such that $h(0) = 0$ and $h'(0) = \ln s$. Suppose that

$$h(x) = (\ln s)x + c_2x^2 + c_3x^3 + \ldots \quad \text{for} \quad x \in [0, 1-s)$$

yields a solution of (35). A simple calculation shows that

$$c_2 = \frac{\ln s}{s^2 - s} \quad \text{and} \quad c_3 = \frac{-2\ln s}{s(s^3 - s)}.$$

Thus we get the existence of a positive $b < 1-s$ such that $h'' > 0$ and $h''' < 0$ on $(0, b)$, i.e., h is convex but not 2-convex on $(0, b)$. On account of Theorem 2, f possesses a convex but not 2-convex iteration group.

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Mappings onto circle-like continua

by

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Abstract. The main object of the present paper is to give a characterization of continua which can be mapped onto non-planar circle-like curves. This result is then applied to show that certain classes of continua cannot be mapped onto such curves. These results extend several well-known facts in this field.

The term compactum is used to mean a compact metric space. A connected compactum is called a continuum. By a curve we mean a one-dimensional continuum. The terms map and mapping will be used interchangingly to mean a continuous function. A map $f\colon X \to Y$ is said to be an ε-mapping, $\varepsilon > 0$, provided $\operatorname{diam} f^{-1}(y) < \varepsilon$ for every $y \in Y$. Throughout the paper we denote by S the unit circle in the complex plane and by I the unit interval $[0, 1]$ of reals. A continuum X is called *circle-like* (*snake-like*) if for every $\varepsilon > 0$ there exists an ε-mapping of X onto S (onto I, respectively). Clearly, any circle-like or snake-like continuum is a curve. The above classes of curves have been extensively studied by several authors. Known results show an important difference between the class of circle-like curves which can be embedded in the plane and the others. This difference will also be underlined by the results of this paper. Our main result gives a characterization of continua which can be mapped onto non-planar circle-like curves. This result solves a problem raised by Henderson in [7], and extends his result in this direction. We obtain also generalizations of the results of Ingram [8].

1. Some remarks on Abelian groups. Let G be an Abelian group. Denote by N the set of natural numbers, $N = \{1, 2, \ldots\}$. We say that $g \in G$ is *divisible* by a natural number n, notation: n/g, if $g = n \cdot g'$ for some $g' \in G$. For every $g \in G$ we define

$$d(g) = \sup\{n \in N\colon\ n/g\}.$$

Clearly, $d(0) = \infty$. If $d(g) < \infty$, then we say that g is *finitely divisible*; otherwise g is called *infinitely divisible*. If every element of G different from the neutral element 0 is finitely divisible, then we simply say that G is finitely divisible. Notice that every free Abelian group is finitely divisible.

1.1. *If $m, n \in N$ are relatively prime, $g \in G$, m/g and n/g, then $m \cdot n/g$.*

1.2. *Let g be an infinitely divisible element of an Abelian group G. Then there exists a sequence $p_1, p_2, \ldots$ of prime numbers* (1) *such that $p_1 \cdot \ldots \cdot p_j | g$ for every $j \geqslant 1$.*

Proof (2). There can be two cases: 1) there exists a sequence of prime numbers $p_1, p_2, \ldots$ such that $p_i \neq p_j$ for $i \neq j$ and $p_n | g$ for every $n \geqslant 1$. By 1.1 this sequence satisfies the conclusion of 1.2, 2) for some prime number p we have $p^n | g$ for every $n \geqslant 1$. In this case the sequence $p, p, \ldots$ satisfies the conclusion of 1.2.

A direct sequence $\underline{G} = \{G_n, h_{nm}\}$, $h_{nm}\colon G_n \to G_m$, of groups is called *movable* if for every $n \geqslant 1$ there exists an index $n_0 \geqslant n$ such that for every $m \geqslant n$ there exists a homomorphism $h\colon G_m \to G_{n_0}$ such that

$$(*) \qquad h_{nn_0} = h \circ h_{nm}.$$

1.3. *Let $\underline{G} = \{G_n, h_{nm}\}$ be a direct sequence with limit G^∞ such that $G_n = Z$, the group of integers, for every $n \geqslant 1$. Then the following conditions are equivalent:*

(i) *$\underline{G}$ is movable,*

(ii) *$G^\infty \approx 0$ or Z,*

(iii) *G^∞ is finitely divisible,*

(iv) *for every $l \geqslant 1$ there exists an index $m \geqslant l$ such that for every $n > m$ we have $|h_{mn}(1)| \leqslant 1$.*

Proof. (i)$\Rightarrow$(ii). Suppose $G^\infty \neq 0$. We have to show that $G^\infty \approx Z$. Let $g \neq 0$ be an element of G^∞. Let $\eta_n\colon G_n \to G^\infty$ denote the natural projection. Hence

$$(1) \qquad \eta_n = \eta_m \circ h_{nm} \quad \text{for} \quad m \geqslant n.$$

There exist an index n and an element $g_n \in G_n$ such that

$$(2) \qquad g = \eta_n(g_n).$$

Since $g, g_n \neq 0$, by (1) and (2) we obtain $h_{nm}(g_n) \neq 0$. Since $g, g_n = g_n \cdot 1$, we have $0 \neq h_{nm}(g) = g_n \cdot h_{nm}(1)$; therefore

$$(3) \qquad h_{nm}(1) \neq 0 \quad \text{for every } m \geqslant n.$$

Let $n_0 \geqslant n$ be chosen as in the definition of movability. We shall show that

$$(4) \qquad h_{n_0m}(1) = \pm 1 \quad \text{for every } m > n_0.$$

Indeed, let $h\colon G_m \to G_{n_0}$ be a homomorphism satisfying $(*)$. Since $h_{nm} = h_{n_0m} \circ h_{nn_0}$, by $(*)$ we obtain

$$h_{nn_0}(1) = h \circ h_{n_0m} \circ h_{nn_0}(1) = h(1) \cdot h_{n_0m}(1) \cdot h_{nn_0}(1).$$

Since $n_0 \geqslant n$, the last equality and condition (3) imply

$$h_{n_0m}(1) \cdot h(1) = 1.$$

But $h_{n_0m}(1)$ and $h(1)$ are integers, and hence $h_{n_0m}(1) = \pm 1$.

(1) Recall that 1 is not considered as prime.

(2) This simple proof is due to Professor J. Mioduszewski.

It follows from condition (4) that for every $m > n_0$ the function h_{n_0m} is an isomorphism between G_{n_0} and G_m. Therefore $G^\infty \approx G_{n_0} = Z$, which proves the implication.

Implication (ii)$\Rightarrow$(iii) is obvious.

(iii)$\Rightarrow$(iv). Suppose condition (iv) does not hold. Hence there exists an increasing sequence of natural numbers $n_1 < n_2 < n_3 < \ldots$ such that

$$(5) \qquad |h_{n_in_{i+1}}(1)| > 1 \quad \text{for every } i \geqslant 1.$$

Now we show that

$$(6) \qquad g = \eta_{n_1}(1) \neq 0,$$

where η_n is the projection defined at the beginning of the proof. Suppose $g = 0$; then there exists an index $n > n_1$ such that $h_{n_1n}(1) = 0$. Let $n_i > n$. Then we have

$$h_{n_1n_i}(1) = h_{nn_i} \circ h_{n_1n}(1) = 0,$$

contrary to (5).

Hence to finish the proof we need only to show that for every natural number k there exists an $l \geqslant k$ such that $l | g$. By (5) there exists an index $j > 1$ such that $|h_{n_1n_j}(1)| > k$. Put $l = |h_{n_1n_j}(1)|$. Then we have by (1) and (6),

$$g = \eta_{n_1}(1) = \eta_{n_j} \circ h_{n_1n_j}(1) = l \cdot [(\operatorname{sign} h_{n_1n_j}(1)) \cdot \eta_{n_j}(1)].$$

This implies that g is infinitely divisible, contrary to (iii).

(iv)$\Rightarrow$(i). Let n be a given natural number. If there exists an index $k > n$ such that $h_{nk}(1) = 0$, then put $n_0 = k$. Otherwise $|h_{nk}(1)| > 0$ for every $k \geqslant n$ and by (iv) there exists an index $n_0 \geqslant n$ such that

$$(7) \qquad \text{for every } m > n_0 \text{ we have } |h_{n_0m}(1)| = 1.$$

We have to prove that for $m \geqslant n$ there is a homomorphism $h\colon G_m \to G_{n_0}$ such that $(*)$ is fulfilled. If $m \leqslant n_0$, it suffices to set $h = h_{mn_0}$. Assume $m > n_0$. If $h_{nn_0}(1) = 0$, put $h = 0$, the null-homomorphism. If $h_{nn_0}(1) \neq 0$, then condition (7) is fulfilled. Hence h_{n_0m} is an isomorphism. So there is a homomorphism h such that $h \circ h_{n_0m} = 1_{G_{n_0}}$. It is easy to check that in both cases condition $(*)$ is fulfilled. This completes the proof.

1.4. *If each factor of a movable direct sequence of groups $\underline{G} = \{G_n, h_{nm}\}$ is finitely divisible, then the limit G^∞ is also finitely divisible.*

Proof. Suppose, to the contrary, that some element $g \neq 0$ of G^∞ is infinitely divisible. For the natural projections $\eta_n\colon G_n \to G^\infty$ we have

$$(1) \qquad \eta_n = \eta_m \circ h_{nm} \quad \text{for every } 1 \leqslant n \leqslant m.$$

There exist an index n and an element g_n of G_n such that

$$(2) \qquad \eta_n(g_n) = g.$$

Let n_0 be chosen as in the definition of movability. Since $g \neq 0$, by (1) and (2) we have $h_{nn_0}(g_n) \neq 0$. Since G_{n_0} is finitely divisible there exists a natural number k such that

(3) if $l \geqslant k$, then $h_{nn_0}(g_n)$ is not divisible by l.

By our assumption there exist an integer $l \geqslant k$ and an element $g' \in G^\infty$ such that $g = l \cdot g'$. There exist an index r and an element $g_r \in G_r$ such that $\eta_r(g_r) = g'$. Since $\eta_r(l \cdot g_r) = l \cdot g' = g$, by (2) there is an index $m \geqslant n, r$ such that

(4) $$h_{nm}(g_n) = h_{rm}(l \cdot g_r) .$$

Let h: $G_m \to G_{n_0}$ be a homomorphism satisfying (*). So by (*) and (4), $h_{nn_0}(g_n) = h \circ h_{nm}(g_n) = h \circ h_{rm}(l \cdot g_r) = l \cdot h \circ h_{rm}(g_r)$, which contradicts (3). This completes the proof. (Let us note that the proof is valid for arbitrary movable systems.)

2. Bruschlinsky's theorem and its consequences. Consider the unit circle S as an Abelian group with multiplication of complex numbers with module one as a group operation in S. Let X be a compactum and let f, g: $X \to S$. As usual we define $f \cdot g$: $X \to S$ by the formula $f \cdot g(x) = f(x) \cdot g(x)$ for every $x \in X$. It is evident that if $f \simeq g$ and $f' \simeq g'$, then $f \cdot f' \simeq g \cdot g'$. In this way the above operation induces a group operation in the set of homotopy classes of maps from X into S. This set with the induced group operation is denoted by $\pi^1(X)$ and is called the *Bruschlinsky group* of X. If Y is a compactum, then $H^1(Y)$ is used in this paper to denote the first Čech cohomology group of Y with integers Z as the coefficient group. If f is a map from X into Y, then by f^* we denote the induced homomorphism f^*: $H^1(Y) \to H^1(X)$. By γ we will denote a generator of the group $H^1(S) \approx Z$. Let $a \in \pi^1(X)$ be an element with a representative f, i.e., $a = [f]$. To the map f: $X \to S$, corresponds the element $f^*(\gamma) \in H^1(X)$ and it is easy to check that this element does not depend on the choice of a particular map f representing a. In this way we obtain a function χ: $\pi^1(X) \to H^1(X)$ defined by $\chi([f]) = f^*(\gamma)$. An important fact about χ is contained in the following *Bruschlinsky theorem*:

2.1. *The function* χ: $\pi^1(X) \to H^1(X)$ *is an isomorphism* [4, p. 226].

We say that a compactum X is *contractible with respect* to S, notation: cr$S$, if every map f from X into S is null-homotopic, $f \simeq 0$. By 2.1 we obtain the following known corollary

2.2. *A compactum X is* cr$S$ *iff* $H^1(X) = 0$.

If f: $S \to S$ is a map, then the degree of f, notation: deg$f$, is defined as the unique number such that $f^*(\gamma) = (\deg f) \cdot \gamma$. From now on by p_n we will denote a map from S into itself defined by $p_n(z) = z^n$, $n \geqslant 1$. It is well known that $\deg p_n = n$ ([5], p. 306), and p_n is a covering map. Now we shall prove the following proposition:

2.3. *Let f be a mapping of a compactum X into S. Let $g = f^*(\gamma)$ and suppose that $g = n \cdot \tilde{g}$ for some $n \geqslant 1$ and $\tilde{g} \in H^1(X)$. Then there exists a map $\tilde{f}$: $X \to S$ such that $\tilde{f}^*(\gamma) = \tilde{g}$ and $f = p_n \circ \tilde{f}$.*

Proof. By the Bruschlinsky theorem there exists a mapping k: $X \to S$ such that $k^*(\gamma) = \tilde{g}$. Since $p_n^*(\gamma) = n \cdot \gamma$, we have

$$(p_n \circ k)^*(\gamma) = k^*(n \cdot \gamma) = n \cdot \tilde{g} = g .$$

Again by 2.1 we see that $p_n \circ k \simeq f$. Let h: $X \times I \to S$ be a homotopy joining these maps, that is: $h(x, 0) = p_n \circ k(x)$ and $h(x, 1) = f(x)$ for every $x \in X$. Since p_n is a covering map, it has the homotopy lifting property [16, p. 67]. Hence there exists a map $\tilde{h}$: $X \times I \to S$ such that $h = p_n \circ \tilde{h}$ and $\tilde{h}(x, 0) = k(x)$. Setting

$$\tilde{f}(x) = \tilde{h}(x, 1), \quad \text{for } x \in X ,$$

we obtain the required map. Indeed, $f = p_n \circ \tilde{f}$ and since $\tilde{h}$ is a homotopy joining k and $\tilde{f}$, we have $\tilde{f}^*(\gamma) = k^*(\gamma) = \tilde{g}$.

2.4. *If X is a compactum, the group $H^1(X)$ is torsion-free* (comp. [11, p. 409]).

Proof. Suppose $\tilde{g} \in H^1(X)$, $\tilde{g} \neq 0$ and $n \cdot \tilde{g} = 0$ for some $n > 1$. Let f: $X \to S$ be a constant map, say $f(X) = (s_0)$. Then $f^*(\gamma) = 0$ and by 2.3 there exists a map $\tilde{f}$: $X \to S$ such that $\tilde{f}^*(\gamma) = \tilde{g}$ and $f = p_n \circ \tilde{f}$. Hence $\tilde{f}(X) \subset p_n^{-1}(s_0)$. Since the fiber $p_n^{-1}(s_0)$ consists of n points, we infer in particular that $\tilde{f}(X)$ is a proper subset of S, and therefore $\tilde{f} \simeq 0$. Thus $\tilde{g} = \tilde{f}^*(\gamma) = 0$, contrary to our assumption.

The following proposition is an immediate consequence of 2.3 and the definition of the symbol $d(\cdot)$ (see § 1).

2.5. *If f is a mapping of a compactum X into S, then*

$$d\big(f^*(\gamma)\big) = \sup\{n \in N:\ \bigvee_{\tilde{f}:\, X \to S} f = p_n \circ \tilde{f}\} .$$

2.6. *Let X be a compactum and let $n_1, n_2, \ldots$ be a sequence of natural numbers such that for some $g_0 \in H^1(X)$ we have*

(1) $$n_1 \cdot n_2 \cdot \ldots \cdot n_j / g_0 \quad \textit{for every } j \geqslant 1 .$$

Then there exists a sequence of maps $\{f_i$: $X \to S\}$ such that $f_1^*(\gamma) = g_0$ and $f_i = p_{n_i} \circ f_{i+1}$ for every $i \geqslant 1$.

Proof. Observe that by (1) and 2.4 there exists a sequence $g_1, g_2, \ldots$ of elements of $H^1(X)$ such that

$$g_j = n_{j+1} \cdot g_{j+1} \quad \text{for} \quad j \geqslant 0 .$$

By 2.1 there exists a map f_1: $X \to S$ such that $f_1^*(\gamma) = g_0$. Hence applying 2.3 infinitely many times, we can construct all the other maps f_i with the required properties.

2.7. *Let X be a continuum and let f: $X \to S$ be a map such that $f \not\simeq 0$. Suppose X can be represented as the union, $X = A \cup B$, of its two proper subcontinua A and B such that $f_A = f|A \simeq 0 \simeq f|B = f_B$. Then $f^*(\gamma)$ is a finitely divisible element of $H^1(X)$, i.e., $d\big(f^*(\gamma)\big) < \infty$.*

Proof. Let R be the real line and let $\varphi\colon R\to S$ be defined by $\varphi(t) = e^{2\pi it}$. Since φ is a covering map, it has the homotopy lifting property. Hence there exist two maps $\tilde{f}_A\colon A\to R$ and $\tilde{f}_B\colon B\to R$ such that $f_A = \varphi\circ\tilde{f}_A$ and $f_B = \varphi\circ\tilde{f}_B$. Put

$$l = \max(\operatorname{diam}\tilde{f}_A(A),\ \operatorname{diam}\tilde{f}_B(B)).$$

Choose a natural number m so large that

$$(1)\qquad l/m<\tfrac{1}{2}.$$

Suppose, to the contrary, that $g = f^*(\gamma)$ is infinitely divisible. Then there exist a natural number n and an element $\tilde{g}\in H^1(X)$ such that

$$(2)\qquad g = n\cdot\tilde{g}\quad\text{and}\quad n\geqslant m.$$

By 2.3 there exists a map $\tilde{f}\colon X\to S$ such that

$$(3)\qquad f = p_n\circ\tilde{f}\quad\text{and}\quad \tilde{f}^*(\gamma) = \tilde{g}.$$

Since p_n is a covering map and $\tilde{f}$ is a lifting of f, we infer that $k_A = \tilde{f}/A\simeq 0\simeq\tilde{f}/B = k_B$. By the same argument as at the beginning of the proof, there exist maps $\tilde{k}_A\colon A\to R$ and $\tilde{k}_B\colon B\to R$ such that $k_A = \varphi\circ\tilde{k}_A$ and $k_B = \varphi\circ\tilde{k}_B$. Let $\tilde{p}_n\colon R\to R$ be defined by $\tilde{p}_n(t) = n\cdot t$. Observe that $\varphi\circ\tilde{p}_n = p_n\circ\varphi$; hence we have

$$\varphi\circ(\tilde{p}_n\circ\tilde{k}_A) = p_n\circ\varphi\circ\tilde{k}_A = p_n\circ k_A = f_A.$$

It follows that $\tilde{p}_n\circ\tilde{k}_A$ and $\tilde{f}_A$ are two liftings of f_A. Since A is connected, there exists an integer c such that $\tilde{p}_n\circ\tilde{k}_A(x) = \tilde{f}_A(x)+c$ for every $x\in A$ [11, p. 406]. In particular, the sets $\tilde{p}_n\circ\tilde{k}_A(A)$ and $\tilde{f}_A(A)$ are congruent. In the same way we prove that $\tilde{p}_n\circ\tilde{k}_B(B)$ and $\tilde{f}_B(B)$ are congruent sets. It follows that

$$(4)\qquad \max(\operatorname{diam}\tilde{p}_n\circ\tilde{k}_A(A),\ \operatorname{diam}\tilde{p}_n\circ\tilde{k}_B(B)) = l.$$

Since for every subset M of R we have $\operatorname{diam}\tilde{p}_n(M) = n\cdot\operatorname{diam}M$, by (1), (2) and (4) we obtain

$$(5)\qquad \operatorname{diam}\tilde{k}_A(A)<\tfrac{1}{2}\quad\text{and}\quad \operatorname{diam}\tilde{k}_B(B)<\tfrac{1}{2}.$$

Since A and B intersect, there is a point $z_0\in\tilde{f}(A)\cap\tilde{f}(B)\subset S$. Since $\tilde{f}(A) = \varphi\circ\tilde{k}_A(A)$ and $\tilde{f}(B) = \varphi\circ\tilde{k}_B(B)$, by (5) we infer that $-z_0\notin\tilde{f}(A)\cup\tilde{f}(B)$. Hence we obtain in turn: $\tilde{f}(X) = \tilde{f}(A)\cup\tilde{f}(B)\neq S$, $\tilde{f}\simeq 0$ and finally $\tilde{f}^*(\gamma) = 0$. On the other hand, since $f\not\simeq 0$, we have $g = f^*(\gamma)\neq 0$ by the Bruschlinsky theorem. Hence by (2) we see that $\tilde{g}\neq 0$, and by (3) we have $\tilde{f}^*(\gamma)\neq 0$, contrary to the previous conclusion. This completes the proof of 2.7.

3. Movable compacta. If $X\in$ ANR, then X is homotopically dominated by a polyhedron P. It follows that $H^1(X)$ is a direct summand of $H^1(P)$. Since $H^1(P)$ is a finitely generated torsion free group (see 2.4) and thus is a free group; we conclude that

3.1. *If X is an* ANR-*set, then $H^1(X)$ is a free group. In particular, $H^1(X)$ is finitely divisible.*

An inverse sequence $\underline{X} = \{X_n, f_{nm}\}$ is called an ANR-*sequence* if $X_n\in$ ANR for every $n\geqslant 1$. $\underline{X}$ is called *movable* if for every number $n\geqslant 1$ there exists a number $n'\geqslant n$ such that for every $m\geqslant n$ there exists a mapping $f\colon X_{n'}\to X_m$ such that $f_{nn'}\simeq f_{nm}\circ f$. We say that $\underline{X}$ is associated with a compactum X if $X = \operatorname{invlim}\underline{X}$. The compactum X is called *movable* if there exists a movable ANR-sequence associated with X [13]. It is known that every ANR-sequence associated with a movable compactum is movable [13]. The notion of movability was introduced by K. Borsuk [2].

Now we shall prove the following proposition:

3.2. *If X is a movable compactum, then $H^1(X)$ is finitely divisible.*

Proof. By the classical result of Freudenthal [6] there exists an ANR-sequence $\underline{X} = \{X_n, f_{nm}\}$ associated with X. By the continuity of the Čech cohomology we may assume that $H^1(X) = \operatorname{dirlim}\{H^1(X_n), f^*_{nm}\}$. Since X is movable, the sequence $\underline{X}$ is movable [13]. This implies that the direct sequence of groups $\{H^1(X_n), f_{nm}\}$ is movable. Hence 3.2 follows from 3.1 and 1.4.

By a result of K. Borsuk [2] all plane compacta are movable. Combining this result with 3.2, we obtain

3.3. *If X is a plane compactum, then $H^1(X)$ is finitely divisible.*

4. Main results. In this section we give a characterization of continua which can be mapped onto non-planar circle-like curves. First we establish the following result:

4.1. *Let X be a circle-like continuum and let $\underline{X} = \{X_n, f_{nm}\}$ be an inverse sequence associated with X such that $X_n = S$ for every $n\geqslant 1$ (see* [12] *for the existence of such sequence). The following conditions are equivalent:*

(i) *X can be embedded in the plane (into a 2-manifold),*

(ii) *$H^1(X)\approx 0$ or Z,*

(iii) *$H^1(X)$ is finitely divisible,*

(iv) *X is movable,*

(v) *for every $l\geqslant 1$ there exists an $n\geqslant l$ such that $|\deg f_{nm}|\leqslant 1$ for every $m>n$.*

In the case $H^1(X) = 0$, X is snake-like [15, p. 324] *and it is either indecomposable or the union of two of its proper indecomposable subcontinua* [9] (¹).

Proof. The equivalence (i)⇔(ii) was established by Mc Cord [15, p. 323]. By the continuity of Čech cohomology [5, p. 261] we may identify $H^1(X)$ with the limit of the direct sequence $\underline{G} = \{H^1(X_n), f^*_{nm}\}$. If X is movable, then $\underline{X}$ is movable and this in turn implies that $\underline{G}$ is movable. Thus Proposition 1.3 implies all the other equivalences.

4.2. *If a circle-like continuum is not movable, it is indecomposable* (comp. [8]).

(¹) Added in proof. The second fact has first been obtained by C. E. Burgess in his paper: *Chainable continua and indecomposability*, Pacific J. Math. 9 (1959), pp. 653–660.

Proof. Let X be a non-movable circle-like continuum. By 4.1 there exists an infinitely divisible element $g \neq 0$ of $H^1(X)$. By the Bruschlinsky theorem there is a map $f\colon X \to S$ such that $f^*(\gamma) = g$. Suppose X can be represented in the form $X = A \cup B$, where A and B are proper subcontinua of X. Then A and B are snake-like and therefore $f/A \simeq 0 \simeq f/B$. Since $g \neq 0$, $f \not\simeq 0$. Using 2.7, we see that g is finitely divisible, a contradiction.

Remark. If a continuum X is the limit of an inverse sequence of the form

$$S \xleftarrow{p_{n_1}} S \xleftarrow{p_{n_2}} \ldots,$$

where n_i is a prime number for every $i \geq 1$, then X is called a solenoid. Since $n_i > 1$, we see by 4.1 and 4.2 that solenoids are non-movable [2], indecomposable, circle-like curves not embeddable in the plane. Using the notion of shape (see [1] and [14]), we can easily see that every non-planar circle-like curve has the shape of a solenoid.

We say (following Mazurkiewicz and Knaster, Fund. Math. 21 (1933), pp. 85–90) that a continuum X is λ-connected if every two points of X can be joined by a hereditarily decomposable subcontinuum of X. Let us note that

4.3. *No λ-connected continuum X can be mapped onto an indecomposable continuum Y.*

Proof. Suppose $f(X) = Y$ and let $a, b \in Y$ be two points from distinct composants of Y (see [11, p. 208] for the notion of a composant). Let $f(a') = a$, $f(b') = b$ and let C be a subcontinuum of X joining a' and b'. Then $f(C) = Y$ because Y is irreducible between a and b. According to [11, p. 208] there exists an indecomposable subcontinuum of C. Hence X is not λ-connected, contrary to our assumption.

Combining 4.1, 4.2 and 4.3 we obtain

4.4. *No λ-connected continuum can be mapped onto any non-planar circle-like continuum. In particular, the same conclusion holds for every hereditarily decomposable or arcwise connected continuum.*

4.5. *If X is a compactum such that $H^1(X)$ contains an infinitely divisible element $g \neq 0$, then X can be mapped onto some non-planar circle-like curve.*

Proof. According to 1.2 there exists a sequence of natural numbers $n_1, n_2, \ldots$ such that $n_1 \cdot n_2 \cdot \ldots \cdot n_j / g$ and

(1) $$n_j > 1 \quad \text{for every } j \geq 1.$$

Applying 2.6, we obtain a sequence of maps $\{f_j\colon X \to S\}$ such that

(2) $$g = f_1^*(\gamma),$$

(3) $$f_j = p_{n_j} \circ f_{j+1} \quad \text{for every } j \geq 1.$$

Since $g \neq 0$, the Bruschlinsky theorem and (2) imply $f_1 \not\simeq 0$. Hence by (3) we see that no map f_j is homotopic to a constant map. In particular, we obtain

(4) f_j is a mapping onto S for every $j \geq 1$.

Let Y be the limit of the inverse sequence

$$S \xleftarrow{p_{n_1}} S \xleftarrow{p_{n_2}} \ldots$$

Hence Y is a circle-like continuum. Moreover, because of 4.1 (v) and (1), Y cannot be embedded in the plane. Using (3), we see that the maps f_j induce a map $f\colon X \to Y$. Finally, condition (4) implies that f is onto Y, which completes the proof.

Before we state our next result, let us recall the following facts established in [10].

4.6. *One-dimensional image of a snake-like curve is movable.*

4.7. *Let $Y = \operatorname{invlim}\{Y_n, \varphi_{nm}\}$, where $Y_n = S$ for every $n \geq 1$. Let $\pi_n\colon Y \to Y_n$ denote the projection. Suppose f is a map of a continuum X onto Y such that $\pi_n \circ f \simeq 0$ for every $n \geq 1$. Then Y can be represented as the image of a snake-like curve.*

Now we are ready to prove the following theorem:

4.8. *If a continuum X can be mapped onto a non-planar circle-like curve, then $H^1(X)$ contains an infinitely divisible element $g \neq 0$.*

Proof. Let Y be a non-planar circle-like curve and let f be a map of X onto Y. We may regard Y as the limit of an inverse sequence $Y = \{Y_n, \varphi_{nm}\}$ such that $Y_n = S$ for every $n \geq 1$ [12]. Let $f_n = \pi_n \circ f$, where $\pi_n\colon Y \to Y_n$ is the projection. Hence we have

(1) $$f_n = \varphi_{nm} \circ f_m \quad \text{for every } m \geq n.$$

Using 4.1 (v) we see that there exists an index l_0 such that for every $n \geq l_0$ there exists $m > n$ such that $|\deg \varphi_{nm}| > 1$. Hence there exists an increasing sequence of natural numbers $k_1 < k_2 < \ldots$ such that $|\deg \varphi_{k_i k_{i+1}}| > 1$ for every $i \geq 1$. Choosing if necessary a subsequence of Y we may assume that

(2) $$n_i = |\deg \varphi_{i,i+1}| > 1 \quad \text{for every } i \geq 1.$$

We claim that $f_n \not\simeq 0$ for some $n \geq 1$. Indeed, otherwise by 4.7 we could obtain Y as the image of a snake-like curve, and by 4.6 the continuum Y would be movable because $\dim Y = 1$, contrary to our assumption and 4.1. Without loss of generality we may assume that $f_1 \not\simeq 0$. By the Bruschlinsky theorem $g = f_1^*(\gamma) \neq 0$. To finish the proof we need only to show that for every $k \geq 1$ there exists an $l \geq k$ such that l/g. By condition (2) there exists a $j \geq 1$ such that $l = n_1 \cdot \ldots \cdot n_j \geq k$. Hence by (1) we have in succession

$$\begin{aligned} g = f_1^*(\gamma) &= (\varphi_{1,j+1} \circ f_{j+1})^*(\gamma) = f_{j+1}^* \circ \varphi_{j,j+1}^* \circ \ldots \circ \varphi_{12}^*(\gamma) \\ &= f_{j+1}^*(\deg \varphi_{j,j+1} \cdot \ldots \cdot \deg \varphi_{12} \cdot \gamma) \\ &= l \cdot [\operatorname{sign}(\deg \varphi_{j,j+1} \cdot \ldots \cdot \deg \varphi_{12}) \cdot f_{j+1}^*(\gamma)], \end{aligned}$$

which completes the proof.

Combining 4.3 and 4.6, we obtain the following theorem characterizing continua which cannot be mapped onto non-planar circle-like curves.

4.9 (1). *Continuum X cannot be mapped onto any non-planar circle-like curve iff $H^1(X)$ is finitely divisible.*

This result provides an answer to a problem of G. W. Henderson [7].

5. Final conclusions. By 3.2, 3.3 and 4.8 we obtain

5.1. *No movable continuum can be mapped onto a non-planar circle-like curve. In particular, the same holds for plane continua* [8] *and continua with trivial shape* [10].

Combining 4.4 with 4.5, we obtain

5.2. *If X is a λ-connected continuum, then $H^1(X)$ is finitely divisible.*

Hence the same conclusion holds for hereditarily decomposable and arcwise connected continua.

As a particular case of 4.9 we have by 2.2 the following proposition:

5.3. *If X is a* cr$S$ *continuum, then X cannot be mapped onto a non-planar circle-like continuum.*

The Case–Chamberlin curve [3] is cr$S$. Hence 5.3 implies;

5.4 ([10]). *The Case–Chamberlin curve cannot be mapped onto any non-planar circle-like curve.*

Let X be a connected and simply connected ANR-set. Then the fundamental group $\pi_1(X)$ of X is trivial. Hence, by the lifting theorem [16, p. 76], for every map $f\colon X\to S$ there exists a map $\varphi\colon X\to R$ such that $f(x) = e^{2\pi i\varphi(x)}$. In particular, $f\simeq 0$, i.e., X is cr$S$.

Now we show that

5.5. *If a continuum X is the limit of an inverse sequence $\{X_n, f_{nm}\}$ of connected and simply connected* ANR*-sets, then X is* cr$S$.

Hence by 5.3 it cannot be mapped onto a non-planar circle-like curve [7].

Proof. Let $f\colon X\to S$ and let $\pi_n\colon X\to X_n$ be the projection. According to [12] there exist an index n and a map $g\colon X_n\to S$ such that $f\simeq g\circ\pi_n$. Hence $f\simeq 0$ because $g\simeq 0$, which completes the proof.

Remark. If a continuum X is fundamentally dominated (see [1] for the definition) by a continuum Y, then $H^1(X)$ is isomorphic to a subgroup of $H^1(Y)$ [14]. Hence if $H^1(Y)$ is finitely divisible, then so is $H^1(X)$. Thus if Y cannot be mapped onto a non-planar circle-like curve, the same holds for X.

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(1) After submitting the paper to the editors I received from Prof. J. T. Rogers, Jr. a preprint of his paper. A cohomological characterization of pre-images of non-planar circle-like continua, to appear in Proc. Amer. Math. Soc., in which he obtained an equivalent result.

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