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Equivalence of fundamental and approaching groups of movable pointed compacta

by

J. Brendan Quigley (Dublin)

Abstract. If (X, x) is a pointed compact space then K. Borsuk defined, for each n, the fundamental homotopy group, $\underline{\pi}_n(X, x)$ which captures n-dimensional spherical holes in X, which escape the (standard) homotopy group $\pi_n(X, x)$, especially if X is not a well behaved space. For each n the fundamental homotopy group $\underline{\pi}_n(X, A, x)$ may be defined for a pair of compacta. Then a long fundamental homotopy sequence $\underline{\pi}(X, A, x)$ exists, but is not exact, (compare the case of Čech homology).

The approaching homotopy groups $\underline{\underline{\pi}}_n(X, x)$ share the advantages of the fundamental homotopy groups. But also, and it is proven here, the long approaching homotopy sequence of a pair of compacta is exact.

K. Borsuk defined and studied the movable compacta. It is proven here, that for these movable compacta the fundamental and approaching homotopy groups agree. Thus the long fundamental homotopy sequence of a movable pointed pair of compacta is exact.

Introduction. In the first section of this paper we draw the reader's attention to some results in approaching theory and give proofs of certain of these where there is no easily accessible reference. We define the approaching homotopy groups $\underline{\underline{\pi}}_n$ and the inward homotopy groups I_n, both for single pointed compacta (X, x) and for pointed pairs of compacta (X, A, x) contained in the Hilbert cube. There is the following exact sequence, 0.1, from the nth fundamental homotopy group $\underline{\pi}_n(X, x)$ to the $(n-1)$-st fundamental homotopy group $\underline{\pi}_{n-1}(X, x)$.

$$0 \to \underline{\pi}_n(X, x) \to I_n(X, x) \to I_n(X, x) \to \underline{\underline{\pi}}_{n-1}(X, x) \to \underline{\pi}_{n-1}(X, x) \to 0\,. \tag{0.1}$$

The above sequence is defined and the exactness proven in [6]. The corresponding exact sequence for pairs, (0.2) below, is described in detail

$$0 \to \underline{\pi}_n(X, A, x) \to I_n(X, A, x) \to I_n(X, A, x) \to \underline{\underline{\pi}}_{n-1}(X, A, x) \to \underline{\pi}_{n-1}(X, A, x) \to 0\,. \tag{0.2}$$

Finally the long approaching sequence $\underline{\underline{\pi}}(X, A, x)$, (0.3) below, is described in detail and a full proof of exactness of this sequence is given.

$$\to \underline{\underline{\pi}}_{n+1}(X, A, x) \xrightarrow{\delta} \underline{\underline{\pi}}_n(A, x) \xrightarrow{i} \pi_n(X, x) \xrightarrow{j} \underline{\underline{\pi}}_n(X, A, x) \xrightarrow{\delta} \underline{\underline{\pi}}_{n-1}(A, x) \to . \tag{0.3}$$

In the second section we study the relationship between the approaching homotopy groups $\underline{\underline{\pi}}_n$ and the fundamental homotopy groups $\underline{\pi}_n$ both for the case of a single movable pointed compactum (X, x) and a movable pointed pair of compacta (X, A, x). In fact we show that in both cases the approaching and fundamental groups are isomorphic in all dimensions. Using (0.3) and these isomorphisms we deduce exactness of the long fundamental homotopy sequence $\underline{\pi}(X, A, x)$, (0.4) below, for a *movable* pointed pair of compacta (X, A, x).

$$\underline{\pi}_{n+1}(X, A, x) \xrightarrow{\delta} \underline{\pi}_n(A, x) \xrightarrow{i} \underline{\pi}_n(X, x) \xrightarrow{j} \underline{\pi}_n(X, A, x) \xrightarrow{\delta} \underline{\pi}_{n-1}(A, x) \to . \tag{0.4}$$

Section 1. Basic results in approaching theory.

Remark 1.1. Throughout this paper I denotes the closed unit interval $[0, 1]$ and I^ω the Hilbert Cube. (E^n, S^{n-1}, p_0) denotes the n-dimensional ball with its boundary sphere and base point $(1, 0, 0, \ldots, 0)$. R is the real numbers, J the non-negative integers, $R^+ = \{x|\ x \in R, x \geqslant 0\}$.

Definition 1.2. *The approaching group* $\underline{\pi}_n(X, x)$.

Let $(X, x) \subset (I^\omega, x)$ be a pointed compactum. A continuous mapping ξ from $R^+ \times (S^n, p_0)$ to (I^ω, x) is said to be an *approximative approaching mapping* from (S^n, p_0) to (X, x) if the following condition is satisfied:

(1.3) given $U \in \mathrm{Nhd}(X)$ there is an $s \in R^+$ such that $\xi([s, \infty) \times S^n) \subset U$.

Two approximative approaching mappings ξ, η are said to be *homotopic* (we write $\xi \simeq \eta$) iff there is a continuous mapping Φ from $R^+ \times (S^n, p_0) \times I$ to (I^ω, x) such that the following conditions are satisfied:

$$ {}_0\Phi = \xi , \quad {}_1\Phi = \eta , \tag{1.4}$$

(1.5) given $U \in \mathrm{Nhd}(X)$ there is an $s \in R^+$ such that $\Phi([s, \infty) \times S^n \times I) \subset U$.

Homotopy is an equivalence relation on the set of approximative approaching mappings from (S^n, p_0) to (X, x) and the set of equivalence classes is denoted $\underline{\pi}_n(X, x)$ and the class of ξ is written $[\xi]$. For $n \geqslant 1$, $\underline{\pi}_n(X, x)$ is a group, called the *n-th approaching group of* (X, x), with multiplication induced by the comultiplication of the homotopy cogroup (S^n, p_0).

Definition 1.6. *The approaching group* $\underline{\pi}_n(X, A, x)$.

Let $(X, A, x) \subset (I^\omega, I^\omega, x)$ is said to be an *approximative approaching mapping* from (E^n, S^{n-1}, p_0) to (X, A, x) iff the following condition is satisfied:

(1.7) given $(U, U') \in \mathrm{Nhd}(X, A)$ there is an $s \in R^+$ such that $\xi([s, \infty) \times (E^n, S^{n-1})) \subset (U, U')$.

Two approximative approaching mappings ξ, η are said to be *homotopic* (we write $\xi \simeq \eta$) iff there is a continuous mapping Φ from $R^+ \times (E^n, S^{n-1}, p_0) \times I$ to (I^ω, I^ω, x) such that the following conditions are satisfied:

$$ {}_0\Phi = \xi, \quad {}_1\Phi = \eta , \tag{1.8}$$

(1.9) given $(U, U') \in \mathrm{Nhd}(X, A)$ there is an $s \in R^+$ such that $\Phi([s, \infty) \times (E^n, S^{n-1}) \times I) \subset (U, U')$.

Homotopy is an equivalence relation on the set of approximative approaching mappings from (E^n, S^{n-1}, p_0) to (X, A, x) and the set of equivalence classes is denoted $\underline{\pi}_n(X, A, x)$ which, for $n \geqslant 2$, is a group, called the *n-th approaching group of* (X, A, x), with multiplication induced by the comultiplication of the homotopy cogroup (E^n, S^{n-1}, p_0).

Definition 1.10. *The inward group* $I_n(X, x)$.

In Definition 1.2, replacing R^+ and $[s, \infty)$ everywhere they appear by J and $J \cap [s, \infty)$ respectively, we get the concept of an *inward* mapping from (S^n, p_0) to (X, x) and the equivalence relation of homotopy between inward mappings. The class of ξ is denoted by $[\xi]$ and the set of such classes by $I_n(X, x)$. For $n \geqslant 1$, $I_n(X, x)$ is the group called the *n-th inward group of* (X, x).

Definition 1.11. *The inward group* $I_n(X, A, x)$.

In Definition 1.6, replacing R^+ and $[s, \infty)$ everywhere they appear by J and $J \cap [s, \infty)$ respectively, we get the concept of an *inward* mapping from (E^n, S^{n-1}, p_0) to (X, A, x) and the equivalence relation of homotopy between such inward mappings. The class of ξ is denoted $[\xi]$ and the set of such classes by $I_n(X, A, x)$. For $n \geqslant 2$, $I_n(X, A, x)$ is a group called the *n-th inward group of* (X, A, x).

Remark 1.12. At this point we find it convenient to describe the endomorphism from $I_n(X, x)$ to $I_n(X, x)$ in the exact sequence of (0.1). If $\xi = \{\xi_j\}_{j \geqslant 0}$ is an inward mapping from (S^n, p_0) to (X, x) then $A_n(\xi)$ is that inward mapping from (S^n, p_0) to (X, x) such that $(A_n(\xi))_j = \xi_{j+1}$, for all $j \geqslant 0$. There is an endomorphism A_n; $I_n(X, x) \to I_n(X, x)$ which carries $[\xi] \in I_n(X, x)$ to $[A_n(\xi)] \in I_n(X, x)$ and this is well defined since $\xi \simeq \eta$ implies $A_n(\xi) \simeq A_n(\eta)$. Finally the endomorphism of the sequence (0.1) is written

$$\mathrm{Id}_n - A_n;\ I_n(X, x) \to I_n(X, x) , \quad n \geqslant 1$$

and carries $[\xi] \in I_n(X, x)$ to $[\xi] * [A_n(\xi)]^{-1} \in I_n(X, x)$. In other words $(\mathrm{Id}_n - A_n)([\xi]) = [\eta]$ where $\eta_j = \xi_j * \xi_{j+1}^{-1}$ where inversion of ξ_{j+1} and multiplication of the two continuous mappings ξ_j, ξ_{j+1} from (S^n, p_0) to (I^ω, x) is induced by inversion and comultiplication in the homotopy cogroup (S^n, p_0).

Remark 1.13. If (X, A, x) is a pointed pair of compacta then analogous to (0.1) there is an exact sequence.

$$0 \to \underline{\pi}_n(X, A, x) \xrightarrow{i} I_n(X, A, x) \xrightarrow{\mathrm{Id}_n - A_n} I_n(X, A, x) \xrightarrow{\delta} \underline{\pi}_{n-1}(X, A, x) \xrightarrow{r} \underline{\pi}_{n-1}(X, A, x) \to 0 . \tag{0.2}$$

We now describe the homomorphisms i, $\mathrm{Id}_n - A_n$, δ, r. First we remark that $\underline{\pi}_n(X, A, x)$ can be described as that subset of $I_n(X, A, x)$ such that $[\xi] \in \underline{\pi}_n(X, A, x)$ iff there is an approximative approaching mapping ν from (E^n, S^{n-1}, p_0) such that $\nu|_{J^+ \times (E^n, S^{n-1})} = \xi$. The homomorphism i is then simply an inclusion mapping. $\mathrm{Id}_n - A_n$ is described as in 1.12, the homomorphism r takes $[\xi] \in \underline{\pi}_{n-1}(X, A, x)$

to $\xi|_{J^+\times(E^n,S^{n-1})}\in\underline{\underline{\pi}}_{n-1}(X,A,x)$. Next we describe δ. If $[\xi]\in I_n(X,A,x)$ then $\delta([\xi])=[\eta]\in\underline{\underline{\pi}}_{n-1}(X,A,x)$ is described by the following diagram.

$$\begin{tikzcd}[column sep=small]
{[j,j+1]\times(E^{n-1},S^{n-2},p_0)} \arrow[d, "f_j"'] \arrow[rrr, "\eta|_{[j,j+1]\times(E^{n-1},S^{n-2})}"] & & & {} \arrow[ddd] \\
{[0,1]\times(E^{n-1},S^{n-2},p_0)} \arrow[d, "\sigma"'] & & & \\
{(E^n,S^{n-1},p_0)} \arrow[d, "\xi_j"'] & & & \\
{(I^\omega,I^\omega,x)} & & &
\end{tikzcd}$$

where σ is an identification mapping from $[0,1]\times(E^n,S^{n-1},p_0)$ regarded as the reduced suspension of (E^{n-1},S^{n-2},p_0), and $f_j(s,e)=(s-j,e)$. Exactness of (0.2) now follows by the methods used in [6] to prove exactness of (0.1).

Remark 1.14. As mentioned in the introduction it is proven in [5] and in [7] that the long approaching sequence $\underline{\underline{\pi}}(X,A,x)$ of a pointed pair of compacta (X,A,x) is exact. We now describe $\underline{\underline{\pi}}(X,A,x)$ and prove the exactness as in [5].

We describe the homomorphisms i,j,δ of $\underline{\underline{\pi}}(X,A,x)$, (see (0.3)). If $[\xi]\in\underline{\underline{\pi}}_n(A,x)$, $i([\xi])=[\xi]\in\underline{\underline{\pi}}_n(X,x)$. Let ϱ be a continuous mapping from (E^n,S^{n-1},p_0) to (S^n,p_0,p_0) whose restriction is a homeomorphism from E^n-S^{n-1} to $S^n-\{p_0\}$, if $[\xi]\in\underline{\underline{\pi}}_n(X,x)$ then $j([\xi])=[\xi\circ(\mathrm{Id}_{R^+},\varrho)]\in\underline{\underline{\pi}}_n(X,A,x)$. If $[\xi]\in\underline{\underline{\pi}}_n(X,A,x)$ then $\delta([\xi])=[\xi|_{R^+\times S^{n-1}}]\in\underline{\underline{\pi}}_{n-1}(A,x)$.

We prove $i\circ\delta=0$. Let H be a continuous mapping from $(E^n,p_0)\times I$ to (E^n,p_0) such that ${}_0H$ is the identity mapping and ${}_1H(E^n)=\{p_0\}$. Let $\tilde{H}$ denote the restriction of H to a mapping from $(S^{n-1},p_0)\times I$ to (E^n,p_0). Let $[\xi]\in\underline{\underline{\pi}}_n(X,A,x)$. Then $\xi\circ(\mathrm{Id}_{R^+},\tilde{H})$ is a homotopy between $\xi|_{R^+\times S^{n-1}}$ and the constant mapping c, $c(R^+\times S^{n-1})=\{x\}$, both approximative approaching mappings from (S^{n-1},p_0) to (X,x). Thus $i\circ\delta([\xi])=i([\xi|_{R^+\times S^{n-1}}])=[\xi|_{R^+\times S^{n-1}}]=[c]=0\in\underline{\underline{\pi}}_{n-1}(X,x)$.

We prove $j\circ i=0$. Let $[\xi]\in\underline{\underline{\pi}}_n(A,x)$. Since ξ is an approximative approaching mapping from (S^n,p_0) to (A,x), $\xi\circ(\mathrm{Id}_{R^+},\varrho)\circ(\mathrm{Id}_{R^+},H)$ is a homotopy from $\xi\circ(\mathrm{Id}_{R^+},\varrho)$ to the constant mapping c, $c(R^+\times E^n)=\{x\}$, both approximative approaching mappings from (E^n,S^{n-1},p_0) to (X,A,x). Therefore $j\circ i([\xi])=[\xi\circ(\mathrm{Id}_{R^+},\varrho)]=[c]=0\in\underline{\underline{\pi}}_n(X,A,x)$.

We prove $\delta\circ j=0$. Let $[\xi]\in\underline{\underline{\pi}}_n(X,x)$,

$$\begin{aligned}\delta\circ j([\xi]) &= \delta([\xi\circ(\mathrm{Id}_{R^+},\varrho)])=[(\xi\circ(\mathrm{Id}_{R^+},\varrho))|_{R^+\times S^{n-1}}]\\ &= [\xi|_{R^+\times\{p_0\}}\circ(\mathrm{Id}_{R^+},\varrho|_{S^{n-1}})]=[c\circ(\mathrm{Id}_{R^+},\varrho|_{S^{n-1}})]\\ &= [c]=0\in\underline{\underline{\pi}}_{n-1}(A,x).\end{aligned}$$

We show $\mathrm{Ker}(i)\subset\mathrm{Im}(\delta)$. Let $[\xi]\in\underline{\underline{\pi}}_n(A,x)$, $[\xi]\in\mathrm{Ker}(i)$. Then there is a continuous mapping Φ from $R^+\times(S^n,p_0)\times I$ to (I^ω,x) which is a homotopy from ξ to c, both approximative approaching mappings from (S^n,p_0) to (X,x). Let η be the unique continuous mapping from $R^+\times E^{n+1}$ to I^ω such that $\eta\circ(\mathrm{Id}_{R^+},H)=\Phi$. Then η is an approximative approaching mapping from (E^{n+1},S^n,p_0) to (X,A,x). But $\delta([\eta])=[\eta|_{R^+\times S^n}]=[{}_0\Phi]=[\xi]$.

We show $\mathrm{Ker}(j)\subset\mathrm{Im}(i)$. Let K be a continuous mapping from $E^n\times I$ to $E^n\times I$ such that, $K(e,t)=(e,0)$ for all $(e,t)\in(S^{n-1}\times I)\cup(E^n\times\{0\})$ and $K(E^n\times\{1\})=(S^{n-1}\times I)\cup(E^n\times\{1\})$. Assume $[\xi]\in\underline{\underline{\pi}}_n(X,x)$ and $j([\xi])=0\in\underline{\underline{\pi}}_n(X,A,x)$. Then there is a homotopy Φ from $\xi\circ(\mathrm{Id}_{R^+},\varrho)$ to c, both approximative approaching mappings from (E^n,S^{n-1},p_0) to (X,A,x). Now the continuous mapping $\Phi\circ(\mathrm{Id}_{R^+},K)$ from $R^+\times E^n\times I$ to I^ω carries $R^+\times S^{n-1}\times I$ to $\{x\}$. Thus there is an unique continuous mapping Ψ from $R^+\times S^n\times I$ to I^ω such that $\Psi\circ(\mathrm{Id}_{R^+},\varrho,\mathrm{Id}_I)=\Phi$, Ψ is a homotopy from $\xi={}_0\Psi$ to ${}_1\Psi$, both approximative approaching mappings from (S^n,p_0) to (X,x). But since Φ takes $(r,e,t)\in R^+\times S^{n-1}\times I$ close to $A\subset I^\omega$ for large r (see (1.9)) and since $K(E^n\times\{1\})=(S^n\times I)\cup(E^n\times\{1\})$, ${}_1\Psi$ is an approximative approaching mapping to A. Thus $i([{}_1\Psi])=[{}_1\Psi]=[\xi]$.

We show $\mathrm{Ker}(\delta)\subset\mathrm{Im}(j)$. Let $[\xi]\in\underline{\underline{\pi}}_n(X,A,x)$ and assume $\delta([\xi])=[\xi|_{R^+\times S^{n-1}}]=0\in\underline{\underline{\pi}}_{n-1}(A,x)$. Then there is a homotopy Φ from $\xi|_{R^+\times S^{n-1}}$ to c, both approximative approaching mappings to (A,x). Let χ be that continuous mapping from $R^+\times((E^n\times\{0\})\cup(S^{n-1}\times I))$ to I^ω, whose restriction to $R^+\times E^n\times\{0\}$ is ξ and whose restriction to $S^{n-1}\times I$ is Φ. Let h be a retraction from $E^n\times I$ to $(E^n\times\{0\})\cup(S^{n-1}\times I)$. Then $\Psi\circ(\mathrm{Id}_{R^+},h)$ is a homotopy from ξ to ${}_1(\Psi\circ(\mathrm{Id}_{R^+},h))$ both approximative approaching mappings from (E^n,S^{n-1},p_0) to (X,A,x). But ${}_1(\Psi\circ(\mathrm{Id}_{R^+},h))(R^+\times S^{n-1})=\{x\}$. Thus there is an unique approximative approaching mapping η from (S^n,p_0) to (X,x) such that $\eta\circ(\mathrm{Id}_{R^+},\varrho)={}_1\Psi$. Therefore $j([\eta])=[\eta\circ(\mathrm{Id}_{R^+},\varrho)]=[{}_1\Psi]=[{}_0\Psi]=[\xi]$.

The latter 6 paragraphs prove exactness of the long approaching sequence $\underline{\underline{\pi}}(X,A,x)$ for any pointed pair of compacta (X,A,x).

Section 2. Proof of the main results.

DEFINITION 2.1. *Movable compactum.*

A pointed compactum $(X,x)\subset(I^\omega,x)$ is said to be *movable* iff for each $U\in\mathrm{Nhd}(X)$ there exists $U_0\in\mathrm{Nhd}(X)$ such that, for each $V\in\mathrm{Nhd}(X)$ there is a continuous mapping L from $U_0\times I$ to U satisfying the following conditions:

(2.2) ${}_0L$ is the inclusion mapping $U_0\subset U$,

(2.3) $${}_1L(U_0)\subset V,$$

(2.4) $${}_tL(x)=x\,,\quad\text{for all } t\in I$$

(by $\mathrm{Nhd}(X)$ we mean the set of all W such that $X\subset W\subset I^\omega$ and W is open, and by ${}_tL$ we mean the mapping from U_0 to U carrying u to ${}_tL(u)=L(u,t)$). This definition was given originally by K. Borsuk in [3] 3.1. For more information about movable compacta see [2], [3], [4], [8] and [9].

DEFINITION 2.5. *Movable pairs of compacta.*

A pointed pair of compacta $(X,A,x)\subset(I^\omega,I^\omega,x)$ is said to be *movable* iff for each pair $(U,U')\in\mathrm{Nhd}(X,A)$ there exists a pair $(U_0,U_0')\in\mathrm{Nhd}(X,A)$ such that for each pair $(V, V') \in \mathrm{Nhd}(X, A)$ there is a continuous mapping L from $(U_0, U_0') \times I$ to (U, U') satisfying the following conditions:

(2.6) ${}_0L$ is the inclusion mapping $(U_0, U_0') \subset (U, U')$,

(2.7) $${}_1L(U_0, U_0') \subset (V, V'),$$

(2.8) $${}_tL(x) = x, \quad \text{for all } t \in I$$

(by $\mathrm{Nhd}(X, A)$ we mean the set of pairs (W, W') with $W' \subset W$ and $W \in \mathrm{Nhd}(X)$ and $W' \in \mathrm{Nhd}(A)$).

Remark 2.9. It is apparent that if (X, A, x) is a movable pointed pair of compacta then each of (X, x) and (A, x) is a movable pointed compactum. However a pair of movable pointed compacta need not be a movable pointed pair of compacta. Indeed there is an example in [8] by R. H. Overton of a pair of movable (non-pointed) compacta which is not a movable (non-pointed) pair. Overton also points out how to alter these compacta to get the same result in the pointed case.

Remark 2.10. We will show that when the pointed compactum (X, x) is movable, the endomorphism of (0.1)

$$\mathrm{Id}_n - A_n;\ I_n(X, x) \to I_n(X, x)$$

is an epimorphism for all $n \geq 1$. We now indicate the intuitive reasoning behind the proof.

Let $n \geq 2$ and $[\xi] \in I_n(X, x)$. Define $\eta_0 = c$, the constant mapping $c(S^n) = x \in I^\omega$, and for each $i \geq 1$ define $\eta_i = (\xi_0 * \xi_1 * \xi_2 * \xi_3 * \ldots * \xi_{i-2} * \xi_{i-1})^{-1}$. Compounding these η_i. $i \geq 0$, we get a continuous mapping η; $J^+ \times S^n \to I^\omega$ and it might appear that $\mathrm{Id}_n - A_n([\eta]) = \eta * (A([\eta]))^{-1} = \xi$ and that we have proven that $\mathrm{Id}_n - A_n$ is an epimorphism. This naive argument fails since η may not be an inward n-mapping. Indeed if η were an inward n-mapping, given any $V \in \mathrm{Nhd}(X)$, $\eta_i(S^n) \subset V$ for some large i. Thus $\xi_0(S^n) \subset \eta_i(S^n) \subset V$, for each $V \in \mathrm{Nhd}(X)$. Thus $\xi_0(S^n) \subset X$, which is not in general true.

Since (X, x) is movable we can replace this incorrect argument with a correct argument which is essentially the same but also involves "moving" $\xi_0, \xi_1, \xi_2, \ldots$ etc. arbitrarily close to X. The next lemma is the technical tool which accomplishes this.

LEMMA 2.11. *Let $(X, x) \subset (I^\omega, x)$ be a movable pointed compactum. Corresponding to each integer $p \geq -1$ there is a neighbourhood V_p of X and corresponding to each integer $p \geq 0$ there is a continuous mapping H^p; $R^+ \times V_p \to I^\omega$ satisfying the following conditions*:

(i) $V_{-1} = V_0 = I^\omega$,

(ii) $V_p \supset V_{p+1}$, *for all* $p \geq -1$,

(iii) $\bigcap_{p \geq -1} V_p = X$,

(iv) $H_0^p = \mathrm{inc}$; $V_p \subset V_{p-1}$, *all* $p \geq 0$,

(v) $H^p([j, j+1] \times V_p) \subset V_{p-1+j}$, *for all* $p, j \geq 0$,

(vi) $H^p(\{j\} \times V^p) \subset V_{p+j}$, *for all* $p, j \geq 0$,

(vii) $H^p(R^+ \times \{x\}) = \{x\}$, *for all* $p \geq 0$.

Proof. Let ϱ be the usual metric on I^ω. Let $W_p = \{w;\ w \in I^\omega, \varrho(w, X) \leq 1/p\}$, for all integers $p \geq 1$. We will inductively define $V_p, p \geq -1$, to satisfy (α), (β), (γ), (δ) below

(α) $V_{-1} = V_0 = I^\omega$,

(β) $V_p \supset V_{p+1}, \quad p \geq -1$,

(γ) $V_p \subset W_p, \quad p \geq -1$.

(δ) Given $Q \in \mathrm{Nhd}(X)$ and $p \geq 0$, there is a continuous mapping F; $V_p \times I \to V_{p-1}$ such that ${}_0F$ is the inclusion mapping $V_p \subset V_{p-1}$, ${}_1F(V_p) \subset Q$ and $F(x, t) = x$, $0 \leq t \leq 1$.

Define $V_0 = V_{-1} = I^\omega$. Since I^ω is contractible (δ) is satisfied for $p = 0$. Now assume $V_{-1}, V_0, V_1, \ldots, V_p$ have been defined to satisfy (α), (β), (γ), (δ) above. Since (X, x) is movable there is a $U \in \mathrm{Nhd}(X)$ such that, given $Q \in \mathrm{Nhd}(X)$ there is a continuous mapping K; $U \times I \to V_p$ such that ${}_0K$ is the inclusion mapping $U \subset V_p$, ${}_1K(U) \subset Q$, $K(x, t) = x$, $0 \leq t \leq 1$. Define $V_{p+1} = W_{p+1} \cap U$, (β) and (γ) are automatically satisfied and given $Q \in \mathrm{Nhd}(X)$ a continuous mapping F from $V_{p+1} \times I$ to V_p which shows (δ) to be satisfied in degree $p+1$ is, $F = K|_{V_{p+1} \times I}$.

By (δ) above, for each $p \geq 0$, there is a continuous mapping G^p; $V_p \times I \to V_{p-1}$ such that ${}_0G^p$ is the inclusion mapping $V_p \subset V_{p-1}$, ${}_1G^p(V_p) \subset V_{p+1}$ and $G^p(x, t) = x$, $0 \leq t \leq 1$.

We now define, for all $p \geq 0$, H^p; $R^+ \times V_p \to I^\omega$. Let $j \in J^+$, $p \in J^+$, for all $(s, y) \in [j, j+1] \times V_p$, define

$$H^p(s, y) = G^{p+j}({}_1G^{p+j-1} \circ {}_1G^{p+j-2} \circ \ldots \circ {}_1G^{p+1} \circ {}_1G^p(y), s-j).$$

This defines H^p; $R^+ \times V_p \to I^\omega$, for all $p \geq 0$.

Now (i) is satisfied by (α), (ii) by (β) and (iii) by (γ) and the definition of W_p, $p \geq 1$. By the definition of H^p, ${}_0H^p = {}_0G^p = \mathrm{inc}$; $V_p \subset V_{p-1}$, for all $p \geq 0$, thus (iv) is satisfied. Again by the definition of H^p, $H^p([j, j+1] \times V_p) \subset \mathrm{Image}\, G^{p+j} \subset V_{p+j-1}$, thus (v) is satisfied. Again by the definition of H^p, $H^p(\{j\} \times V_p) \subset \mathrm{Image}\, {}_1G^{p+j-1} \subset V_{p+j}$ and so (vi) is satisfied. (vii) is also satisfied since $G^p(x, t) = x$, all $p \geq 0$, $0 \leq t \leq 1$. This completes the proof of the lemma. Q.E.D.

Remark 2.12. We can also show that if (X, A, x) is a movable pointed pair of compacta then the endomorphism of (0.2)

$$\mathrm{Id}_n - A_n;\ I_n(X, A, x) \to I_n(X, A, x)$$

is an epimorphism for all $n \geq 2$. The next lemma is the technical tool which accomplishes this.

LEMMA 2.13. *Let $(X, A, x)\subset(I^\omega, I^\omega, x)$ be a movable pointed pair of compacta. Corresponding to each integer $p\geqslant -1$ there is a neighbourhood pair (V_p, V'_p) of (X, A) and corresponding to each integer $p\geqslant 0$ there is a continuous mapping*

$$H^p;\ R^+\times(V_p, V'_p)\to(I^\omega, I^\omega)$$

satisfying the following conditions:

(i) $(V_{-1}, V'_{-1}) = (V_0, V'_0) = (I^\omega, I^\omega)$,

(ii) $(V_p, V'_p)\supset(V_{p+1}, V'_{p+1})$, *for all* $p\geqslant -1$,

(iii) $\bigcap\limits_{p\geqslant -1}(V_p, V'_p) = (X, A)$,

(iv) $H^p_0 = \mathrm{inc}$; $(V_p, V'_p)\subset(V_{p-1}, V'_{p-1})$, *for all* $p\geqslant 0$,

(v) $H^p([j, j+1]\times(V_p, V'_p))\subset(V_{p-1+j}, V'_{p-1+j})$, *for all* $p, j\geqslant 0$,

(vi) $H^p(\{j\}+(V_p, V'_p))\subset(V_{p+j}, V'_{p+j})$, *for all* $p, j\geqslant 0$,

(vii) $H^p(R^+\times\{x\}) = \{x\}$, *for all* $p, j\geqslant 0$.

Proof. Exactly as in Lemma 2.11 above but replacing single compacta and neighbourhoods by pairs of compacta and neighbourhood pairs throughout.

LEMMA 2.14. *Let (X, x) be a movable pointed compactum. Then*

$$\mathrm{Id}_n - A_n;\ I_n(X, x)\to I_n(X, x)$$

is surjective for all $n\geqslant 1$.

Proof. Let $\{V_p\}_{p\geqslant -1}$ and $\{H^p\}_{p\geqslant 0}$ be as in Lemma 2.11. If $\{\alpha_r;\ (S^n, p_0)\to(I^\omega, x)\}_{r=0}^{i}$ are continuous, where $n\geqslant 1$ and $i\geqslant 0$ are integers, then we denote

$$\alpha_0 * \alpha_1 * \alpha_2 * \ldots * \alpha_{i-1} * \alpha_i$$

by $\mathop{*}\limits_{r=0}^{i}\alpha_r$. Let $[\xi]\in I_n(X, x)$ where $n\geqslant 1$. Since ξ is an inward n-mapping of (X, x) there is an increasing function $k;\ J^+\to J^+$ such that $k(i)$ tends to infinity as i tends to infinity and such that $\xi_i(S^n)\subset V_{k(i)}$, for all i.

Let η_0 be the constant mapping $\eta_0(S^n) = \{x\}\subset I$ and for each $i\geqslant 1$ define a continuous mapping η_i from (S^n, p_0) to (I^ω, x) by $\eta_i = \big(\mathop{*}\limits_{r=0}^{i-1} H^{k(r)}_{k(i-1)-k(r)}\circ\xi_r\big)^{-1}$. For each $i\geqslant 1$ we have

(2.15)
$$\begin{aligned}
\eta_i(S^n) &\subset \bigcup_{r=0}^{i-1}\big(H^{k(r)}_{k(i-1)-k(r)}\circ\xi_r\big)(S^n), && \text{by definition of } \eta_i\\
&\subset \bigcup_{r=0}^{i-1} H^{k(r)}_{k(i-1)-k(r)}(V_{k(r)}), && \text{by definition of } k\\
&\subset V_{k(r)+k(i-1)-k(r)}, && \text{by (vi) Lemma 2.11}\\
&= V_{k(i-1)}.
\end{aligned}$$

Compounding the η_i, $i\geqslant 0$, we get a continuous mapping $\eta;\ J^+\times S^n\to I^\omega$. Since $k(i)$ tends to infinity with i, by (2.15) η is an inward n-mapping of (X, x).

We will show that $\eta * \big(A(\eta)\big)^{-1}$ is homotopic to ξ. Passing over this point for the present, we have $(\mathrm{Id}_n - A_n)([\eta]) = [\eta * (A(\eta))^{-1}] = [\xi]$, for all $n\geqslant 1$. Thus $\mathrm{Id}_n - A_n$ is surjective for all $n\geqslant 1$ and the theorem is proven.

We introduce some technicalities useful in showing that $\eta * \big(A(\eta)\big)^{-1}\simeq\xi$ (inwardly). Let $i_0\geqslant 1$ be an integer such that $k(i_0)>1$. Let $i>i_0$. Define,

$$\theta_r = H^{k(r)}_{k(i-1)-k(r)}, \qquad \text{for all } 0\leqslant r\leqslant i-1,$$

$$\varphi_r = H^{k(r)}_{k(i)-k(r)}, \qquad \text{for all } 0\leqslant r\leqslant i.$$

By Lemma 2.11 part (v),

$$H^{k(r)}([k(i-1)-k(r), k(i)-k(r)]\times V_{k(r)})\subset V_{k(r)+(k(i-1)-k(r))-1} = V_{k(i-1)-1},$$
$$\text{for all } 0\leqslant r\leqslant i-1.$$

Thus θ_r is homotopic to φ_r, rel$x$, in $V_{k(i-1)-1}$, for all $0\leqslant r\leqslant i-1$. Thus $\theta_r\circ\xi_r$ is homotopic to $\varphi_r\circ\xi_r$, relp_0, in $V_{k(i-1)-1}$, for all $0\leqslant r\leqslant i-1$. Thus

(2.16) $(\theta_r\circ\xi_r)^{-1} * (\varphi_r\circ\xi_r)$ is homotopic to the constant mapping, rel to p_0, in $V_{k(i-1)-1}$.

Now

$$\begin{aligned}
\big(\eta * (A(\eta))^{-1}\big)_i &= \eta_i * \eta_{i+1}^{-1}\\
&= \big(\mathop{*}_{r=0}^{i-1} H^{k(r)}_{k(i-1)-k(r)}\circ\xi_r\big)^{-1} * \big(\mathop{*}_{r=0}^{i} H^{k(r)}_{k(i)-k(r)}\circ\xi_r\big)\\
&= \big(\mathop{*}_{r=0}^{i-1}\theta_r\circ\xi_r\big)^{-1} * \big(\mathop{*}_{r=0}^{i}\varphi_r\circ\xi_r\big)\\
&= \big(\mathop{*}_{r=0}^{i-1}(\theta_{i-1-r}\circ\xi_{i-1-r})^{-1}\big) * \big(\mathop{*}_{r=0}^{i}\varphi_r\circ\xi_r\big) \quad \text{in } V_{k(i-1)-1}.
\end{aligned}$$

Applying (2.16) above to $(\theta_r\circ\xi_r)^{-1} * (\varphi_r\circ\xi_r)$ for each r such that $0\leqslant r\leqslant i-1$ we see that $\eta_i * \eta_{i+1}^{-1}$ is homotopic to $\varphi_i\circ\xi_i$, rel to p_0, in $V_{k(i-1)-1}$. Therefore, for each $i>i_0$ there is a continuous mapping $\Psi_i;\ S^n\times I\to V_{k(i-1)-1}$ such that

$${}_0\Psi_i = \eta_i * \eta_{i+1}^{-1},$$

$${}_1\Psi_i = \varphi_i\circ\xi_i = H^{k(i)}_{k(i)-k(i)}\circ\xi_i = H^{k(i)}_0\circ\xi_i = \xi_i,$$

by Lemma 2.11 part (iv), and such that $\Psi_i(p_0, t) = x$, for all $t\in I$. When $i\leqslant i_0$ let Ψ_i be any homotopy from $\eta_i * \eta_{i+1}^{-1}$ to ξ_i, relp_0 (such homotopies exist since I^ω is contractible). Compounding these Ψ_i, $i\geqslant 0$, we obtain a continuous mapping $\Psi;\ J^+\times S^n\times I\to I^\omega$ with ${}_0\Psi = \eta * \big(A(\eta)\big)^{-1}$, ${}_1\Psi = \xi$, $\Psi(J^+\times p_0\times I) = \{x\}$. Given any $V \in \mathrm{Nhd}(X)$, since $k(i)$ tends to infinity with i, by Lemma 2.11 part (iii), we may choose $N \in J^+$, such that $N > i_0$ and $V_{k(N-1)-1} \subset V$. If $i > N > i_0$,

$$\Psi_i(S^n \times I) \subset V_{k(i-1)-1} \subset V_{k(N-1)-1} \subset V.$$

Thus $\Psi;\ \xi \simeq \eta * (A(\eta))^{-1}$ (inwardly). Q.E.D.

LEMMA 2.17. *Let (X, A, x) be a movable pointed pair of compacta. Then*

$$\mathrm{Id}_n - A_n;\ I_n(X, A, x) \to I_n(X, A, x)$$

is surjective for all $n \geqslant 2$.

Proof. As in Lemma 2.14 above but using Lemma 2.13 in place of Lemma 2.11.

THEOREM 2.18. *If (X, x) is a movable pointed compactum then*

$$\underline{\underline{\pi}}_n(X, x) \cong \underline{\pi}_n(X, x), \quad \text{for all } n \geqslant 0 .$$

If (X, A, x) is a movable pointed pair of compacta then

$$\underline{\underline{\pi}}_n(X, A, x) \cong \underline{\pi}_n(X, A, x), \quad \text{for all } n \geqslant 0 .$$

Proof. For all $n \geqslant 0$ apply Lemma 2.14 to the exact sequence (0.1). For all $n \geqslant 0$ apply Lemma 2.17 to the exact sequence (0.2).

COROLLARY 2.19. *If (X, A, x) is a movable pointed pair of compacta the long fundamental homotopy sequence $\underline{\pi}(X, A, x)$ of* (0.4) *is exact.*

Proof. Apply Theorem 2.18 to the long approaching sequence (0.3).

Remark 2.20. Let (X, x) be a pointed compactum where X is an ANR. By [3] 2.4 (X, x) is movable. By [1], 14.6, $\underline{\pi}_n(X, x) \equiv \pi_n(X, x)$. By Theorem 2.18 above and these remarks $\underline{\underline{\pi}}_n(X, x) \equiv \pi_n(X, x)$. This result is proven in a more general form by a direct method in [5], Section 2, and also in [7].

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