On stability and products *

by

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Abstract. We investigate which properties connected with stability are preserved by product-type operation on theories.

It is known ([2], [3], [20]) that if $T_1 = \mathrm{Th}(\mathfrak{A})$ and $T_2 = \mathrm{Th}(\mathfrak{B})$, then $\mathrm{Th}(\mathfrak{A}\times\mathfrak{B})$ is determined by T_1 and T_2; we denote this theory by $T_1 \times T_2$. Similarly, if $\mathscr{F}$ is a filter ever I and $\mathscr{G}$ is a filter ever $I\times I$, then T_1 and $\mathrm{Th}(2^I_{\mathscr{F}}|\mathscr{G})$ determines $\mathrm{Th}(\mathfrak{A}^I_{\mathscr{F}}|\mathscr{G})$. We denote the last theory by $T_{1\mathscr{F}}^{\,I}|\mathscr{G}$.

In the paper we characterize those product operations of direct and reduced products which preserve stability of theories. We also give a similar characterization for Keisler's finite cover property and other notions related to stability. The results of the paper were announced in [19].

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§ 0. **Preliminaries.** We use the standard notation. T (with indices if necessary) always denotes a complete, countable first order theory in a language L with equality. We assume that every theory under consideration has infinite models only. 2 denotes the two-element Boolean algebra. $\mathscr{P}(A)$ is the power set of A. $A^0 = A$ and A^1 is the complement of A. The letters k, l, n, N are reserved for natural numbers and $\varkappa, \lambda$ for infinite cardinals. $\bar{x}, \bar{y}, \bar{a}, \bar{b}$ denote finite sequences, $\mathrm{lh}(\bar{x})$ is the length of $\bar{x}$. ${}^{\omega>}A$ is the set of all finite sequences of elements of A. If $\mathfrak{A}$ is a model of T, then $S(\mathfrak{A})$ is the set of all 1-types over $\mathfrak{A}$. Let φ be a formula of L with exactly $x_0, \ldots, x_n$ free variables and let $a_1, \ldots, a_n \in A$. Then

$$\varphi^{\mathfrak{A}}[a_1, \ldots, a_n] = \{a \in A|\ \mathfrak{A} \models \varphi[a, a_1, \ldots, a_n]\}.$$

THEOREM 0.1 ([14]). *Let* $\mathrm{St}(T) = \{\varkappa|\ T \textit{ is } \varkappa\textit{-stable}\}$. *Then* $\mathrm{St}(T)$ *is equal to one of the following classes*:

(i) 0 (*in this case we refer to* T *as unstable*),

(ii) $\{\varkappa|\ \varkappa^{\omega} = \varkappa\}$,

(iii) $\{\varkappa|\ \varkappa \geqslant 2^{\omega}\}$,

(iv) *the class of all infinite cardinals.*

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* Most of the problems solved in this paper were stated by B. Węglorz.

Theories which satisfy (ii) or (iii) or (iv) are called *stable*, and those which satisfy (iii) or (iv) — *superstable*.

THEOREM 0.2 ([13], [14]). *If T is unstable and $\varkappa>\omega$, then T has a saturated model of power $\varkappa$ iff $\varkappa = \sum_{\lambda<\varkappa}\varkappa^{\lambda}$.*

THEOREM 0.3 ([9], [14], [18]). *If T is stable and $\varkappa>\omega$, then T has a saturated model of power $\varkappa$ iff T is $\varkappa$-stable.*

We shall use Galvin's autonomous systems [3]. Let φ be a formula with exactly $x_0, \ldots, x_n$ free variables, $\mathfrak{A} \in \mathrm{Mod}(T_1)$, $\mathfrak{B} \in \mathrm{Mod}(T_2)$, $q_1 \in S(\mathfrak{A})$, $q_2 \in S(\mathfrak{B})$, $a_1, \ldots, a_n \in A$, $b_1, \ldots, b_n \in B$. Let $\boldsymbol{S}_\varphi = \langle S_\varphi, \pi^\varphi\rangle$ be an autonomous system for φ. Then we define

$$q_1(\varphi, \langle a_1, \ldots, a_n\rangle) = \psi_1(x_0, \ldots, x_n)\,,$$
$$q_2(\varphi, \langle b_1, \ldots, b_n\rangle) = \psi_2(x_0, \ldots, x_n)$$

where $\psi_1, \psi_2 \in S_\varphi$ and $\psi_1(x_0, a_1, \ldots, a_n) \in q_1$ and $\psi_2(x_0, b_1, \ldots, b_n) \in q_2$.

DEFINITION 0.4.

$$\pi(q_1, q_2)(\varphi, \langle\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle\rangle) = \pi^\varphi(q_1(\varphi, \langle a_1, \ldots, a_n\rangle), q_2(\varphi, \langle b_1, \ldots, b_n\rangle))\,.$$

$q_1 \times q_2$ is the type over $\mathfrak{A}\times\mathfrak{B}$ which contains all $\pi(q_1, q_2)(\varphi, \langle\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle\rangle)$ for all formulas φ of L and all $\langle\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle\rangle \in {}^{|\mathrm{Fr}(\varphi)|-1}A\times B$.

We shall need the following lemma, which is a slight extension of Lemma 1.2 of [17].

LEMMA 0.5. *If $\mathfrak{A} \in \mathrm{Mod}(T_1)$, $\mathfrak{B} \in \mathrm{Mod}(T_2)$ and $p \in S(\mathfrak{A}\times\mathfrak{B})$, then there are $q_1 \in S(\mathfrak{A})$ and $q_2 \in S(\mathfrak{B})$ such that $p = q_1 \times q_2$.*

§ **1. Stability.** In this section we prove that finite products preserve stability, superstability, and ω-stability of theories. We also give an example which shows that unstability is not preserved under finite powers.

THEOREM 1.1. *Suppose that T_1 and T_2 are stable theories. Then $T_1\times T_2$ is also stable.*

Proof. Since $(\beth_{\omega_1})^\omega = \beth_{\omega_1}$, by Theorem 0.1 T_1 and T_2 are $\beth_{\omega_1}$-stable. Hence, by Theorem 0.2, T_1 and T_2 have saturated models of power $\beth_{\omega_1}$, say $\mathfrak{A}$ and $\mathfrak{B}$. By a theorem of Waszkiewicz and Węglorz [17] 1.5, $\mathfrak{A}\times\mathfrak{B}$ is a saturated model of $T_1\times T_2$ of power $\beth_{\omega_1}$. Since $\sum_{\lambda<\omega_1}(\beth_{\omega_1})^\lambda > \beth_{\omega_1}$, by Theorem 0.2 $T_1\times T_2$ is stable.

THEOREM 1.2. *Suppose that T_1 and T_2 are superstable theories. Then $T_1\times T_2$ is also superstable.*

Proof. Let $2^\omega = \omega_\alpha$. Since T_1 and T_2 are superstable, they are $\omega_{\alpha+\omega}$-stable. So, by Theorem 0.2., T_1 and T_2 have saturated models of power $\omega_{\alpha+\omega}$, say $\mathfrak{A}$ and $\mathfrak{B}$. As before, $\mathfrak{A}\times\mathfrak{B}$ is a saturated model of $T_1\times T_2$ of power $\omega_{\alpha+\omega}$. By Theorem 1.1 $T_1\times T_2$ is stable and has a saturated model of power $\omega_{\alpha+\omega}$. Hence, by Theorem 0.3, $T_1\times T_2$ is $\omega_{\alpha+\omega}$-stable. Consequently, by Theorem 0.1, $T_1\times T_2$ is superstable.

THEOREM 1.3. *Suppose that T_1 and T_2 are ω-stable. Then $T_1\times T_2$ is also ω-stable* (¹).

Proof. By [9] T_1 and T_2 have countable saturated models, say $\mathfrak{A}$ and $\mathfrak{B}$. By Lemma 0.5, $|S(\mathfrak{A}\times\mathfrak{B})| \leqslant |S(\mathfrak{A})|\cdot|S(\mathfrak{B})| = \omega$. Let $\mathfrak{C}$ be a countable model of $T_1\times T_2$. $\mathfrak{A}\times\mathfrak{B}$ is a saturated model of $T_1\times T_2$ ([17] 1.4), and hence a universal one ([10]). So we may assume that $\mathfrak{C} \prec \mathfrak{A}\times\mathfrak{B}$. But then $|S(\mathfrak{C})| \leqslant |S(\mathfrak{A}\times\mathfrak{B})|$. So $T_1\times T_2$ is ω-stable.

Remark 1.4. Theorems 1.1 and 1.2 can be proved in the same way as Theorem 1.3, but our proofs show the consequences of the existence of saturated models for stability.

EXAMPLE 1.5. Now we shall construct a structure $\mathfrak{B}$ such that $\mathrm{Th}(\mathfrak{B})$ is unstable but $\mathrm{Th}(\mathfrak{B}\times\mathfrak{B})$ is stable. Let $\mathfrak{B}$ be the following structure:

$$\mathfrak{B} = \langle \omega \cup (\omega\times\omega\times\omega), W, C, D, R\rangle$$

where

W is a unary relation, and $W(a)$ iff $a \in \omega$,
C is a unary relation, and $C(a)$ iff $a \notin \omega$,
D is a ternary relation, and $D(a, b, c)$ iff

$$W(a)\ \&\ W(b)\ \&\ C(c)\ \&\ \exists k \in \omega\ c = \langle a, b, k\rangle,$$

R is a ternary relation, and $R(a, b, c)$ iff

$$D(a, b, c)\ \&\begin{cases} a<b \to \exists k \in \omega\ c = \langle a, b, k\rangle\ \&\ k \text{ even} \\ a\geqslant b \to \exists k \in \omega\ c = \langle a, b, k\rangle\ \&\ k \neq 0\,. \end{cases}$$

Let $\varphi(x_0, x_1)$ be the following formula:

$$W(x_0)\wedge W(x_1)\wedge \exists y_1, y_2[\neg(y_1 = y_2)\wedge D(x_0, x_1, y_1)\wedge D(x_0, x_1, y_2)\wedge \wedge\neg R(x_0, x_1, y_1)\wedge\neg R(x_0, x_1, y_2)]\,.$$

By the construction of the structure $\mathfrak{B}$ we have: $\mathfrak{B} \models \varphi[a, b]$ iff $a, b \in \omega$ and $a<b$. So $\mathfrak{B}$ has an infinite linear ordering, and hence by [14] 2.13 $\mathrm{Th}(\mathfrak{B})$ is unstable.

On the other hand, it is not too difficult to show that:
(i) $\mathrm{Th}(\mathfrak{B}\times\mathfrak{B})$ has the elimination of quantifiers,
(ii) $\mathrm{Th}(\mathfrak{B}\times\mathfrak{B})$ is ω-categorical,
(iii) $S(\mathfrak{B}\times\mathfrak{B})$ is countable,
(iv) $\mathrm{Th}(\mathfrak{B}\times\mathfrak{B})$ is ω-stable.

Indeed, (i) follows from the Quantifier Elimination Theorem (see e.g. [15]), (ii) follows by finding an isomorphism between any two countable models of $\mathrm{Th}(\mathfrak{B}\times\mathfrak{B})$; (iii) follows by listing all 1-types over $\mathfrak{B}\times\mathfrak{B}$. From (ii) and (iii) we obtain (iv).

(¹) After submitting this paper, the author heard through B. Węglorz that the same result had been announced in [7] and proved in a slightly stronger version in [8].

§ 2. Some lemmas. Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures in the same language L with predicates $\{R_i\}_{i\in I}$ and without functions and constants. We assume that $A\cap B=0$. We define

$$\mathfrak{A}\oplus\mathfrak{B} = \langle A\cup B, A, B, R_i^{\mathfrak{A}}\cup R_i^{\mathfrak{B}}\rangle_{i\in I}.$$

We shall use two auxiliary languages $L^*(\mathfrak{A})$ and $L^*(\mathfrak{B})$. The set of formulas of $L^*(\mathfrak{A})$ contains atomic formulas of the form $\bigwedge_{i<k_n} A(x_i)\wedge R_s(x_0,\dots,x_{k_n})$ and is closed under the usual connectives and quantifiers restricted to A. $L^*(\mathfrak{B})$ is defined in a similar way. $L(\mathfrak{A})$ denotes the usual language of $\mathfrak{A}$. If $\varphi\in L^*(\mathfrak{A})$, then $\bar\varphi$ denotes the natural translation of φ into $L(\mathfrak{A})$. For the sake of simplicity we shall write $\mathfrak{A}\models\varphi$ instead of $\mathfrak{A}\models\bar\varphi$.

Let us notice that:

2.1. *If* $\varphi\in L^*(\mathfrak{A})$ *and* $a_0,\dots,a_n\in A$, *then*

$$\mathfrak{A}\oplus\mathfrak{B}\models\varphi[a_0,\dots,a_n] \quad \textit{iff} \quad \mathfrak{A}\models\varphi[a_0,\dots,a_n].$$

LEMMA 2.2. *Let* $\varphi(\bar x)\in L(\mathfrak{A}\oplus\mathfrak{B})$. *Then there are formulas* $\varphi_1,\dots,\varphi_k\in L^*(\mathfrak{A})$, $\psi_1,\dots,\psi_l\in L^*(\mathfrak{B})$, *and a Boolean polynomial* P *such that*:

$$\mathfrak{A}\oplus\mathfrak{B}\models\varphi(\bar x)\leftrightarrow P(\varphi_1,\dots,\varphi_k,\psi_1,\dots,\psi_l)(\bar x).$$

Proof. We proceed by induction on the complexity of φ. The case of atomic formulas and logical connectives is easy.

Suppose that $\varphi(\bar x)=\exists y\psi(\bar x,y)$. By the induction hypothesis we may assume that

$$\psi(\bar x,y)=\bigvee_{i<k}\psi_{i0}(\bar x,y)\wedge\psi_{i1}(\bar x,y) \tag{1}$$

where, for $i<k$, $\psi_{i0}(\bar x,y)\in L^*(\mathfrak{A})$ and $\psi_{i1}(\bar x,y)\in L^*(\mathfrak{B})$.

Let $\tilde\psi_{i1}(\bar x)$ be the formula obtained from $\psi_{i1}(\bar x,y)$ by replacing each atomic subformula of it which contains y by a false sentence of $L^*(\mathfrak{B})$. We define the formulas $\tilde\psi_{i0}(\bar x)$ similarly. For $i<k$ we have:

$$\mathfrak{A}\oplus\mathfrak{B}\models[A(y)\wedge\tilde\psi_{i1}(\bar x)]\leftrightarrow[A(y)\wedge\psi_{i1}(\bar x,y)]. \tag{2}$$

$$\mathfrak{A}\oplus\mathfrak{B}\models[B(y)\wedge\tilde\psi_{i0}(\bar x)]\leftrightarrow[B(y)\wedge\psi_{i0}(\bar x,y)]. \tag{3}$$

Note that $\tilde\psi_{i0}(\bar x)$ and $\tilde\psi_{i1}(\bar x)$ do not contain y. So using (1), (2), (3) by an easy calculation we obtain:

$$\mathfrak{A}\oplus\mathfrak{B}\models\varphi(\bar x)\leftrightarrow\bigvee_{i<k}\{[\exists y(A(y)\wedge\psi_{i0}(\bar x,y))\wedge\tilde\psi_{i1}(\bar x)]\vee$$
$$\vee[\exists y(B(y)\wedge\psi_{i1}(\bar x,y))\wedge\tilde\psi_{i0}(\bar x)]\}.$$

This formula has the required form, and thus the proof is complete.

COROLLARY 2.3 ([2]). (i) *If* $\mathfrak{A}\equiv\mathfrak{A}_0$ *and* $\mathfrak{B}\equiv\mathfrak{B}_0$, *then* $\mathfrak{A}\oplus\mathfrak{B}\equiv\mathfrak{A}_0\oplus\mathfrak{B}_0$.
(ii) *If* $\mathfrak{A}\prec\mathfrak{A}_0$ *and* $\mathfrak{B}\prec\mathfrak{B}_0$, *then* $\mathfrak{A}\oplus\mathfrak{B}\prec\mathfrak{A}_0\oplus\mathfrak{B}_0$.

Proof. Let θ denote $\forall x(A(x)\vee B(x))\wedge\neg\exists x(A(x)\wedge B(x))$. The corollary follows from the fact that Lemma 2.2 can be reformulated by replacing everywhere "$\mathfrak{A}\oplus\mathfrak{B}\models$" by "$\theta\vdash$".

LEMMA 2.4. *If* $\varphi\in L^*(\mathfrak{B})$ *and* $\bar c\in{}^{\omega>}A\cup B$, *then either* $\varphi^{\mathfrak{A}\oplus\mathfrak{B}}[\bar c]\cap A=0$, *or* $\varphi^{\mathfrak{A}\oplus\mathfrak{B}}[\bar c]\cap A=A$.

Proof. We proceed by induction on the complexity of φ. We prove it only for the case $\varphi=\exists y(B(y)\wedge\psi(z,y,\bar x))$. Fix $\bar c\in{}^{\omega>}A\cup B$. Suppose

$$\mathfrak{A}\oplus\mathfrak{B}\models\exists y(B(y)\wedge\psi[a,y,\bar c]) \quad \text{for some } a\in A.$$

Hence $\mathfrak{A}\oplus\mathfrak{B}\models\psi[a,b,\bar c]$ for some $a\in A$ and $b\in B$. Now by the induction hypothesis we have $\psi^{\mathfrak{A}\oplus\mathfrak{B}}[b,\bar c]\cap A=A$. But

$$\psi^{\mathfrak{A}\oplus\mathfrak{B}}[b,\bar c]\subseteq\exists y(B(y)\wedge\psi[y,\bar c])^{\mathfrak{A}\oplus\mathfrak{B}},$$

and so $\varphi^{\mathfrak{A}\oplus\mathfrak{B}}[\bar c]\cap A=A$.

LEMMA 2.5. *For every formula* $\varphi(x,\bar y)\in L(\mathfrak{A}\oplus\mathfrak{B})$ *there are formulas* $\varphi_A(x,\bar y,\bar z)\in L^*(\mathfrak{A})$ *and* $\varphi_B(x,\bar y,\bar z)\in L^*(\mathfrak{B})$ *and functions* $f_{A,\varphi}, f_{B,\varphi}$ *such that for every sequence* $\bar c\in{}^{\mathrm{lh}(\bar y)}(A\cup B)$ *we have*:

(i) $\mathrm{lh}(f_{A,\varphi}(\bar c))=\mathrm{lh}(f_{B,\varphi}(\bar c))=\mathrm{lh}(\bar y)+\mathrm{lh}(\bar z)$,

(ii) $f_{A,\varphi}(\bar c)\in{}^{\omega>}A$ *and* $f_{B,\varphi}(\bar c)\in{}^{\omega>}B$,

(iii) $\varphi_A^{\mathfrak{A}}[f_{A,\varphi}(\bar c)]=\varphi^{\mathfrak{A}\oplus\mathfrak{B}}[\bar c]\cap A$, $\varphi_B^{\mathfrak{B}}[f_{B,\varphi}(\bar c)]=\varphi^{\mathfrak{A}\oplus\mathfrak{B}}[\bar c]\cap B$.

Proof. We shall assume that there are two distinct elements a_0, a_1 of A and two distinct elements b_0, b_1 of B. It is obvious how to generalize the proof to the other case. We define φ_A and $f_{A,\varphi}$. The definition of φ_B and $f_{B,\varphi}$ is similar.

Let h be an enumeration of all subsets of $\mathrm{lh}(\bar y)$, i.e., $h\colon(2^{\mathrm{lh}(\bar y)}+1)\setminus\{0\}\to\mathscr{P}(\mathrm{lh}(\bar y))$. By Lemma 2.2 there are $\varphi_1,\dots,\varphi_k\in L^*(\mathfrak{A})$, $\psi_1,\dots,\psi_l\in L^*(\mathfrak{B})$ and a Boolean polynomial P such that

$$\mathfrak{A}\oplus\mathfrak{B}\models\varphi(x,\bar y)\leftrightarrow P(\varphi_1,\dots,\varphi_k,\psi_1,\dots,\psi_l)(x,\bar y). \tag{4}$$

We can always assume that the bounded variables of $\varphi(x,\bar y)$ are different from $\bar y=\langle y_0,\dots,y_{\mathrm{lh}(\bar y)-1}\rangle$. Let $N=2^{\mathrm{lh}(\bar y)}$, and let $\bar z=\langle z_0,\dots,z_{N+1}\rangle$ be a sequence of distinct variables not occurring in $\varphi(x,\bar y)$. For $1\leqslant s\leqslant l$ we put

$$\psi_s'(z_0,z_{N+s})=((z_0=z_{N+s})\wedge A(z_0)\wedge A(z_{N+s})).$$

Let $1\leqslant s\leqslant k$ and $\sigma\subseteq\mathrm{lh}(\bar y)$. Then φ_s^σ is the formula φ in which every atomic subformula θ such that $y_i\in\mathrm{Fr}(\theta)$ and $i\in\sigma$ is replaced by $\neg(z_0=z_{h^{-1}(\sigma)})\wedge A(z_0)\wedge\wedge A(z_{h^{-1}(\sigma)})$.

For $1\leqslant s\leqslant k$ we put

$$\varphi_s'(x,\bar y,z_0,\dots,z_N)=\bigvee_{\sigma\in\mathscr{P}(\mathrm{lh}(\bar y))}(\varphi_s^\sigma(x,\bar y,z_0,z_{h^{-1}(\sigma)})\wedge(z_0=z_{h^{-1}(\sigma)})\wedge A(z_0)\wedge A(z_{h^{-1}(\sigma)})).$$

Now we define

$$\varphi_A(x,\bar y,\bar z)=P(\varphi_1',\dots,\varphi_k',\psi_1',\dots,\psi_l')(x,\bar y,\bar z). \tag{5}$$

Note that $\varphi_A(x, \bar{y}, \bar{z}) \in L^*(\mathfrak{A})$. Let $\bar{c} \in {}^{\mathrm{lh}(\bar{y})}(A \cup B)$. We are going to define $f_{A,\varphi}(\bar{c})$.

$$(f_{A,\varphi}(\bar{c}))_i = \begin{cases} (\bar{c})_i & \text{if} \quad (\bar{c})_i \in A \text{ and } i<\mathrm{lh}(\bar{y}), \\ a_0 & \text{if} \quad (\bar{c})_i \notin A \text{ and } i<\mathrm{lh}(\bar{y}), \\ a_0 & \text{if} \quad i = \mathrm{lh}(\bar{y}), \\ a_0 & \text{if} \quad i = \mathrm{lh}(\bar{y})+s \text{ for some } 1\leqslant s\leqslant N \text{ and } h(s) = \{j|\ (\bar{c})_j \in B\}, \\ a_1 & \text{if} \quad i = \mathrm{lh}(\bar{y})+s \text{ for some } 1\leqslant s\leqslant N \text{ and } h(s) \neq \{j|\ (\bar{c})_j \in B\}, \\ a_0 & \text{if} \quad i = \mathrm{lh}(\bar{y})+N+s \text{ for some } 1\leqslant s\leqslant l \text{ and } \mathfrak{A} \oplus \mathfrak{B} \models \\ & \qquad \models \exists x(A(x) \wedge \psi_s(x, \bar{c})), \\ a_1 & \text{if} \quad i = \mathrm{lh}(\bar{y})+N+s \text{ for some } 1\leqslant s\leqslant l \text{ and } \mathfrak{A} \oplus \mathfrak{B} \models \\ & \qquad \models \neg\exists x(A(x) \wedge \psi_s(x, \bar{c})). \end{cases}$$

It is seen that the conditions (i) and (ii) hold. Now we are going to verify (iii). Fix $1\leqslant s\leqslant l$. By Lemma 2.4 and the definition of $f_{A,\varphi}(\bar{c})$ for $a \in A$ we have

$$\mathfrak{A} \oplus \mathfrak{B} \models \psi_s[a, \bar{c}] \quad \text{iff} \quad (f_{A,\varphi}(\bar{c}))_{\mathrm{lh}(\bar{y})+N+s} = a_0,$$

$$\mathfrak{A} \oplus \mathfrak{B} \models \psi_s[a, \bar{c}] \quad \text{iff} \quad \mathfrak{A} \models \psi_s'[a, f_{A,\varphi}(\bar{c})]. \tag{6}$$

Now let us fix $1\leqslant s\leqslant k$ and put $\sigma_0 = \{j|\ (\bar{c})_j \in B\}$. Using 2.1 and the definition of $\varphi_s^{\sigma_0}$, for $a \in A$, we obtain

$$\mathfrak{A} \oplus \mathfrak{B} \models \varphi_s[a, \bar{c}] \quad \text{iff} \quad \mathfrak{A} \oplus \mathfrak{B} \models \varphi_s^{\sigma_0}[a, f_{A,\varphi}(\bar{c})] \wedge a_0 = (f_{A,\varphi}(\bar{c}))_{\mathrm{lh}(\bar{y})+h^{-1}(\sigma_0)},$$

(7) if $\mathfrak{A} \oplus \mathfrak{B} \models \varphi_s[a, \bar{c}]$, then $\mathfrak{A} \models \varphi_s'[a, f_{A,\varphi}(\bar{c})]$.

To prove the implication converse to (7) it suffices to show that

if $\mathfrak{A} \models (\varphi_s^{\sigma}(x, \bar{y}) \wedge (z_{\mathrm{lh}(\bar{y})} = z_{\mathrm{lh}(\bar{y})+h^{-1}(\sigma)}))[a, f_{A,\varphi}(\bar{c})]$ then $\sigma = \{j|(\bar{c})_j \in B\}$.

But the last implication follows from the definition of the sequence $f_{A,\varphi}(\bar{c})$. So for $a \in A$ we have

$$\mathfrak{A} \oplus \mathfrak{B} \models \varphi_s[a, \bar{c}] \quad \text{iff} \quad \mathfrak{A} \models \varphi_s'[a, f_{A,\varphi}(\bar{c})]. \tag{8}$$

Now by (4), (5), (6), and (8) for $a \in A$ we obtain

$$\mathfrak{A} \oplus \mathfrak{B} \models \varphi[a, \bar{c}] \quad \text{iff} \quad \mathfrak{A} \oplus \mathfrak{B} \models \varphi_A[a, f_{A,\varphi}(\bar{c})].$$

This completes the proof.

§ 3. The finite cover property. In this section we prove that the finite product of theories does not have the f.c.p., provided they do not have the f.c.p. (²).

Let us recall some notions concerning the f.c.p.

(i) A formula $\varphi(\bar{x}, \bar{y})$ has the *n-cover property* (*n-c.p.*) with respect to T if there are: a model $\mathfrak{A}$ of T and $\bar{a}_0, \ldots, \bar{a}_{n-1}$ ($\bar{a}_i \in {}^{\omega>}A$ for $i<n$) such that

$$\mathfrak{A} \models \neg\exists \bar{x} \bigwedge_{i<n} \varphi[\bar{x}, \bar{a}_i]$$

(²) The notion of the finite cover property was introduced by Keisler in [4].

but for every $j<n$

$$\mathfrak{A} \models \exists \bar{x} \bigwedge_{\substack{i<n \\ i\neq j}} \varphi[\bar{x}, \bar{a}_i].$$

(ii) A formula $\varphi(\bar{x}, \bar{y})$ has the *f.c.p.* with respect to T if $\varphi(\bar{x}, \bar{y})$ has the n-c.p. for arbitrarily large $n \in \omega$.

(iii) T has the *f.c.p.* if there is a formula $\varphi(x, \bar{y})$ which has the f.c.p. with respect to T.

Remark 3.1 (Shelah [14], 3.10). If a formula $\varphi(\bar{x}, \bar{y})$ has the f.c.p. with respect to T, then some formula $\psi(x, \bar{y})$ has the f.c.p. with respect to T.

LEMMA 3.2. *If* $\mathrm{Th}(\mathfrak{A} \oplus \mathfrak{B})$ *has the f.c.p., then either* $\mathrm{Th}(\mathfrak{A})$ *or* $\mathrm{Th}(\mathfrak{B})$ *has the f.c.p.*

Proof. Suppose that $\varphi(x, \bar{y})$ has the f.c.p. with respect to $\mathrm{Th}(\mathfrak{A} \oplus \mathfrak{B})$. Fix $N \in \omega$ and let $k = \min\{l\geqslant 2N|\ \varphi(x, \bar{y})$ has the l-c.p. with respect to $\mathrm{Th}(\mathfrak{A} \oplus \mathfrak{B})\}$. Let $\bar{c}_0, \ldots, \bar{c}_{k-1}$ be as in the definition of the k-c.p. and let $\varphi_A, \varphi_B, f_{A,\varphi}, f_{B,\varphi}$ be formulas and sequences which can be obtained from Lemma 2.5. We shall write f_A, f_B instead of $f_{A,\varphi}, f_{B,\varphi}$.

Let us define:

$$F = \{j<k|\ \mathfrak{A} \models \exists x \bigwedge_{\substack{i<k \\ i\neq j}} \varphi_A[x, f_A(\bar{c}_i)]\}$$

and

$$F' = \{j<k|\ \mathfrak{B} \models \exists x \bigwedge_{\substack{i<k \\ i\neq j}} \varphi_B[x, f_B(\bar{c}_i)]\}.$$

If $F = k$ or $F' = k$, then the Lemma is proved. Otherwise either $|F|\geqslant N$ or $|F'|\geqslant N$. Without loss of generality we assume that $F = \{0, 1, \ldots, l-1\}$ for some $l\geqslant N$. Now by induction we define sequences $D = \langle D_s|\ s\leqslant\alpha\rangle$, $C = \langle C_s|\ s\leqslant\alpha\rangle$ of subsets of k, such that for $0\leqslant s<\alpha$ the following holds:

(i) $D_s \subseteq D_{s+1}$ and $C_s \subseteq C_{s+1}$.

(ii) If $j \in D_s$, then

$$\mathfrak{A} \models \exists x \bigwedge_{\substack{i\notin C_s \\ i\neq j}} \varphi_A[x, f_A(\bar{c}_i)].$$

(iii) $\mathfrak{A} \models \neg\exists x \bigwedge_{i\notin C_s} \varphi_A[x, f_A(\bar{c}_i)]$.

Let $D_0 = \{0, 1, \ldots, l-1\}$, $C_0 = 0$, $C_1 = \{l\}$, and

$$D_1 = \{j<k|\ \mathfrak{A} \models \exists x \bigwedge_{\substack{i\notin C_1 \\ i\neq j}} \varphi_A[x, f_A(\bar{c}_i)]\}.$$

Then, as is easy to show, the conditions (i)-(iii) hold. Suppose that C_s and D_s have been defined. Put:

$$E = \{t<k|\ \mathfrak{A} \models \neg\exists x \bigwedge_{\substack{i\notin C_s \\ i\neq t}} \varphi_A[x, f_A(\bar{c}_i)] \ \&\ t \notin C_s\}.$$

If $E \neq 0$, then $C_{s+1} = C_s \cup \{\min E\}$ and

$$D_{s+1} = \{j<k|\ \mathfrak{A} \models \exists x \bigwedge_{\substack{i \notin C_{s+1} \\ i \neq j}} \varphi_A[x, f_A(\bar{c}_i)]\} .$$

Otherwise, if $E = 0$ then $D = \langle D_i|\ i \leqslant s\rangle$ and $C = \langle C_i|\ i \leqslant s\rangle$ (i.e., $\alpha = s$). By the definition of α and the conditions (i)-(iii) we have: $D_\alpha \cap C_\alpha = 0$, $D_\alpha \cup C_\alpha = k$ and $|D_\alpha| \geqslant N$. So we obtain:

$$\mathfrak{A} \models \neg \exists x \bigwedge_{i \in D_\alpha} \varphi_A[x, f_A(\bar{c}_i)] ,$$

but for $j \in D_\alpha$

$$\mathfrak{A} \models \exists x \bigwedge_{\substack{i \in D_\alpha \\ i \neq j}} \varphi_A[x, f_A(\bar{c}_i)] .$$

Therefore the formula $\varphi_A(x, \bar{y}, \bar{z})$ has the $|D_\alpha|$-c.p. and $|D_\alpha| \geqslant N$. Let

$$G_A = \{n \in \omega|\ \varphi_A(x, \bar{y}, \bar{z}) \text{ has the } n\text{-c.p.}\} ,$$
$$G_B = \{n \in \omega|\ \varphi_B(x, \bar{y}, \bar{z}) \text{ has the } n\text{-c.p.}\} .$$

Then G_A or G_B is infinite. Hence, $\varphi_A(x, \bar{y}, \bar{z})$ or $\varphi_B(x, \bar{y}, \bar{z})$ has the finite cover property. This completes the proof.

LEMMA 3.3. *If* $\mathrm{Th}(\mathfrak{A} \times \mathfrak{B})$ *has the f.c.p., then* $\mathrm{Th}(\mathfrak{A} \oplus \mathfrak{B})$ *has the f.c.p.*

Proof. We may assume that L contains only relational symbols. Let $\varphi(x, \bar{y})$ be a formula of L and assume that $\varphi(x, \bar{y})$ has the f.c.p. with respect to $\mathrm{Th}(\mathfrak{A} \times \mathfrak{B})$. Let $S = \langle S, \pi\rangle$ be an autonomous system for $\varphi(x, \bar{y})$, and let

$$\vdash \varphi(x, \bar{y}) \leftrightarrow \bigvee_{1 \leqslant i \leqslant n} \varphi_i(x, \bar{y}) \quad \text{where} \quad \varphi_i \in S .$$

For $1 \leqslant i \leqslant n$ put:

$$P_i = \{\langle \psi, \eta\rangle|\ \psi, \eta \in S \text{ and } \pi(\psi, \eta) = \varphi_i\} .$$

Then for $k = \mathrm{lh}(\bar{y})+1$ and $a_0, \ldots, a_k \in A$, and $b_0, \ldots, b_k \in B$ we have:

$$\mathfrak{A} \times \mathfrak{B} \models \varphi[\langle a_0, b_0\rangle, \ldots, \langle a_k, b_k\rangle] \quad \text{iff}$$
$$\bigvee_{1 \leqslant i \leqslant n} \bigvee_{p \in P_i} \mathfrak{A} \models (p)_0[a_0, \ldots, a_k] \ \&\ \mathfrak{B} \models (p)_1[b_0, \ldots, b_k]$$

and

$$(*) \quad \mathfrak{A} \times \mathfrak{B} \models \varphi[\langle a_0, b_0\rangle, \ldots, \langle a_k, b_k\rangle] \quad \text{iff}$$
$$\mathfrak{A} \oplus \mathfrak{B} \models \bigvee_{1 \leqslant i \leqslant n} \bigvee_{p \in P_i} ((p)'_0[a_0, \ldots, a_k] \wedge (p)'_1[b_0, \ldots, b_k])$$

where $(p)'_0(x, y_1, \ldots, y_k)$ and $(p)'_1(x', y'_1, \ldots, y'_k)$ are the corresponding translations of $(p)_0$ and $(p)_1$ from the languages $L(\mathfrak{A})$, $L(\mathfrak{B})$ into $L^*(\mathfrak{A})$ and $L^*(\mathfrak{B})$, respectively.

By $(*)$ the following formula has the f.c.p. with respect to $\mathrm{Th}(\mathfrak{A} \oplus \mathfrak{B})$:

$$\psi(x_0, x'_0, \bar{y}, \bar{y}') = \bigvee_{1 \leqslant i \leqslant n} \bigvee_{p \in P_i} [(p)'_0(x_0, \bar{y}) \wedge A(x_0) \wedge (p)'_1(x'_0, \bar{y}') \wedge B(x'_0)] .$$

So by Remark 3.1 $\mathrm{Th}(\mathfrak{A} \oplus \mathfrak{B})$ has the f.c.p.

THEOREM 3.4. *If* T_1 *and* T_2 *do not have the f.c.p., then* $T_1 \times T_2$ *does not have the f.c.p.* (3).

Proof. By Lemmas 3.3 and 3.2.

Remark 3.5. $\mathrm{Th}(\mathfrak{B})$ defined in Example 1.5 is unstable, and so by Shelah's Theorem ([14] 3.8.) has the f.c.p. On the other hand, $\mathrm{Th}(\mathfrak{B} \times \mathfrak{B})$ does not have the f.c.p.

§ 4. The strict order property and the independence property. In this section we prove, for the strict order property and the independence property (4) theorems analogous to Theorem 3.4. Let us recall some definitions.

(i) A formula $\varphi(\bar{x}, \bar{y})$ has the *n-strict order property* with respect to T if there are: a model $\mathfrak{A}$ of T, and $\bar{c}_0, \ldots, \bar{c}_{n-1}$ ($\bar{c}_i \in {}^{\omega>}A$ for $i<n$) such that

$$\mathfrak{A} \models \exists \bar{x}(\neg \varphi[\bar{x}, \bar{c}_i] \wedge \varphi[\bar{x}, \bar{c}_j]) \quad \text{iff} \quad i<j<n .$$

(ii) A formula $\varphi(\bar{x}, \bar{y})$ has the *strict order property* with respect to T if it has the n-strict order property with respect to T for all $n \in \omega$.

(iii) T has the *strict order property* if there is a formula $\varphi(x, \bar{y})$ which has the strict order property with respect to T.

(iv) A formula $\varphi(\bar{x}, \bar{y})$ has the *n-independence property* with respect to T if there are: a model $\mathfrak{A}$ of T, and $\bar{c}_0, \ldots, \bar{c}_{n-1}$ ($\bar{c}_i \in {}^{\omega>}A$) such that for all $w \subseteq n$

$$\mathfrak{A} \models \exists \bar{x}(\bigwedge_{i \in w} \varphi[\bar{x}, \bar{c}_i] \wedge \bigwedge_{i \in n-w} \neg \varphi[\bar{x}, \bar{c}_i]) .$$

(v) A formula $\varphi(\bar{x}, \bar{y})$ has the *independence property* with respect to T if it has the n-independence property with respect to T for all $n \in \omega$.

(vi) T has the *independence property* if there is a formula $\varphi(x, \bar{y})$ which has the independence property with respect to T.

Let us note the following:

PROPOSITION 4.1. (i) *If* $\varphi(\bar{x}, \bar{y})$ *has the n-strict order property n-independence property and* $m<n$, *then* $\varphi(\bar{x}, \bar{y})$ *has the m-strict order property (m-independence property).*

(ii) *If* $\varphi(\bar{x}, \bar{y})$ *has the n-strict order property (n-independence property) for arbitrarily large* $n \in \omega$, *then* $\varphi(\bar{x}, \bar{y})$ *has the strict order property (the independence property).*

We shall use the following theorem, part (i) of which is due to Lachlan ([6]) and part (ii) — to Shelah ([14] 4.6).

(3) After the theorem had been proved the author was informed by B. Węglorz about the following result of Shelah: T does not have the f.c.p. iff T is Δ-minimal (Δ is Keisler's ordering from [4]). From this Theorem 3.4 easily follows.

(4) The notions of the strict order property and the independence property were introduced by Shelah in [14].

THEOREM 4.2. (i) *If* $\varphi(\bar{x}, \bar{y})$ *has the strict order property with respect to* T, *then some formula* $\psi(x, \bar{y})$ *has the strict order property with respect to* T.

(ii) *If* $\varphi(\bar{x}, \bar{y})$ *has the independence property with respect to* T, *then some formula* $\psi(x, \bar{y})$ *has the independence property with respect to* T.

LEMMA 4.3. (i) *If* $\mathrm{Th}(\mathfrak{A}\times\mathfrak{B})$ *has the strict order property, then* $\mathrm{Th}(\mathfrak{A}\oplus\mathfrak{B})$ *has the strict order property.*

(ii) *If* $\mathrm{Th}(\mathfrak{A}\times\mathfrak{B})$ *has the independence property, then* $\mathrm{Th}(\mathfrak{A}\oplus\mathfrak{B})$ *has the independence property.*

Proof. Since the proof is similar to that of Lemma 3.3, we omit it. (The main difference is that one uses Theorem 4.2 instead of 3.1).

LEMMA 4.4. *If* $\mathrm{Th}(\mathfrak{A}\oplus\mathfrak{B})$ *has the strict order property, then either* $\mathrm{Th}(\mathfrak{A})$ *or* $\mathrm{Th}(\mathfrak{B})$ *has the strict order property.*

Proof. Let $\varphi(x, \bar{y})$ have the strict order property with respect to $\mathrm{Th}(\mathfrak{A}\oplus\mathfrak{B})$. Fix $N\in\omega$ and let $\bar{c}_0, \ldots, \bar{c}_{2N-1}$ be as in the definition of the $2N$-strict order property. Let us note that the definition implies (*):

(*) If $i<j<2N$, then $\varphi^{\mathfrak{A}\oplus\mathfrak{B}}[\bar{c}_i]\subsetneq\varphi^{\mathfrak{A}\oplus\mathfrak{B}}[\bar{c}_j]$.

Let us define for $i<2N-1$:

$$C_i = (\varphi^{\mathfrak{A}\oplus\mathfrak{B}}[\bar{c}_{i+1}]\setminus\varphi^{\mathfrak{A}\oplus\mathfrak{B}}[\bar{c}_i])\cap A,$$
$$D_i = (\varphi^{\mathfrak{A}\oplus\mathfrak{B}}[\bar{c}_{i+1}]\setminus\varphi^{\mathfrak{A}\oplus\mathfrak{B}}[\bar{c}_i])\cap B,$$
$$C = \{i|\ C_i\neq 0\},\qquad D = \{i|\ D_i\neq 0\}.$$

Then by (*) either $|C|\geqslant N$ or $|D|\geqslant N$. Without loss of generality we assume that $|C| = N_0\geqslant N$. Let $C = \{k_0, \ldots, k_{N_0-1}\}$. Let $\varphi_A(x, \bar{y}, \bar{z})$ be a formula and let $f_{A,\varphi}$ be a function which satisfy the conclusion of Lemma 2.5. Then, of course, $\varphi_A(x, \bar{y}, \bar{z})$ and $f_{A,\varphi}(\bar{c}_{k_0}), \ldots, f_{A,\varphi}(\bar{c}_{k_{N_0-1}})$ satisfy the definition of the N_0-strict order property. Let:

$$G_A = \{n\in\omega|\ \varphi_A(x, \bar{y}, \bar{z}) \text{ has the } n\text{-strict order property}\},$$
$$G_B = \{n\in\omega|\ \varphi_B(x, \bar{y}, \bar{z}) \text{ has the } n\text{-strict order property}\}.$$

Then either G_A or G_B is infinite, and so by Proposition 4.1(ii) either $\varphi_A(x, \bar{y}, \bar{z})$ or $\varphi_B(x, \bar{y}, \bar{z})$ has the strict order property.

THEOREM 4.5. *If* T_1 *and* T_2 *do not have the strict order property, then* $T_1\times T_2$ *does not have the strict order property.*

Proof. By Lemmas 4.3(i) and 4.4.

LEMMA 4.6. *If* $\mathrm{Th}(\mathfrak{A}\oplus\mathfrak{B})$ *has the independence property, then either* $\mathrm{Th}(\mathfrak{A})$ *or* $\mathrm{Th}(\mathfrak{B})$ *has the independence property.*

Proof. Assume that $\varphi(x, \bar{y})$ has the independence property with respect to $\mathrm{Th}(\mathfrak{A}\oplus\mathfrak{B})$. Fix $N\in\omega$ and let $\bar{c}_0, \ldots, \bar{c}_{2N-1}$ be as in the definition of the $2N$-independence property. For $i<2N$, let us define $C_i = \varphi^{\mathfrak{A}\oplus\mathfrak{B}}[\bar{c}_i]$ and

$$k = \{\min l|\ \exists i_0, \ldots, i_{l-1}<2N\,\exists f\in {}^{l}2(C_{i_0}^{f(0)}\cap\ldots\cap C_{i_{l-1}}^{f(l-1)}\cap A = 0)\vee$$
$$\vee(C_{i_0}^{f(0)}\cap\ldots\cap C_{i_{l-1}}^{f(l-1)}\cap B = 0).$$

If $k>N$, then $C_0\cap A, \ldots, C_N\cap A$ are independent subsets of A. Hence, if $\varphi_A(x, \bar{y}, \bar{z})$ and $f_{A,\varphi}$ are taken from Lemma 2.5, then $\varphi_A(x, \bar{y}, \bar{z})$ and $f_{A,\varphi}(\bar{c}_0), \ldots, f_{A,\varphi}(\bar{c}_N)$ satisfy the definition of the $(N+1)$-independence property.

On the other hand, if $k\leqslant N$, then take $C_{i_0}, \ldots, C_{i_{k-1}}$ such that $C_{i_0}^{f(0)}\cap\ldots\cap C_{i_{k-1}}^{f(k-1)} = D\subseteq A$ for some $f\in{}^{k}2$. For $l\notin\{i_0, \ldots, i_{k-1}\}$ define sets $D_l = D\cap C_l$. One can easily see that $D_l\subseteq A$ and $\{D_l|\ l\in 2N\setminus\{i_0, \ldots, i_{k-1}\}\}$ are independent sets. Since $k\leqslant N$, we have $|2N\setminus\{i_0, \ldots, i_{k-1}\}|\geqslant N$. So again $\varphi_A(x, \bar{y}, \bar{z})$ and $\{f_{A,\varphi}(\bar{c}_l)|\ l\in 2N\setminus\{i_0, \ldots, i_{k-1}\}\}$ satisfy the definition of the $|2N\setminus\{i_0, \ldots, i_{k-1}\}|$-independent property.

We complete the proof in the same way as that of Lemma 4.4.

THEOREM 4.7. *If* T_1 *and* T_2 *do not have independence property then* $T_1\times T_2$ does *not have the independence property.*

Proof. By Lemmas 4.3(ii) and 4.6.

Remark 4.8. One can see that $\mathrm{Th}(\mathfrak{B})$ from Example 1.5 has the strict order property. On the other hand, $\mathrm{Th}(\mathfrak{B}\times\mathfrak{B})$ is stable, and so by Shelah's theorem ([14] 4.1) does not have the strict order property.

Remark 4.9. Using the same method as in Example 1.5, one can construct a theory T, such that T has the independence property, but $T\times T$ does not have the independence property.

§ 5. Morley rank.

In this section we estimate the Morley rank of $p\times q$. We write CB-rank for the Cantor–Bendixon rank, rank for the Morley rank and d^α, D^α for the corresponding αth derivatives (see e.g. [12]). If x has no CB-rank (rank), then we write $\text{CB-rank}(x) = \infty$ ($\mathrm{rank}(x) = \infty$).

We need some topological facts:

PROPOSITION 5.1. *Let* X, Y *be compact Hausdorff spaces. Then*:

(i) *If* $f\colon X\to Y$ *is a continuous onto map and* α *is an ordinal number, then* $d^\alpha(Y)\subseteq f(d^\alpha(X))$.

(ii) ([16]) Thm. 2) *If* $x\in X$ *and* $y\in Y$, *then* $\text{CB-rank}(\langle x, y\rangle) = \text{CB-rank}(x)(+)\text{CB-rank}(y)$ ([5]).

LEMMA 5.2. *Let* $f\colon S(\mathfrak{A})\times S(\mathfrak{B})\to S(\mathfrak{A}\times\mathfrak{B})$ *be a map defined by* $f(\langle p, q\rangle) = p\times q$. *Then* f *is a continuous and onto map.*

Proof. By Lemma 0.5 f is onto. Let $U = \{p\in S(\mathfrak{A}\times\mathfrak{B})|\ \varphi(x, \langle a_0, b_0\rangle, \ldots)\in p\}$ be a basic open set of $S(\mathfrak{A}\times\mathfrak{B})$. Let $S = \langle S, \pi\rangle$ be an autonomous system for φ. Without loss of generality we assume that $\varphi\in S$. Let

$$P = \{\langle\eta, \psi\rangle|\ \eta, \psi\in S \text{ and } \pi(\eta, \psi) = \varphi\}.$$

Then

$$f^{-1}(U) = \bigcup_{\langle\eta,\psi\rangle\in P}(\{q_1\in S(\mathfrak{A})|\ \eta(x, a_0, \ldots)\in q_1\}\times\{q_2\in S(\mathfrak{B})|\ \psi(x, b_0, \ldots)\in q_2\}$$

is an open set in $S(\mathfrak{A})\times S(\mathfrak{B})$.

([5]) (+) denotes the natural sum of ordinals, see e.g. [5].

THEOREM 5.3. *If* $p \in S(\mathfrak{A}\times\mathfrak{B})$, *then*

$$\mathrm{rank}(p) \leqslant \mathrm{l.u.b.}\{\mathrm{rank}(q_1)(+)\mathrm{rank}(q_2)|\ q_1 \in S(\mathfrak{A}) \text{ and } q_2 \in S(\mathfrak{B}) \text{ and } q_1 \times q_2 = p\}.$$

Proof. First assume that $\mathfrak{A}$ and $\mathfrak{B}$ are ω-saturated. By [17] 1.5 $\mathfrak{A}\times\mathfrak{B}$ is also ω-saturated. So by [12] 31.3

(∗) if $p \in S(\mathfrak{A})(S(\mathfrak{B}), S(\mathfrak{A}\times\mathfrak{B}))$, then $\mathrm{rank}(p) = \mathrm{CB\text{-}rank}(p)$.

Let $p \in S(\mathfrak{A}\times\mathfrak{B})$. Then by (∗) and Proposition 5.1 we have:

$$\begin{aligned}\mathrm{rank}(p) = \mathrm{CB\text{-}rank}(p) &\leqslant \mathrm{l.u.b.}\{\mathrm{CB\text{-}rank}(\langle q_1, q_2\rangle)|\ q_1\times q_2 = p\}\\ &= \mathrm{l.u.b.}\{\mathrm{CB\text{-}rank}(q_1)(+)\mathrm{CB\text{-}rank}(q_2)|\ q_1\times q_2 = p\}\\ &= \mathrm{l.u.b.}\{\mathrm{rank}(q_1)(+)\mathrm{rank}(q_2)|\ q_1\times q_2 = p\}.\end{aligned}$$

Now we are going to eliminate the assumption that $\mathfrak{A}$ and $\mathfrak{B}$ are ω-saturated. There are $\mathfrak{C}\succ\mathfrak{A}$ and $\mathfrak{B}\succ\mathfrak{B}$ such that $\mathfrak{C}$ and $\mathfrak{B}$ are ω-saturated. Fix $q \in S(\mathfrak{A}\times\mathfrak{B})$. By the basic properties of the Morley rank (see [9] or [12]) there is a $p \in S(\mathfrak{C}\times\mathfrak{B})$ such that $p \supseteq q$ and $\mathrm{rank}(p) = \mathrm{rank}(q)$. Therefore

$$\begin{aligned}\mathrm{rank}(q) = \mathrm{rank}(p) &\leqslant \mathrm{l.u.b.}\{\mathrm{rank}(q_1)(+)\mathrm{rank}(q_2)|\ q_1\times q_2 = p\}\\ &\leqslant \mathrm{l.u.b.}\{\mathrm{rank}(q_1|\mathfrak{A})(+)\mathrm{rank}(q_2|\mathfrak{B})|\ q_1\times q_2 = p\}\\ &\leqslant \mathrm{l.u.b.}\{\mathrm{rank}(q_1)(+)\mathrm{rank}(q_2)|\ q_1\times q_2 = q\}\end{aligned}$$

where $q_1|\mathfrak{A}$ $(q_2|\mathfrak{B})$ denotes the subtype of $q_1(q_2)$ which belongs to $S(\mathfrak{A})$ $(S(\mathfrak{B}))$.

COROLLARY 5.4. $\alpha_{T_1\times T_2} < \alpha_{T_1}(+)\alpha_{T_2}$ (6) *provided that* T_1 *and* T_2 *are* ω-*stable*.

In [9] Morley asked what model-theoretical conditions on T imply that α_T is finite. In [1] Baldwin proved that if T is ω_1-categorical, then α_T is finite. By Corollary 5.4 we have the following result:

COROLLARY 5.5. (i) *If T is a finite product of ω_1-categorical theories, then* α_T *is finite.*

(ii) *The class of all theories for which* α_T *is finite is closed under finite products.*

EXAMPLE 5.6. Let $\mathfrak{A} = \langle[0,2], R_1, R_2\rangle$, where $R_1(a)$ iff $a \in [0,1]$ and $R_2(a)$ iff $a \in [1,2]$. Let $T = \mathrm{Th}(\mathfrak{A})$. It is not too difficult to show that T is ω-stable, $\alpha_T = 2$ and T is not a finite product of ω_1-categorical theories.

PROPOSITION 5.7. *For each ordinal* $0<\alpha, \beta<\omega_1$, *there are theories* T_α *and* T_β *such that*: (i) T_α *and* T_β *are* ω-*stable*, (ii) $\alpha_{T_\alpha} = \alpha+1$ *and* $\alpha_{T_\beta} = \beta+1$, (iii) $\alpha_{T_\alpha\times T_\beta} = \alpha_{T_\alpha}(+)\alpha_{T_\beta}-1$.

Proof. We prove this only for $\alpha, \beta \geqslant \omega$. We consider Morley's Example III from [9]. Let X and Y be closed subsets of the Cantor set such that $\alpha+1 = \min\{\xi|\ d^\xi(X) = 0\}$ and $\beta+1 = \min\{\xi\ |d^\xi(Y) = 0\}$. Let T_α be a theory constructed by using X, and let T_β be a theory constructed by using Y. Note that if $\mathfrak{A} \models T_\alpha$ and $\mathfrak{B} \models T_\beta$, then for $K_0, K_1 \subseteq \omega$ $\big(\bigwedge_{i\in K_1} R_i \wedge \bigwedge_{j\in K_0} \neg R_j\big)^{\mathfrak{A}\times\mathfrak{B}}$ is infinite iff we have $\langle f, g\rangle \in X\times Y$ such that

$$\langle f(i), g(i)\rangle = \langle 1, 1\rangle \quad \text{for} \quad i \in K_1$$

and

$$\langle f(i), g(i)\rangle \neq \langle 1, 1\rangle \quad \text{for} \quad i \in K_0.$$

By this, reasoning as in Example III [9] and Proposition 5.1 (ii) we have

$$\alpha_{T_\alpha\times T_\beta} = \min\{\xi|\ d^\xi(X\times Y) = 0\} = (\alpha_{T_\alpha}(+)\alpha_{T_\beta})-1.$$

(6) For the definition of α_T see [9].

§ 6. Infinite powers. In this section we investigate properties connected with the stability of theories of reduced powers.

THEOREM 6.1. *Let $\mathcal{F}$ be a filter over I. Then the following conditions on the filter $\mathcal{F}$ are equivalent*:

(I) $2^I_{\mathcal{F}}$ *is finite.*

(II) *For any theory T, if T satisfies* (∗), *then* $T^I_{\mathcal{F}}$ *satisfies* (∗), *where* (∗) *denotes one of the following conditions*:

A) *T is stable,*

B) *T is superstable,*

C) *T is ω-stable,*

D) *T does not have the f.c.p.,*

E) *T does not have the strict order property,*

F) *T does not have the independence property.*

Proof. The implication (I) ⇒ (II) follows from Theorems 1.1, 1.2, 1.3, 3.4, 4.5 and 4.7. To obtain the converse implication take $T = \mathrm{Th}(\mathfrak{A})$, where $\mathfrak{A} = \langle\{\mathbf{0}, \mathbf{1}\}\cup\omega, <\rangle$ and $x<y$ iff $x = \mathbf{0}$ and $y = \mathbf{1}$. Now note that if $\mathfrak{B}$ is a Boolean algebra, then $\mathfrak{B}$ has property (∗) iff $\mathfrak{B}$ is finite.

THEOREM 6.2. *Let $\mathcal{F}$ be a filter over I. Then the following conditions on the filter $\mathcal{F}$ are equivalent*:

(I) *$\mathcal{F}$ is an ultrafilter.*

(II) *For any theory T, if T satisfies* (∗∗), *then* $T^I_{\mathcal{F}}$ *satisfies* (∗∗), *where* (∗∗) *denotes one of the following conditions*:

A) *T is unstable,*

B) *T has the f.c.p.,*

C) *T has the strict order property,*

D) *T has the independence property.*

Proof. The implication (I) ⇒(II) is obvious. It is a matter of easy calculation to show that if $|2^I_{\mathcal{F}}|>2$ and $\mathfrak{B}$ is from Example 1.5, then $\mathfrak{B}^I_{\mathcal{F}} \equiv \mathfrak{B}^2$. Hence if $T = \mathrm{Th}(\mathfrak{B})$, then $T^I_{\mathcal{F}}$ is unstable iff $\mathcal{F}$ is an ultrafilter. For the other cases the proofs are similar; for the independence property see Remark 4.9.

Remark 6.3. Since $T^I_{\mathcal{F}}|\mathcal{G}$ is determined by T and $2^I_{\mathcal{F}}|\mathcal{G}$, we have the same characterization for limit reduced powers.

§ 7. **Final remarks.** The product-type operations depend on the choice of the language. We are going to show that, if we expand the language, then unstability and other properties are preserved under reduced powers and finite products.

We say that T is neat (7) if for every formula φ of the language of T there is a predicate P_φ such that $T \vdash \varphi \leftrightarrow P_\varphi$.

PROPOSITION 7.1. *Let T_1 and T_2 be neat theories. Then*:

(i) *If T_1 and T_2 satisfy* (**), *then $T_1 \times T_2$ satisfies* (**).

(ii) *If T_1 satisfies* (**), *and $\mathscr{F}$ is a filter over I, then $T_{\mathscr{F}}^I$ satisfies* (**), *where* (**) *denotes one of the following conditions*:

A) *T is unstable,*

B) *T has the f.c.p.,*

C) *T has the strict order property,*

D) *T has the independence property.*

Since the proofs are easy we leave them for the reader.

It is not difficult to notice that for most of the results the assumption of countability of L is not necessary.

We close this section by mentioning an open problems similar to Problem 7 in [9].

PROBLEM. Find a possibly small class K of theories such that the class of all finite products of elements from K is equal to the class of all ω-stable theories T for which α_T is finite.

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