOREM 6.1. *If a T_2-space X is the union of open subspaces each having a base closurewise of countable order then every subspace of X has a base closurewise of countable order.*

THEOREM 6.2. *A space X is essentially T_1 and has a base closurewise of countable order if and only if there exists a sequence $(\mathscr{G}_n)_{n\in N}$ of bases for X such that if $x \in X$ and $(G_n)_{n\in N}$ is a decreasing representative of $(\mathscr{G}_n)_{n\in N}$ such that if $x \in \bigcap\{\overline{G}_n\colon n \in N\}$ then $(G_n)_{n\in N}$ converges to x.*

THEOREM 6.3. *If a space X has a base closurewise of countable order, then any base for X includes a base closurewise of countable order.*

THEOREM 6.4. *If X is pararegular and has a base of countable order, then X has a base closurewise of countable order.*

Proof. There exists a sequence $(\mathscr{B}_n)_{n\in N}$ of bases for X such that if $(B_n)_{n\in N}$ is a decreasing representative of $(\mathscr{B}_n)_{n\in N}$ and $x \in \bigcap\{B_n\colon n \in N\}$, then $\{B_n\colon n \in N\}$ is a base at x. This follows from Theorem 2 of [23]. By pararegularity and Lemma 2.1 each $B \in \bigcup\{\mathscr{B}_n\colon n \in N\}$ has a primitive p-sequence $P(B)$. Consider the bases $\mathscr{B}_n$ as well-ordered and obtain with the use of Lemma 1.5 a primitive sequence $(\mathscr{H}_i)_{i\in N}$ of X with the properties in the statement of the lemma. From $(\mathscr{H}_i)_{i\in N}$ a sequence $(\mathscr{G}_n)_{n\in N}$ as described in Lemma I.2.3 may be obtained. Each $\mathscr{H}_n \subset \mathscr{B}_n$. Therefore if $(H_n)_{n\in N}$ is a decreasing representative of $(\mathscr{H}_n)_{n\in N}$ and $x \in \bigcap\{H_n\colon n \in N\}$ then $\{H_n\colon n \in N\}$ is a base at x. It follows readily from this and the construction in Lemma I.2.3 that each term of $(\mathscr{G}_n)_{n\in N}$ is a base for X. Suppose $(G_n)_{n\in N}$ is a decreasing representative of $(\mathscr{G}_n)_{n\in N}$ and $x \in \bigcap\{\overline{G}_n\colon n \in N\}$. By Lemma I.2.3 there exists a decreasing representative $(H_n)_{n\in N}$ of $(\mathscr{H}_n)_{n\in N}$ such that for each n the first element of $\mathscr{H}_n$ that includes a term of $(G_n)_{n\in N}$ is H_n. Suppose $x \in \pi(H_{n+1}, \mathscr{H}_{n+1})$. If $x \in \pi(H, \mathscr{H}_n)$, then $H \supset H_{n+1}$. Because H_{n+1} includes a term of $(G_n)_{n\in N}$, it follows that H does not precede H_n. Therefore $x \in \pi(H_n, \mathscr{H}_n)$, because $H_n \supset H_{n+1}$. By Lemma 1.5 there exists a decreasing representative $(G_{ji})_{i\in N}$ of $P(H_j)$ such that for each $n \in N$, G_{jn} is the first element of $P(H_j)_n$ that includes a term of $(H_i)_{i\in N}$. Because $\bigcap\{\overline{G}_n\colon n \in N\} \subset \bigcap\{\overline{H}_n\colon n \in N\}$ and $\bigcap\{\overline{H}_n\colon n \in N\} \subset \bigcap\{\overline{G}_{jn}\colon n \in N\} \subset H_j$ (because $P(H_j)$ is a primitive p-sequence), it follows that $x \in \bigcap\{H_j\colon j \in N\}$. If U is open and $x \in U$, then there exists some $H_n \subset U$. But H_n includes some G_j. Thus, by Theorem 6.2, X has a base closurewise of countable order.

A proof of the following theorem may be obtained from the proof of Theorem 6.4.

THEOREM 6.5. *Suppose $(\mathscr{B}_n)_{n\in N}$ is a sequence of well-ordered bases for a pararegular space X. There exists a primitive sequence $(\mathscr{H}_n)_{n\in N}$ of X in itself such that each $\mathscr{H}_n \subset \mathscr{B}_n$ and if $(H_n)_{n\in N}$ is a decreasing representative of $(\mathscr{H}_n)_{n\in N}$ such that $\pi(H_n, \mathscr{H}_n) \cap \pi(H_{n+1}, \mathscr{H}_{n+1}) \neq \emptyset$ for all $n \in N$, then $\bigcap\{\overline{H}_n\colon n \in N\} = \bigcap\{\overline{H}_n\colon n \in N\}$.*

The following theorem may be obtained directly from the method used to prove Theorem 2 of [20].

THEOREM 6.6. *If X is an essentially T_1-space having a base closurewise of countable order and Y is an essentially T_1 open continuous uniformly monotonically complete image of X, then Y has a base of countable order. If Y is pararegular, then it has a base closurewise of countable order.*

THEOREM 6.7. *A monotonically complete base closurewise of countable order is a λ-base.*

7. Some characterizations of basic completeness.

THEOREM 7.1. *A space is basically complete if and only if it is T_1, locally monotonically complete, and has a base closurewise of countable order.*

Proof. The necessity of the conditions follows from Theorem 4.2 and the fact that a T_2-space having a λ-base has a base closurewise of countable order.

If X satisfies the condition and U is open in X, then U is the union of open subspaces having monotonically complete bases closurewise of countable order by Theorems 6.1 and 6.3. Hence U has a λ-base by Theorems 6.6 and I.3.1. Therefore X has λ-bases locally. It is easy to see that X is T_2.

THEOREM 7.2. *A space is basically complete if and only if it is a pararegular monotonically complete T_0-space having a base of countable order.*

Proof. The necessity follows from Theorems 3.1, 4.2, and the definitions involved. If X is a pararegular T_0-space, then X is T_2 and by Theorem 6.4, X has a base closurewise of countable order. Therefore X has a λ-base by Theorem 6.6. By Theorem 3.2, X has λ-bases locally.

THEOREM 7.3. *Suppose X is a pararegular T_0-space having a base of countable order. Then the following conditions are equivalent:*

(a) *X is countably monotonically complete.*
(b) *X is monotonically complete.*
(c) *X is subcompact.*
(d) *X is basically complete.*

Proof. (a) implies (b) by Theorem 4.5, (b) implies (d) by Theorem 7.2 and clearly (d) implies (a). Also (c) implies (b). Suppose (d) holds. Then X has a monotonically complete base $\mathscr{B}$ closurewise of countable order. Suppose $\mathscr{F} \subset \mathscr{B}$ is a filter base. Either some member of $\mathscr{F}$ is a subset of every member of $\mathscr{F}$ or there is a perfectly decreasing monotonic subcollection $\mathscr{C} \subset \mathscr{F}$. There exists $x \in \bigcap \{\bar{C}\colon C \in \mathscr{C}\}$. Suppose U is open and $x \in U$. Then there exists $C \in \mathscr{C}$ such that $C \subset U$ since $\mathscr{B}$ is closurewise of countable order. If $A \in \mathscr{F}$ then $A \cap C \neq \emptyset$. Therefore $x \in \bar{A}$. Hence $\bigcap \{\bar{A}\colon A \in \mathscr{F}\} \neq \emptyset$. Thus X is subcompact. Therefore (d) implies (c).

Corollary (De Groot [4]). *In a metrizable space the following conditions are equivalent:*
(a) *countable subcompactness,*
(b) *subcompactness,*
(c) *metric topological completeness.*

References

[1] P. S. Aleksandrov (≡ P. S. Alexandroff), *Sur les ensembles de la première classe et les espaces abstraits*, C. R. Acad. Sci. Paris 178 (1924), pp. 185–187.
[2] — and P. Uryson, *Mémoire sur les espaces topologiques compacts*, Verh. Nederl. Akad. Wetensch. Afd. Natuurk. Sect. I, vol. 14, no. 1 (1929), pp. 1–96.
[3] G. Cantor, *Über eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen*, J. Reine Angew. Math. 77 (1874), pp. 258–262.
[4] J. De Groot, *Subcompactness and the Baire category theorem*, Indag. Math. 25 (1963), pp. 761–767.
[5] R. Engelking, *Outline of General Topology*, Amsterdam 1968.
[6] A. Hajnal and I. Juhász, *A consequence of Martin's axiom*, Indag. Math. 33 (1971), pp. 457–463.
[7] P. R. Halmos, *Naive Set Theory*, Princeton 1960.
[8] F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig 1914. Reprinted New York 1949.
[9] K. Kuratowski and W. Sierpiński, *La théorème de Borel–Lebesgue dans la théorie des ensembles abstraits*, Fund. Math. 2 (1921), pp. 172–178.
[10] N. Levine, *Simple extensions of topologies*, Amer. Math. Monthly 71 (1964), pp. 22–25.
[11] E. Michael, *The product of a normal space and a metric space need not be normal*, Bull. Amer. Math. Soc. 69 (1963), pp. 375–376.
[12] R. L. Moore, *On the most general class L of Fréchet in which the Heine–Borel–Lebesgue theorem holds true*, Proc. Nat. Acad. Sci 5 (1919), pp. 206–210.
[13] — *Foundations of point set theory*, revised edition, Amer. Math. Soc. Coll. Pub. 13 (1962).
[14] J. C. Oxtoby, *Cartesian products of Baire spaces*, Fund. Math. 49 (1961), pp. 157–166.
[15] M. E. Rudin (Estill), *Concerning abstract spaces*, Duke Math. J. 17 (1950), pp. 317–327.
[16] L. A. Steen and J. A. Seebach, Jr., *Counterexamples in Topology*, New York 1970.
[17] M. H. Stone, *Applications of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc. 41 (1937), pp. 375–481.

[18] N. Vedenisov (≡ N. Wedenisoff), *Sur les espaces metriques complets*, Journal de Mathematiques, Ser. 9, vol. 9 (1930), pp. 377–381.
[19] P. S. Uryson (≡ P. S. Urysohn), *Über die Mächtigkeit der zusammenhängenden Mengen*, Math. Ann. 94 (1925), pp. 262–295.
[20] H. H. Wicke and J. M. Worrell, Jr., *Open continuous mappings of spaces having bases of countable order*, Duke Math. J. 34 (1967), pp. 255–272; errata pp. 813–814.
[21] — — *On the open continuous images of paracompact Čech complete spaces*, Pacific J. Math. 37 (1971), pp. 265–275.
[22] — — *Topological completeness of first countable Hausdorff spaces I*, Fund. Math. 75 (1972), pp. 209–222.
[23] — — *Characterizations of developable topological spaces*, Canad. J. Math. 17 (1965), pp. 820–830.

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