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Accepté par la Rédaction le 21. 10. 1974

On bouquets

by

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Abstract. The paper concerns with bouquets of metric continua, in particular with spherical bouquets and usual spherical bouquets. Some results are the following:

Every bouquet of ANR's is movable.

A bouquet is FAR if and only if all its leaves are FAR's.

A bouquet of ANR's is homeomorphic to the inverse limit of a sequence of finite subbouquets with bonding maps being retractions.

Every spherical bouquet is of the same shape as a locally connected spherical bouquet.

For usual n-dimensional spherical bouquets ($n > 1$) two fundamental sequences are homotopic whenever they are homologic.

The classification of spaces, in particular of compact spaces, into classes called shapes is based only on global properties of those spaces, and thus it is far less precise than topological classification. The question arises how to find the singles possible space in each class. E.g., for the class of each plane continuum there exists a representative which is a finite or countable bouquet of 1-spheres, i.e., a set homeomorphic to one of the subsets

$$X_k = \bigcup_{i=1}^{k} \left\{(x, y) \in E^2 \colon \left(x-\frac{1}{i}\right)^2 + y^2 = \left(\frac{1}{i}\right)^2\right\} \quad \text{for} \quad k = 1, 2, \ldots$$

and

$$X = \bigcup_{i=1}^{\infty} \left\{(y, x) \in E^2 \colon \left(x-\frac{1}{i}\right)^2 + y^2 = \left(\frac{1}{i}\right)^2\right\}$$

of the plane E^2.

The aim of this paper is to study the properties of bouquets in more general sense.

1. Basic definitions. In [1]-[5] K. Borsuk introduced the basic notions of shape. We recall some of the basic definitions.

Let X and Y be two compacta lying in an AR($\mathfrak{M}$)-space M.

A sequence of continuous maps $f_k\colon M \to M$ is said to be a *fundamental sequence* from X to Y (notation: $\underline{f} = \{f_k, X, Y\}$) if for every neighborhood V of Y there is a neighborhood U of X such that

$$f_k|_U \simeq f_{k+1}|_U \quad \text{in } V \quad \text{for almost all } k\,.$$

Two fundamental sequences $f=\{f_k, X, Y\}$ and $g=\{g_k, X, Y\}$ are said to be *homotopic* (notation: $f\simeq g$) if for every neighborhood V of Y there is a neighborhood U of X such that

$$f_k|_U \simeq g_k|_U \quad \text{in } V \quad \text{for almost all } k.$$

By the *fundamental identity sequence* from X to X we understand a fundamental sequence $\underline{\mathrm{id}}_X=\{i_k, X, X\}$ such that $i_k(x)=x$ for every $x\in X$ and $k=1,2,\ldots$

By the *composition* $\underline{gf}$ of two fundamental sequences $\underline{f}=\{f_k, X, Y\}$ and $\underline{g}=\{g_k, Y, Z\}$ we understand the fundamental sequence $\{g_k f_k, X, Z\}$.

If there exist two fundamental sequences $\underline{f}=\{f_k, X, Y\}$ and $\underline{g}=\{g_k, Y, X\}$ such that $\underline{gf}\simeq\underline{\mathrm{id}}_X$ and $\underline{fg}\simeq\underline{\mathrm{id}}_Y$, then we say that X and Y have the same *shape* (notation: $\mathrm{Sh}\,X=\mathrm{Sh}\,Y$).

Replacing in those definitions the compacta X, Y by the pointed compacta $(X, x_0)\subset(M, x_0)$, $(Y, y_0)\subset(M, y_0)$ and also the neighborhoods U, V by the pointed neighborhoods (U, x_0), (V, y_0) we get the notions of the *fundamental pointed sequence* $\underline{f}=\{f_k, (X, x_0), (Y, y_0)\}$, of the *homotopy of pointed fundamental sequences* and of the *shape of pointed compacta*.

A closed subset A of a compact set $X\subset M$ is said to be a *fundamental retract* of X if there exists a *fundamental retraction* of X to A, i.e., a fundamental sequence $\underline{r}=\{r_k, X, A\}$ satisfying the condition $r_k(x)=x$ for every point $x\in A$ and $k=1,2,\ldots$

The fundamental retracts of AR-sets are said to be *fundamental absolute retracts* (FAR-sets) and fundamental retracts of ANR-sets are said to be *fundamental absolute neighborhood retracts* (FANR-sets).

All the spaces considered in this paper are metric and separable.

The symbols ϱ, $K(x,\varepsilon)$, $K(A,\varepsilon)$ always denote the metric, the open ball with the centre x and the radius ε and the set of such points y that $\inf_{a\in A}\varrho(y,a)<\varepsilon$, respectively.

2. The notion of a bouquet. The aim of this part is to give the definition of a bouquet and to prove lemmas needed for studying the properties of bouquets.

(2.1) DEFINITION. A pointed continuum (X, a) is said to be a *bouquet* if there exists a family $\mathscr{X}=\{X_\lambda\}_{\lambda\in\Lambda}$ of subcontinua of the space X such that the following conditions are satisfied:

(1) $X=\bigcup_{\lambda\in\Lambda} X_\lambda$,

(2) for every $\lambda\in\Lambda$ the set $X_\lambda-(a)$ is a component of $X-(a)$,

(3) if λ, $\mu\in\Lambda$ are different indexes, then there exist continua C_λ, C_μ such that $X_\lambda\subset C_\lambda$, $X_\mu\subset C_\mu$, $X=C_\lambda\cup C_\mu$, and $C_\lambda\cap C_\mu=(a)$.

The elements of the family $\mathscr{X}$ are called the *leaves* and the point a the *centre* of the bouquet (X, a).

(2.2) DEFINITION. Let (X, a) be a bouquet, $\mathscr{X}=\{X_\lambda\}_{\lambda\in\Lambda}$ the family of its leaves and (Y, a) a pointed continuum such that $Y\subset X$. Then (Y, a) is said to be a *subbouquet* of (X, a) if it is a bouquet and the family of leaves of (Y, a) is contained in $\mathscr{X}$.

It is easy to see that the intersection and the sum of an arbitrary finite family of subbouquets of (X, a) are also subbouquets of (X, a).

(2.3) LEMMA. *If (X, a) is a bouquet and $\mathscr{X}=\{X_\lambda\}_{\lambda\in\Lambda}$ is the family of its leaves, then for every $\lambda\in\Lambda$ and for every neighborhood U of X_λ in X there is a subbouquet (Y, a) of (X, a) such that the set $(X-Y)\cup(a)$ is compact and $X_\lambda\subset Y\subset U$.*

Proof. Let $\lambda\in\Lambda$ be a fixed index and U a neighborhood of X_λ in X. For every $\mu\in\Lambda$, $\mu\neq\lambda$, let us denote by (B_μ, a) and (C_μ, a) subbouquets of (X, a) such that $X_\mu\subset B_\mu$, $X_\lambda\subset C_\mu$, $X=B_\mu\cup C_\mu$ und $C_\mu\cap B_\mu=(a)$.

Since $X_\lambda=\bigcap_{\substack{\mu\in\lambda\\ \mu\neq\lambda}} C_\mu$, there is a sequence $\mu_1,\mu_2,\ldots$ of indexes different from λ such that

$$X_\lambda=\bigcap_{i=1}^{\infty} C_{\mu_i}.$$

Since X_λ is a compact set, there is a natural number n_0 such that $\bigcap_{i=1}^{n_0} C_{\mu_i}\subset U$.

The set $Y=\bigcap_{i=1}^{n_0} C_{\mu_i}$ satisfies the required conditions.

It follows directly from Lemma (2.3) that for every leaf X_λ of a bouquet (X, a) the set $X_\lambda-(a)$ is a quasi-component of $X-(a)$.

(2.4) LEMMA. *Let (X, a) be a bouquet and $\mathscr{X}=\{X_\lambda\}_{\lambda\in\Lambda}$ the family of its leaves. If $\mathscr{U}=\{U_\lambda\}_{\lambda\in\Lambda}$ is a family of open subsets of the space X such that $X_\lambda\subset U_\lambda$ for every $\lambda\in\Lambda$, then there exist finite sequences $\lambda_1,\ldots,\lambda_k$ of indexes and $\{(B_i, a)\}_{i=1}^{k}$ of subbouquets of (X, a) such that the following conditions are satisfied:*

(1) $X_{\lambda_j}\subset B_j\subset U_{\lambda_j}$ *for* $j=1,\ldots,k$,

(2) $X=\bigcup_{j=1}^{k} B_j$,

(3) $B_i\cap B_j=(a)$ *for* $i\neq j$, $i,j=1,\ldots,k$.

Proof. We can find, by Lemma (2.3), a family $\{(Y_\lambda, a)\}_{\lambda\in\Lambda}$ of subbouquets of (X, a) such that for every $\lambda\in\Lambda$ the set $(X-Y_\lambda)\cup(a)$ is compact and $X_\lambda\subset Y_\lambda\subset U_\lambda$.

It is clear that there is a sequence $\mu_1,\mu_2,\ldots$ of elements of Λ such that $X=\bigcup_{i=1}^{\infty} Y_{\mu_i}$.

We define the sets $A_{\mu_1}, A_{\mu_2},\ldots$ by the formulas

$$A_{\mu_1}=Y_{\mu_1}$$

and

$$A_{\mu_i}=\Big(Y_{\mu_i}-\bigcup_{j=1}^{i-1} Y_{\mu_j}\Big)\cup(a) \quad \text{for} \quad i=2,3,\ldots$$

Let us observe that

$$\bigcup_{i=1}^{k} A_{\mu_i}=\bigcup_{i=1}^{k} Y_{\mu_i} \quad \text{for every } k \quad \text{and} \quad X=\bigcup_{i=1}^{\infty} A_{\mu_i}.$$

For every i the pointed continuum (A_{μ_i}, a) is a subbouquet of (X, a), the set $A_{\mu_i}-(a)$ is both closed and open in $X-(a)$ and $(A_{\mu_i}-(a)) \cap (A_{\mu_j}-(a)) = \emptyset$ for $i \neq j$.

Let ε be a positive number such that $A_{\mu_1} \not\subset K(a, \varepsilon) \subset U_{\mu_1}$.

Since $X = K(a, \varepsilon) \cup \bigcup_{i=1}^{\infty} (A_{\mu_i}-(a))$ and X is compact, we infer that $A_{\mu_j} \subset K(a, \varepsilon)$ for almost all j. Let $1 = n_1 < \ldots < n_k$ be a sequence of natural numbers such that $A_{\mu_{n_i}} \not\subset K(a, \varepsilon)$ for $i = 1, \ldots, k$ and $A_{\mu_j} \subset K(a, \varepsilon)$ for $j \neq n_1, \ldots, n_k$.

Let us write $\lambda_i = \mu_{n_i}$ for $i = 1, \ldots, k$. It is clear that $X_{\lambda_j} \subset \bigcup_{i=1}^{j} A_{\lambda_i}$ for every $1 \leqslant j \leqslant k$.

Before we define the bouquets (B_i, a) for $i = 1, \ldots, k$, we shall construct inductively an auxiliary finite sequence (D_l^{j-1}, a), $j = 2, \ldots, k$, $l = 1, \ldots, j$, of subbouquets of (X, a).

If $X_{\lambda_2} \subset A_{\lambda_2}$, then we put $D_1^1 = A_{\lambda_1}$ and $D_2^1 = A_{\lambda_2}$.

If $X_{\lambda_2} \not\subset A_{\lambda_2}$, then $X_{\lambda_2} \subset A_{\lambda_1}$ and we infer by (2.3) that there is a subbouquet (F_2, a) of the bouquet $(A_{\lambda_1} \cap Y_{\lambda_2}, a)$ such that $X_{\lambda_2} \subset F_2$, $X_{\lambda_1} \not\subset F_2$ and the set $(A_{\lambda_1} \cap Y_{\lambda_2} - F_2) \cup (a)$ is compact.

We define $D_1^1 = (A_{\lambda_1} - F_2) \cup (a)$ and $D_2^1 = A_{\lambda_2} \cup F_2$.

Let us suppose that for every $2 \leqslant n \leqslant k$ we have defined a sequence $(D_1^{j-1}, a), \ldots$ $\ldots, (D_j^{j-1}, a)$, $j = 2, \ldots, n$, of subbouquets of (X, a) such that the following conditions are satisfied:

$(\mathrm{a})_n$ if $2 \leqslant j \leqslant n$ and $1 \leqslant i \leqslant j$, then $X_{\lambda_i} \subset D_i^{j-1} \subset U_{\lambda_i}$,

$(\mathrm{b})_n$ if $2 \leqslant j \leqslant n$, $1 \leqslant i, l \leqslant j$ and $i \neq l$, then $D_i^{j-1} \cap D_l^{j-1} = (a)$;

$(\mathrm{c})_n$ for every $2 \leqslant j \leqslant n$ and $1 \leqslant i \leqslant j$ the set $D_i^{j-1}-(a)$ is both closed and open in $X-(a)$;

$(\mathrm{d})_n$ $\bigcup_{i=1}^{j} A_{\lambda_i} = \bigcup_{i=1}^{j} D_i^{j-1}$ for $j = 2, \ldots, n$.

The bouquets $(D_1^n, a), \ldots, (D_{n+1}^n, a)$ we define in the following manner:

If $X_{\lambda_{n+1}} \subset A_{\lambda_{n+1}}$, then we put $D_i^n = D_i^{n-1}$ for $i = 1, \ldots, n$ and $D_{n+1}^n = A_{\lambda_{n+1}}$.

If $X_{\lambda_{n+1}} \not\subset A_{\lambda_{n+1}}$, then $X_{\lambda_{n+1}} \subset \bigcup_{i=1}^{n} D_i^{n-1}$ and there is such an i_0 that $X_{\lambda_{n+1}} \subset D_{i_0}^{i-1}$. There is, by (2.3), a subbouquet (F_{n+1}, a) of the bouquet $(D_{i_0}^{n-1} \cap Y_{\lambda_{n+1}}, a)$ such that $X_{\lambda_{n+1}} \subset F_{n+1}$, $X_{\lambda_{i_0}} \not\subset F_{n+1}$ and the set $(D_{i_0}^{n-1} \cap Y_{\lambda_{n+1}} - F_{n+1}) \cup (a)$ is compact.

We define $D_i^n = D_i^{n-1}$ for $i \neq i_0, n+1$, and $D_{i_0}^n = (D_{i_0}^{n-1} - F_{n+1}) \cup (a)$ and $D_{n+1}^n = A_{\lambda_{n+1}} \cup F_{n+1}$.

The bouquets (D_i^j, a), where $2 \leqslant j \leqslant n+1$, $i = 1, \ldots, j$, satisfy the conditions $(\mathrm{a})_{n+1}$-$(\mathrm{d})_{n+1}$.

The required bouquets (D_i, a) for $i = 1, \ldots, k$ we define by the formulas

$$B_1 = D_1^{k-1} \cup \Big(\bigcup_{j \neq n_1, \ldots, n_k} A_{\mu_j} - \bigcup_{i=2}^{k} D_i^{k-1} \Big) \quad \text{and} \quad B_j = D_j^{k-1} \quad \text{for} \quad j = 2, \ldots, k.$$

It is easy to verify that the bouquets thus defined satisfy conditions (1)-(3) of the lemma.

(2.5) Lemma. *Let (X, a) be a bouquet and $\mathscr{X} = \{X_\lambda\}_{\lambda \in \Lambda}$ the family of its leaves. If $\mathscr{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ is a family of open subsets of X such that $X_\lambda \subset U_\lambda$ for every $\lambda \in \Lambda$ and $\lambda_1, \ldots, \lambda_k$ is a finite sequence of indexes, then there exist finite sequences $\lambda_{k+1}, \ldots, \lambda_n$ of indexes and $\{(B_i, a)\}_{i=1}^{n}$ of subbouquets of (X, a) such that the following conditions are satisfied:*

(1) *$X_{\lambda_i} \subset B_i \subset U_{\lambda_i}$ for $i = 1, \ldots, n$,*

(2) *$X = \bigcup_{i=1}^{n} B_i$,*

(3) *$B_i \cap B_j = (a)$ for $i \neq j$, $i, j = 1, \ldots, n$.*

Proof. We can assume that $X_{\lambda_j} \not\subset U_{\lambda_i}$ for $i \neq j$, $i, j = 1, \ldots, k$.

There exist, by (2.3), subbouquets (Y_{λ_i}, a), $i = 1, \ldots, k$, of the bouquet (X, a) such that for every $1 \leqslant i \leqslant k$ the set $(X - Y_{\mu_i}) \cup (a)$ is compact and $X_{\lambda_i} \subset Y_{\lambda_i} \subset U_{\lambda_i}$.

We define $B_1 = Y_{\lambda_1}$ and $B_i = (Y_{\lambda_i} - \bigcup_{j=1}^{i-1} Y_{\lambda_j}) \cup (a)$ for $i = 2, \ldots, k$.

Let us write $\hat{X} = (X - \bigcup_{i=1}^{k} B_i) \cup (a)$, $\mathscr{M} = \{\lambda \in \Lambda \colon X_\lambda \not\subset \bigcup_{i=1}^{k} B_i\}$, $V_\mu = U_\mu \cap \hat{X}$ for $\mu \in \mathscr{M}$ and $\mathscr{V} = \{V_\mu\}_{\mu \in \mathscr{M}}$.

The bouquet $(\hat{X}, a)$ and the families $\hat{\mathscr{X}} = \{X_\mu\}_{\mu \in \mathscr{M}}$ and $\mathscr{V}$ satisfy the conditions of Lemma (2.4). Then if $\hat{X} \neq (a)$, there exist finite sequences $\lambda_{k+1}, \ldots, \lambda_n$ of elements of $\mathscr{M}$ and $\{(B_i, a)\}_{i=k+1}^{n}$ of subbouquets of $(\hat{X}, a)$ such that $\hat{X} = \bigcup_{i=k+1}^{n} B_i$, $X_{\lambda_i} \subset B_i \subset V_{\lambda_i} \subset U_{\lambda_i}$ for $i = k+1, \ldots, n$ and $B_i \cap B_j = (a)$ for $i \neq j$, $i, j = k+1, \ldots, n$.

All the bouquets (B_i, a), $i = 1, \ldots, n$, are subbouquets of (X, a) and satisfy conditions (1), (2) and (3).

(2.6) Theorem. *A pointed continuum (X, a) is a bouquet if and only if there is a compact set Z and a family $\mathscr{Z} = \{(Z_\mu, a_\mu)\}_{\mu \in \mathscr{M}}$ of pointed continua such that the following conditions are satisfied:*

(1) *for every $\mu \in \mathscr{M}$ the set Z_μ is a component of Z,*

(2) *for every $\mu \in \mathscr{M}$ the set $Z_\mu - (a_\mu)$ is connected,*

(3) *the set $A = \bigcup_{\mu \in \mathscr{M}} (a_\mu)$ is compact,*

(4) *there exists a continuous map $f\colon (Z, A) \to (X, a)$ such that $f|_{Z-A}$ is a homeomorphism of the set $Z-A$ onto $X-(a)$.*

Proof. First, let us assume that there exist: a compact set Z and a family $\mathscr{Z}$ satisfying conditions (1)-(4) of the theorem.

We define $\Lambda = \{\mu \in \mathscr{M} \colon Z_\mu - (a_\mu) \neq \emptyset\}$ and $X_\mu = f(Z_\mu)$ for every $\mu \in \Lambda$. It is easy to verify that conditions (1)-(3) of definition (1.1) are satisfied.

Now let us suppose that (X, a) is a bouquet and $\mathscr{X} = \{X_\lambda\}_{\lambda \in \Lambda}$ is the family of its leaves.

For $i = 1, 2, \ldots$ let us denote by $\mathscr{A}_i = \{A_1^i, \ldots, A_{k_i}^i\}$ a finite family of closed subsets of X such that

(i) $X = \bigcup_{j=1}^{k_l} A_j^l$ for $l = 1, 2, \ldots$,

(ii) $A_j^l \cap A_{j'}^l = (a)$ for $j \neq j'$, $j, j' = 1, \ldots, k_l$, $l = 1, 2, \ldots$,

(iii) (A_j^l, a) is a subbouquet of (X, a) for $j = 1, \ldots, k_l$, $l = 1, 2, \ldots$

It follows from Lemma (2.5) that we can find the sequence $\mathscr{A}_1, \mathscr{A}_2, \ldots$ in such a manner that the following conditions are satisfied:

(α) for every $A \in \mathscr{A}_i$ there is $B \in \mathscr{A}_{i-1}$ such that $A \subset B$,

(β) for every $A \in \mathscr{A}_i$ there is an index $\lambda(A) \in \Lambda$ such that $A \subset K(X_{\lambda(A)}, 1/i)$.

We denote by $B_1^n, \ldots, B_{k_n}^n$ a sequence of closed subsets of the Hilbert cube Q such that the set B_j^n is homeomorphic to A_j^n for $j = 1, \ldots, k_n$ and $B_j^n \cap B_i^n = \emptyset$ for $i \neq j$, $i, j = 1, \ldots, k_n$.

For every natural n the set $Y_n = \bigcup_{j=1}^{k_n} B_j^n$ is compact and there is a continuous map g_n: $Y_n \to X$ such that $g_n|_{B_j^n}$ is a homeomorphism of B_j^n onto A_j^n for $j = 1, \ldots, k_n$.

It follows from conditions (α) and (i)-(iii) that for every $j \in \{1, \ldots, k_{n+1}\}$ there exists exactly one number $l_j^n \in \{1, \ldots, k_n\}$ such that $A_j^{n+1} \subset A_{l_j}^n$.

The maps p_n^{n+1}: $Y_{n+1} \to Y_n$ and $p_n^{n'}$: $Y_{n'} \to Y_n$ defined by the formulas

$$p_n^{n+1}(x) = (g_n|_{B_j^n})^{-1} g_{n+1}(x) \quad \text{for} \quad x \in B_j^{n+1},\ j = 1, \ldots, k_{n+1}$$

and

$$p_n^{n'} = p_n^{n+1} \ldots p_{n'-1}^{n'} \quad \text{for} \quad n' > n$$

are continuous.

Let us write $Z = \varprojlim \{Y_n, p_{n'}^n, N\}$ and denote by p_n: $Z \to Y_n$ the natural projection and by p_0: $Z \to X$ the map defined by the formula

$$p_0(z) = g_1 p_1(z) \quad \text{for every } z \in Z.$$

Let us observe that if $\lambda, \mu \in \Lambda$ and $\lambda \neq \mu$, then the sets $p_0^{-1}(X_\lambda)$ and $p_0^{-1}(X_\mu)$ lie in different components of the space Z and for every component S of Z the intersection $S \cap p_0^{-1}(a)$ contains only one point.

Let $\mathscr{Z} = \{(Z_\mu, a_\mu)\}_{\mu \in \mathscr{M}}$ be the family of pointed components of Z, where a_μ is the point in $Z_\mu \cap p_0^{-1}(a)$ for $\mu \in \mathscr{M}$, and we can assume that $Z_{\mu'} \neq Z_\mu$ if $\mu' \neq \mu$, $\mu', \mu \in \mathscr{M}$.

The map f: $(Z, A) \to (X, a)$ defined by the formula

$$f(z) = p_0(z) \quad \text{for} \quad z \in Z,$$

where $A = \bigcup_{\mu \in \mathscr{M}} (a_\mu)$, is continuous and satisfies the required conditions. Indeed, the map g: $X-(a) \to Z-A$, where $g(x) = (g_1^{-1}(x), (g_1 p_1^2)^{-1}(x), \ldots)$ for every $x \in X-(a)$, is continuous and it is the inverse of $f|_{Z-A}$.

3. Some properties of bouquets.

(3.1) *If a pointed continuum (X, a) is a bouquet and $\mathscr{X} = \{X_\lambda\}_{\lambda \in \Lambda}$ is the family of its leaves, then* $\dim X = \sup_{\lambda \in \Lambda} \dim X_\lambda$.

Proof. It is clear that if $\sup_{\lambda \in \Lambda} \dim X_\lambda = \infty$, then $\dim X = \infty$.

Let us suppose that there is a natural number n such that $\sup_{\lambda \in \Lambda} \dim X_\lambda = n$.

Let us write $Y_m = X - K(a, 1/m)$ for $m = 1, 2, \ldots$ If $Y_m = \emptyset$, then $\dim X_m = -1 < n$, and if $Y_m \neq \emptyset$, then since the dimension of each component of Y_m is not greater than n, we infer ([8], p. 90) that $\dim Y_m \leqslant n$.

It is clear that $X = (a) \cup \bigcup_{k=1}^{\infty} Y_k$, and so we infer ([8], p. 30) that $\dim X \leqslant n$.

Since X contains an n-dimensional subset, we have $\dim X = n$.

It is easy to observe that

(3.2) *If (X, a) is a bouquet, $\mathscr{X} = \{X_\lambda\}_{\lambda \in \Lambda}$ is the family of its leaves and h: $X \to Y$ is a homeomorphism, then $(Y, h(a))$ is a bouquet and $\mathscr{Y} = \{h(X_\lambda)\}_{\lambda \in \Lambda}$ is the family of its leaves.*

It follows directly from (3.2) that if (X, a) and (Y, b) are homeomorphic bouquets, $\mathscr{X} = \{X_\lambda\}_{\lambda \in \Lambda}$ and $\mathscr{Y} = \{Y_\mu\}_{\mu \in \mathscr{M}}$ are the families of leaves of (X, a) and (Y, b), respectively, then there is a one-to-one map φ: $\mathscr{X} \to \mathscr{Y}$ such that for every $\lambda \in \Lambda$ the pointed continuum (X_λ, a) is homeomorphic to $(\varphi(X_\lambda), b)$.

(3.3) Definition. A bouquet (X, a) is said to be *disperse* if for every leaf X_λ of (X, a) the set $X_\lambda - (a)$ is open in X.

Let us observe that a disperse bouquet has a countable family of leaves.

(3.4) Theorem. *Let (X, a) be a bouquet and $\mathscr{X} = \{X_\lambda\}_{\lambda \in \Lambda}$ the family of its leaves. Then (X, a) is disperse if and only if for every $\varepsilon > 0$ the condition $\delta(X_\lambda) < \varepsilon$ is satisfied for almost all λ.*

Proof. Let us suppose that the bouquet (X, a) is disperse and ε is a positive number. It is clear that there exists a finite sequence $\lambda_1, \ldots, \lambda_k$ of indexes such that $X = K(a, \frac{1}{2}\varepsilon) \cup \bigcup_{l=1}^{k} X_{\lambda_l} - (a)$. Then $K(a, \frac{1}{2}\varepsilon)$ contains almost all leaves X_λ.

Now let us suppose that for every $\varepsilon > 0$ the condition $\delta(X_\lambda) < \varepsilon$ is satisfied for almost all $X_\lambda \in \mathscr{X}$.

Let $X_\lambda \in \mathscr{X}$ be any fixed leaf, $x \in X_\lambda - (a)$ a fixed point and $\varepsilon > 0$ a number such that $\varrho(x, a) > 2\varepsilon$. Let $\lambda_1, \ldots, \lambda_k$ be all indexes in Λ such that $X_{\lambda_j} \not\subset K(a, \varepsilon)$ and $\lambda_j \neq \lambda$ for $j = 1, \ldots, k$. The number $r = \min(\varepsilon, \varrho(x, X_{\lambda_1}), \ldots, \varrho(x, X_{\lambda_k}))$ is greater than 0 and $K(x, r) \cap X_\mu = \emptyset$ for $\mu \neq \lambda$. Then $X_\lambda - (a)$ is a neighborhood of x in X.

Then the bouquet (X, a) is disperse.

(3.5) Theorem. *If (X, a) and (Y, b) are disperse bouquets and $\mathscr{X} = \{X_\lambda\}_{\lambda \in \Lambda}$, $\mathscr{Y} = \{Y_\mu\}_{\mu \in \mathscr{M}}$ are the families of leaves of (X, a) and (Y, b), respectively, then (X, a) is homeomorphic to (Y, b) if and only if there exists a one-to-one map φ: $\Lambda \to \mathscr{M}$ such that for every $\lambda \in \Lambda$ the pointed continua (X_λ, a) and $(Y_{\varphi(\lambda)}, b)$ are homeomorphic.*

Proof. Let us assume that there exists a one-to-one map φ: $\Lambda \to \mathscr{M}$ such that for every $\lambda \in \Lambda$ there is a homeomorphism h_λ: $(X_\lambda, a) \to (Y_{\varphi(\lambda)}, b)$.

It is easy to verify that the map $h\colon (X, a) \to (Y, b)$ defined by the formula

$$h(x) = h_\lambda(x) \quad \text{for every } x \in X_\lambda \text{ and } \lambda \in \Lambda$$

is a homeomorphism.

The second part of the theorem we obtain from (3.2)

(3.6) *If (X, a) is a disperse bouquet, then each leaf of (X, a) is a retract of the space X.*

Proof. Let X_λ be any leaf of (X, a). Since the set $X-(X_\lambda-(a))$ is closed in X, the formula

$$r_\lambda(x) = \begin{cases} x & \text{if } x \in X_\lambda, \\ a & \text{if } x \in (X-X_\lambda) \cup (a) \end{cases}$$

defines the retraction $r_\lambda\colon X \to X_\lambda$.

(3.7) *If (X, a) is a disperse bouquet, then X is locally connected if and only if each leaf of (X, a) is locally connected.*

Proof. Let $\mathscr{X} = \{X_\lambda\}_{\lambda \in \Lambda}$ be the family of leaves of (X, a).

If X is locally connected, then it follows from (3.6) that for every $\lambda \in \Lambda$ the leaf X_λ has the same property.

Now let us assume that each leaf of (X, a) is locally connected.

If $x \in X-(a)$ is any point, then $x \in X_\lambda-(a)$ for some $\lambda \in \Lambda$. Since the set $X_\lambda-(a)$ is open in X_λ and in X and X_λ is locally connected, we infer that X is locally connected at the point x.

Let ε be any positive number and let $\lambda_1, \ldots, \lambda_k$ be all the indexes such that $X_{\lambda_i} \not\subset K(a, \varepsilon)$. There are connected sets $U_1, \ldots, U_k$ such that for every $1 \leqslant i \leqslant k$ the set U_i is a neighborhood of a in X_{λ_i} and $U_i \subset K(a, \varepsilon)$.

The set $U = \bigcup_{i=1}^{k} U_i \cup \bigcup_{\lambda \neq \lambda_1, \ldots, \lambda_k} X_\lambda$ is a connected neighborhood of a in X contained in $K(a, \varepsilon)$. Thus X is also locally connected at a.

(3.8) *If $(\underline{Y, a}) = \{(Y_n, a), p_n^{n'}, N\}$ is an inverse sequence of finite bouquets lying in the Hilbert cube and if, for every natural n, $Y_n \subset Y_{n+1}$ and $p_n^{n+1}\colon Y_{n+1} \to Y_n$ is a retraction such that $p_n^{n+1}(Y_{n+1}-Y_n) = (a)$, then the inverse limit $(Y, \hat{a})$ of $(\underline{Y, a})$ is a disperse bouquet and there exists a one-to-one map φ of the family $\mathscr{Y}$ of leaves of $(Y, \hat{a})$ onto the sum $\bigcup_{i=1}^{\infty} \mathscr{Y}_n$ of the families of leaves of (Y_n, a), for $n = 1, 2, \ldots$, such that every leaf $Z \in \mathscr{Y}$ is homeomorphic to $\varphi(Z)$.*

Proof. We can assume that $\mathscr{Y}_n = \{X_1, \ldots, X_n\}$.

Let $i_n^{n'}\colon (Y_n, a) \to (Y_{n'}, a)$, for $n' > n$, be the inclusion, and $i_n\colon (Y_n, a) \to (Y, \hat{a})$ the map defined by the formula:

$$i_n(y) = \begin{cases} (\underbrace{a, \ldots, a}_{n-1}, y, y, \ldots) & \text{for } y \in Y_n - Y_{n-1} \\ i_{n-1}(y) & \text{for } y \in Y_{n-1} \end{cases} \quad \text{for } n > 1$$

$$i_1(y) = (y, y, \ldots) \quad \text{for} \quad y \in Y_1$$

and let $p_n\colon (Y, \hat{a}) \to (Y_n, a)$ be the projection of the inverse limit $(n \geqslant 1)$.

It is clear that for every natural n the set $Z_n = i_n(X_n)$ is homeomorphic to X_n.

Let $y = (y_1, y_2, \ldots) \in Y$ be an arbitrary point.

If $y \neq \hat{a} = (a, a, \ldots)$, then there is an n_0 such that $y_n = y_{n_0} \neq a$ for $n \geqslant n_0$ and $y_n = a$ for $n < n_0$. Thus $y_{n_0} \in X_{n_0}$ and $y = i_{n_0}(y_{n_0}) \in \bigcup_{n=1}^{\infty} Z_n$. If $y = \hat{a}$, then evidently $y \in \bigcup_{n=1}^{\infty} Z_n$.

Then $Y = \bigcup_{n=1}^{\infty} Z_n$.

Since the set $Z_n-(\hat{a})$ is homeomorphic to $X_n-(a)$, it is nonempty and connected.

Let n and m be fixed natural numbers and $n > m$. The sets $C_m = \bigcup_{j=1}^{n-1} Z_j$ and $C_n = \bigcup_{j=n}^{\infty} Z_j$ are continua (C_n is homeomorphic to the inverse limit of the inverse sequence $\{(X_n \cup \ldots \cup X_{n+k}, a), p_k^{k'}|_{(X_n \cup \ldots \cup X_{n+k'}, a)}, N\}$). Since, moreover, $Z_n \subset C_n$, $Z_m \subset C_m$, $Y = C_n \cup C_m$ and $C_n \cap C_m = (\hat{a})$, we infer that $(Y, \hat{a})$ is a bouquet and the sets Z_n, for $n \in N$, are leaves of $(Y, \hat{a})$.

The bouquet $(Y, \hat{a})$ is disperse, because for every Z_n the set $Z_n-(\hat{a}) = Y-(\bigcup_{j=1}^{n-1} Z_j \cup \bigcup_{j=n+1}^{\infty} Z_j)$ is an open subset of Y. The function $\varphi\colon \mathscr{Y} \to \bigcup_{n=1}^{\infty} \mathscr{Y}_n$ defined by $\varphi(Z_n) = X_n$ satisfies the required conditions.

(3.9) *A leaf of a bouquet (X, a) is a retract of the space X if and only if it is a neighborhood retract of X.*

Proof. If the leaf X_λ is a retract of X, then it is evidently a neighborhood retract of X.

Now let us assume that there are: a neighborhood U of X_λ in X and a retraction $\hat{r}\colon U \to X_\lambda$. Let (B, a) be a subbouquet of (X, a) satisfying the conditions of Lemma (2.3). The formula

$$r(x) = \begin{cases} \hat{r}(x) & \text{if } x \in B, \\ a & \text{if } x \in (X-B) \cup (a) \end{cases}$$

defines a retraction $r\colon X \to X_\lambda$.

It follows from (3.9) that

(3.10) *If $X_\lambda \in \mathrm{ANR}$ is a leaf of a bouquet (X, a), then it is a retract of the space X.*

(3.11) Example. Let us write

$$P_0 = \{(x, y, z) \in E^3\colon z = 0\},$$
$$P_n = \{(x, y, z) \in E^3\colon x+y-nz = 0\} \text{ for } n = 1, 2, \ldots,$$
$$I = \{(x, y, z) \in E^3\colon x = 0 \text{ and } 0 \leqslant y \leqslant 2\},$$
$$A_0 = \left\{(x, y, z) \in E^3\colon 0 < x \leqslant 1 \text{ and } y = \sin\frac{1}{x}\right\},$$

$$A_n = \left\{(x, y, z) \in E^3 \colon \frac{1}{n\pi} \leqslant x \leqslant 1 \text{ and } y = 1+\sin\frac{1}{x}\right\} \text{ for } n = 1, 2, \ldots,$$

$$B_n = \left\{(x, y, z) \in E^3 \colon y = 1 \text{ and } 0 \leqslant x \leqslant \frac{1}{n\pi}\right\} \text{ for } n = 1, 2, \ldots,$$

$$Y_n = I \cup A_n \cup B_n \text{ for } n = 1, 2, \ldots,\quad Y_0 = I \cup A_0,\quad X_n = P_n \cap Y_n \text{ for } n = 1, 2, \ldots,$$

$$X = \bigcup_{n=1}^{\infty} X_n.$$

It is easy to verify that the pointed continuum $(X, (0, 0, 0))$ is a bouquet, $\mathscr{X} = \{X_n\}_{n=0}^{\infty}$ is the family of its leaves and the leaf X_0 is not a retract of the space X.

4. On movability of bouquets. K. Borsuk introduced ([4], p. 223) an important notion of movability of pointed compacta.

(4.1) DEFINITION ([4], p. 223). A pointed compactum (X, x_0) lying in the pointed Hilbert cube (Q, x_0) is said to be *movable* if for every neighborhood U of X there is a neighborhood V of X such that for every neighborhood W of X there is a homotopy $\varphi\colon V\times\langle 0, 1\rangle \to U$ such that $\varphi(x, 0) = 1$ and $\varphi(x, 1) \in W$ for every point $x \in V$ and $\varphi(x_0, t) = x_0$ for every $0 \leqslant t \leqslant 1$.

The following question arises: what is the relation between the movability of a bouquet and the movability of its leaves? In order to answer it, let us observe that:

(4.2) *If X is a closed subset of the Hilbert cube Q, $x_0 \in X$ is a fixed point and for every $\varepsilon > 0$ there is a closed subset A_ε of X containing x_0 such that (A_ε, x_0) is movable and there is a continuous map $f_\varepsilon\colon (X, x_0) \to (A_\varepsilon, x_0)$ satisfying the condition $\varrho(f_\varepsilon(x), x) < \varepsilon$ for every $x \in X$, then (X, x_0) is movable.*

(4.3) LEMMA. *Let (X, a) be a bouquet and $\mathscr{X} = \{X_\lambda\}_{\lambda\in\Lambda}$ the family of its leaves. If for every $\lambda \in \Lambda$ the leaf X_λ is a retract of X, then for every $\varepsilon > 0$ there is a finite sequence $X_{\lambda_1}, \ldots, X_{\lambda_n}$ of leaves and there is a retraction $r_\varepsilon\colon X \to \bigcup_{i=1}^{n} X_{\lambda_i}$ satisfying the condition $\varrho(r_\varepsilon(x), x) < \varepsilon$ for every $x \in X$.*

Proof. For every $\lambda \in \Lambda$ there is a neighborhood U_λ of X_λ in X and a retraction $q_\lambda\colon U_\lambda \to X_\lambda$ such that $\varrho(q_\lambda(x), x) < \varepsilon$ for every $x \in U_\lambda$.

There are finite sequences $X_{\lambda_1}, \ldots, X_{\lambda_n}$ of leaves and $\{(B_i, a)\}_{i=1}^{n}$ of subbouquets of (X, a) satisfying conditions (1)-(3) of Lemma (2.4).

The map $r_\varepsilon\colon X \to \bigcup_{i=1}^{n} X_{\lambda_i}$ defined by the formula

$$r_\varepsilon(x) = q_{\lambda_i}(x) \quad \text{for} \quad x \in B_i,\ i = 1, \ldots, n$$

is the required retraction.

From Lemmas (4.2), (4.3) and Borsuk's theorem ([4], p. 224, Th. (2.8)) results the following

(4.4) THEOREM. *Let (X, a) be a bouquet and $\mathscr{X} = \{X_\lambda\}_{\lambda\in\Lambda}$ the family of its leaves. If for every $\lambda \in \Lambda$ the leaf X_λ is a retract of X and (X_λ, a) is movable, then (X, a) is also movable.*

Particularly we get

(4.5) COROLLARY. *If (X, a) is a bouquet and each of its leaves is an* ANR-set, *then (X, a) is movable.*

The movability of a bouquet does not imply the movability of its leaves.

(4.6) EXAMPLE. Let $S_1, S_2, \ldots$ be a sequence of 1-spheres such that $S_i \cap S_j = (a)$ for $i \neq j$, $i, j = 1, 2, \ldots$ and let $\alpha_n^{n+1}\colon (S_{n+1}, a) \to (S_n, a)$, $n = 1, 2, \ldots$ be continuous maps with the degree 2. Let $X_n = S_1 \cup \ldots \cup S_n$, $n = 1, 2, \ldots$, and $p_n^{n+1}\colon (X_{n+1}, a) \to (X_n, a)$ be the continuous map defined by the formula

$$p_n^{n+1}(x) = \begin{cases} \alpha_n^{n+1}(x) & \text{if} \quad x \in S_{n+1}, \\ x & \text{if} \quad x \in S_1 \cup \ldots \cup S_n. \end{cases}$$

The pointed continuum $(X, \hat{a}) = \varprojlim\{(X_n, a), p_n^{n'}, N\}$ is movable ([11], pp. 250-252).

It is easy to verify that $(X, \hat{a})$ is a bouquet and the set $S = \varprojlim\{S_n, \alpha_n^{n'}, N\}$ is one of the leaves of $(X, \hat{a})$.

It is known ([3], p. 138) that S is not movable.

(4.7) THEOREM. *Let (X, a) be a bouquet and $\mathscr{X} = \{X_\lambda\}_{\lambda\in\Lambda}$ the family of its leaves. Then $\mathrm{Sh}(X, a) = \mathrm{Sh}(a, a)$ if and only if $\mathrm{Sh}(X_\lambda, a) = \mathrm{Sh}(a, a)$ for every $\lambda \in \Lambda$.*

Proof. It is known ([6], Chapter VIII, Corollaries (4.6) and (4.7)) and comp. ([2], pp. 72–73, Th. (9.1) and (9.8)) that the following conditions are equivalent:

(1) X is an FAR-set,

(2) for every $a \in X$ and for every neighborhood U of X in Q there is a continuous map $\varphi\colon X\times\langle 0, 1\rangle \to U$ such that $\varphi(x, 0) = x$ and $\varphi(x, 1) = a$ for every $x \in X$,

(3) for every $a \in X$ and every neighborhood U of X in Q there are: a neighborhood V of X in Q and a continuous map $\varphi\colon V\times\langle 0, 1\rangle \to U$ such that $\varphi(x, 0) = x$, $\varphi(x, 1) = a$ for every $x \in V$ and $\varphi(a, t) = a$ for every $t \in \langle 0, 1\rangle$,

(4) $\mathrm{Sh}(X, a) = \mathrm{Sh}(a, a)$.

Let us assume that $\mathrm{Sh}(X, a) = \mathrm{Sh}(a, a)$. Let X_λ be a fixed leaf and let W be an open neighborhood of X_λ in Q. The set $U = W \cap X$ is a neighborhood of X_λ in X. There is subbouquet (Y, a) of (X, a) satisfying the conditions of Lemma (2.3).

The map $r\colon X \to Y$ defined by the formula

$$r(x) = \begin{cases} x & \text{if} \quad x \in Y, \\ a & \text{if} \quad x \in (X - Y) \cup (a) \end{cases}$$

is a retraction.

Then the set Y is an FAR-set and there is a continuous map $\psi\colon Y\times\langle 0, 1\rangle \to U$ such that $\psi(x, 0) = x$, $\psi(x, 1) = a$ for every $x \in Y$ and $\psi(a, t) = a$ for every $t \in \langle 0, 1\rangle$.

The map $\varphi\colon X_\lambda\times\langle 0, 1\rangle \to U$ defined by the formula

$$\varphi(x, t) = \psi(x, t) \quad \text{for every } x \in X_\lambda \text{ and } t \in \langle 0, 1\rangle.$$

is continuous and condition (2) is satisfied.

Then $\mathrm{Sh}(X_\lambda, a) = \mathrm{Sh}(a, a)$.

Now let us assume that $\mathrm{Sh}(X_\lambda, a) = \mathrm{Sh}(a, a)$ for every $\lambda \in \Lambda$ and let U be an arbitrary neighborhood of X in Q.

For every $\lambda \in \Lambda$ there are: a neighborhood U_λ of X_λ in Q and a continuous map φ_λ: $U_\lambda \times \langle 0, 1\rangle \to U$ such that $\varphi_\lambda(x, 0) = x$, $\varphi_\lambda(x, 1) = a$ for every $x \in U_\lambda$ and $\varphi_\lambda(a, t) = a$ for every $t \in \langle 0, 1\rangle$.

There are finite sequences: $\lambda_1, \ldots, \lambda_n$ of indexes and $\{(B_i, a)\}_{i=1}^n$ of subbouquets of (X, a) satisfying conditions (1)-(3) of (2.4).

The map φ: $X \times \langle 0, 1\rangle \to U$ defined by the formula

$$\varphi(x, t) = \varphi_{\lambda_i}(x, t) \quad \text{if} \quad x \in B_i,\ t \in \langle 0, 1\rangle, \quad i = 1, \ldots, n$$

is continuous and satisfies the conditions: $\varphi(x, 0) = x$, $\varphi(x, 1) = a$ for every $x \in X$ and $\varphi(a, t) = a$ for every $t \in \langle 0, 1\rangle$.

Then condition (2) is satisfied and $\mathrm{Sh}(X, a) = \mathrm{Sh}(a, a)$.

5. Bouquets and inverse sequences of finite bouquets. It is convenient in the theory of shape to consider continua as inverse limits of inverse ANR-sequences ([9], pp. 41–42).

We shall prove that some bouquets are homeomorphic to inverse limits of inverse sequences of finite bouquets.

(5.1) LEMMA. *Let (X, a) be a bouquet and $\mathscr{X} = \{X_\lambda\}_{\lambda \in \Lambda}$ the family of its leaves. If for every $\lambda \in \Lambda$ the leaf X_λ is a retract of X, then there are sequences $\lambda_1, \lambda_2, \ldots$ of indexes and $k_1 \leqslant k_2 \leqslant \ldots$ of natural numbers such that for every n there exists a retraction r_n: $X \to \bigcup_{i=1}^{k_n} X_{\lambda_i}$ satisfying the condition $\varrho(x, r_n(x)) < 1/n$ for every $x \in X$.*

Proof. If q_λ: $X \to X_\lambda$, for $\lambda \in \Lambda$, denotes a retraction, then there are neighborhoods W_λ^1, for $\lambda \in \Lambda$, such that $\varrho(q_\lambda(x), x) < 1$ for every $x \in W_\lambda^1$.

There exist finite sequences $\lambda_1, \ldots, \lambda_{k_1}$ of indexes and $\{(B_i^1, a)\}_{i=1}^{k_1}$ of subbouquets of (X, a) satisfying conditions (1)-(3) of (2.4).

The formula

$$r_1(x) = q_{\lambda_i}(x) \quad \text{for every } x \in B_i^1 \text{ and } i = 1, \ldots, k_1$$

defines a retraction r_1: $X \to \bigcup_{i=1}^{k_1} X_{\lambda_i}$.

Let as assume that we have defined natural numbers $k_1 \leqslant \ldots \leqslant k_{n-1}$, leaves $X_{\lambda_1}, \ldots, X_{\lambda_{k_{n-1}}}$ and retractions r_i: $X \to \bigcup_{j=1}^{k_i} X_{\lambda_j}$, $i = 1, \ldots, n-1$.

Let W_λ^n, for $\lambda \in \Lambda$, be a neighborhood of X_λ in X such that $\varrho(q_\lambda(x), x) < 1/n$ for every point $x \in W_\lambda^n$.

If $X - \bigcup_{j=1}^{k_{n-1}} X_{\lambda_j} \neq \emptyset$, then there are: a natural number $k_n \geqslant k_{n-1}$ and sequences $\lambda_{k_{n-1}+1}, \ldots, \lambda_{k_n}$ of indexes and $\{(B_i^n, a)\}_{i=1}^{k_n}$ of subbouquets of (X, a) satisfying conditions (1)-(3) of (2.5).

The formula

$$r_n(x) = q_{\lambda_i}(x) \quad \text{for} \quad x \in B_i^n,\ i = 1, \ldots, k_n,$$

defines the required retraction r_n: $X \to \bigcup_{i=1}^{k_n} X_{\lambda_i}$.

(5.2) LEMMA. *If X is a continuum, $A_1 \subset A_2 \subset \ldots$ is a sequence of subsets of X such that for every n there is a retraction r_n: $X \to A_n$ satisfying the condition $\varrho(r_n(x), x) < 1/n$ for every $x \in X$, then there is a sequence $k_1 < k_2 < \ldots$ of natural numbers such that the space X is homeomorphic to $\varprojlim\{Y_n, p_n^{n'}, N\}$, where $Y_n = A_{k_n}$ and $p_n^{n+1} = r_{k_n}|_{A_{k_{n+1}}}$.*

Proof. We define $k_1 = 1$.

Let $k_2 \geqslant 4$ be a natural number such that

$$\varrho(r_1 r_{k_2}(x), r_1(x)) < 2^{-2} \quad \text{for every } x \in X.$$

Let us assume that we have defined a sequence $k_1 < k_2 < \ldots < k_{n-1}$ of natural numbers such that $k_i \geqslant i^2$ and

$$\varrho(r_{k_i} \ldots r_{k_{j-1}} r_{k_j}(x), r_{k_i} \ldots r_{k_{j-1}}(x)) < j^{-2} \quad \text{for every } x \in X \text{ and } i = 1, 2, \ldots, n-1.$$

There is a natural number $k_n > \max(k_{n-1}, n^2)$ such that

$$\varrho(r_{k_i} \ldots r_{k_{n-1}} r_{k_n}(x), r_{k_j} \ldots r_{k_{n-1}}(x)) < n^{-2} \quad \text{for every } x \in X \text{ and } j = 1, \ldots, n-1.$$

We shall prove that X is homeomorphic to $Y = \varprojlim\{Y_n, p_n^{n'}, N\}$, where $Y_n = A_{k_n}$, $p_n^{n+1} = q_n|_{Y_{n+1}}$ and $q_n = r_{k_n}$ for $n = 1, 2, \ldots$

We define the metric $\hat{\varrho}$ in $\mathop{\boldsymbol{P}}_{n=1}^{\infty} Y_n$ by the formula

$$\hat{\varrho}(x, y) = \sum_{n=1}^{\infty} 2^{-n} \varrho(x_n, y_n) \quad \text{for every } x = (x_n)_{n=1}^{\infty},\ y = (y_n)_{n=1}^{\infty} \in \mathop{\boldsymbol{P}}_{n=1}^{\infty} Y_n.$$

The maps h_n: $X \to \mathop{\boldsymbol{P}}_{k=1}^{\infty} Y_k$, for $n = 1, 2, \ldots$, defined by the formulas

$$h_n(x) = (q_1 \ldots q_{n-1} q_n(x), q_2 \ldots q_{n-1} q_n(x), \ldots, q_{n-1} q_n(x), q_n(x), q_{n+1}(x), \ldots)$$

are continuous.

Since

$$\hat{\varrho}(h_n(x), h_{n+1}(x)) < n^{-2} \quad \text{for every } x \in X \text{ and } n = 1, 2, \ldots,$$

there exists a continuous map h: $X \to \mathop{\boldsymbol{P}}_{k=1}^{\infty} Y_k$ such that $h = \lim_{n\to\infty} h_n$.

We shall prove that h leads X homeomorphically onto Y.

If $x \in X$, then $h(x) = (x_1, x_2, \ldots)$, where $x_k = \lim_{n\to\infty} q_k \ldots q_n(x)$; hence

$$p_k^{k+1}(x_{k+1}) = p_k^{k+1}(\lim_{n\to\infty} q_{k+1} \ldots q_n(x)) = \lim_{n\to\infty} p_k^{k+1}(q_{k+1} \ldots q_n(x)) = x_k,$$

and thus $h(x) \in Y$.

If $y \in Y$, then $y = (y_1, y_2, \ldots)$ and $y_n = p_n^{n+1}(y_{n+1})$ for $n = 1, 2, \ldots$ There is a $y_0 \in X$ such that $y_0 = \lim_{n\to\infty} y_n$, because $\varrho(y_n, y_{n+1}) < n^{-2}$ for every n and X is compact.

The formula

$$g(y_1, y_2, \ldots) = \lim_{n\to\infty} y_n \quad \text{for every } (y_1, y_2, \ldots) \in Y$$

defines the map $g\colon Y \to X$.

It is easy to verify that

$$gh(x) = x \quad \text{for every } x \in X,$$
$$hg(y) = y \quad \text{for every } y \in Y.$$

Then the continuous map $h\colon X \to Y$ is a homeomorphism of X onto Y.

Lemmas (5.1) and (5.2) directly imply

(5.3) THEOREM. *Let (X, a) be a bouquet and $\mathscr{X} = \{X_\lambda\}_{\lambda\in\Lambda}$ the family of its leaves. If for every $\lambda \in \Lambda$ the leaf X_λ is a retract of X, then the bouquet (X, a) is homeomorphic to $(Y, \hat{a}) = \varprojlim\{(Y_n, a), p_n^{n'}, N\}$, where for every n a pointed continuum (Y_n, a) is a finite subbouquet of (X, a), $Y_n \subset Y_{n+1} \subset X$ and p_n^{n+1} is a retraction.*

(5.4) COROLLARY. *If (X, a) is a bouquet and each of its leaves is an ANR-set, then there is an inverse sequence $(\underline{Y, a}) = \{(Y_n, a), p_n^{n'}, N\}$ of finite subbouquets of (X, a) such that $Y_n \subset Y_{n+1}$ for every n, all maps $p_n^{n'}$ are retractions and (X, a) is homeomorphic to $(\varprojlim \underline{Y, a})$.*

6. Spherical bouquets.

(6.1) DEFINITION. A bouquet (X, a) is said to be *spherical* if for every leaf X_λ of (X, a) there is an n_λ such that X_λ is homeomorphic to a subset $S^{n_\lambda} = \{x \in E^{n_\lambda+1}\colon |x| = 1\}$ of the Euclidean space $E^{n_\lambda+1}$.

Corollary (4.5) implies

(6.2) THEOREM. *Every spherical bouquet is a movable pointed continuum.*

(6.3) DEFINITION. Let (X, a) be a spherical bouquet and $\mathscr{X} = \{X_\lambda\}_{\lambda\in\Lambda}$ the family of leaves. The sequence $\underline{n} = (n_1, n_2, \ldots)$ (it may be finite) of all dimensions of leaves of (X, a) is said to be the *type* of (X, a).

By the *character* of (X, a) we understand the sequence $\underline{\mathfrak{M}} = (\mathfrak{M}_1, \mathfrak{M}_2, \ldots)$ of cardinal numbers such that for every i there exist in $\mathscr{X}$ exactly $\mathfrak{M}_i$ n_i-dimensionals leaves.

(6.4) DEFINITION. If all leaves of the bouquet (X, a) are n-spheres, then (X, a) is said to be a *usual n-dimensional spherical bouquet*.

Theorem (3.5) and (3.7) imply

(6.5) THEOREM. *Two locally connected spherical bouquets are homeomorphic if and only if they have the same type and character.*

(6.6) THEOREM. *If (X, a) and (Y, b) are locally connected spherical bouquets, then $\mathrm{Sh}(X, a) = \mathrm{Sh}(Y, b)$ if and only if (X, a) and (Y, b) are homeomorphic.*

Proof. If (X, a) and (Y, b) are homeomorphic, then $\mathrm{Sh}(X, a) = \mathrm{Sh}(Y, b)$. If $\mathrm{Sh}(X, a) = \mathrm{Sh}(Y, b)$, then X and Y have the same Betti numbers and then they have the same type and character. It follows from Theorem (6.5) that (X, a) and (Y, b) are homeomorphic.

(6.7) THEOREM. *Every spherical bouquet has the shape of some locally connected spherical bouquet.*

Proof. Let (X, a) be a spherical bouquet and $\mathscr{X}$ the family of its leaves. If $\mathscr{X}$ is finite, then X is locally connected.

Let us assume that $\mathscr{X}$ is infinite. There exists, by Corollary (5.4), an inverse sequence $(\underline{X, a}) = \{(A_n, a), p_n^{n'}, N\}$ of finite subbouquets of (X, a) such that $A_n \subset A_{n+1}$ for every $n \in N$ and all the maps $p_n^{n'}$ are retractions.

It is easy to observe that we can assume that $A_n \neq A_{n+1}$ and $A_n = S_1 \cup \ldots \cup S_n$ for $n = 1, 2, \ldots$, where $S_i \in \mathscr{X}$, $i = 1, 2, \ldots$

We define the inverse sequence $(\underline{Y, a}) = \{(Y_n, a), q_n^{n'}, N\}$ by the formulas

$$Y_n = A_n,$$

$$q_n^{n+1}(x) = \begin{cases} x & \text{if} \quad x \in Y_n, \\ a & \text{if} \quad x \in (Y_{n+1} - Y_n) \cup (a), \end{cases}$$

and $q_n^{n'} = q_n^{n+1} \ldots q_{n'-1}^{n'}$ for $n' > n$.

It follows from (3.8) that the pointed continuum $(Y, \hat{a}) = \mathrm{Lim}(\underline{Y, a})$ is a locally connected spherical bouquet.

We shall prove that $\mathrm{Sh}(X, a) = \mathrm{Sh}(Y, \hat{a})$. For this purpose we shall define sequences of continuous maps

$$f_n\colon (S_1 \cup \ldots \cup S_n, a) \to (S_1 \cup \ldots \cup S_n, a),$$
$$g_n\colon (S_1 \cup \ldots \cup S_n, a) \to (S_1 \cup \ldots \cup S_n, a)$$

satisfying the conditions

$(1)_n$ $q_n^{n+1} f_{n+1} \simeq f_n p_n^{n+1}$ in $(S_1 \cup \ldots \cup S_n, a)$,

$(2)_n$ $p_n^{n+1} g_{n+1} \simeq g_n q_n^{n+1}$ in $(S_1 \cup \ldots \cup S_n, a)$,

$(3)_n$ $g_n f_n \simeq \mathrm{id}_{(S_1\cup\ldots\cup S_n, a)}$ in $(S_1 \cup \ldots \cup S_n, a)$,

$(4)_n$ $f_n g_n \simeq \mathrm{id}_{(S_1\cup\ldots\cup S_n, a)}$ in $(S_1 \cup \ldots \cup S_n, a)$.

We define $f_1(x) = x$ for every $x \in S_1$.

Let us assume that we have defined maps $f_j\colon (S_1 \cup \ldots \cup S_j, a) \to (S_1 \cup \ldots \cup S_j, a)$ satisfying the conditions $(1)_j$, for $j = 1, \ldots, n-1$. Let us denote by $i_n\colon (S_n, a) \to (S_1 \cup \ldots \cup S_n, a)$ and by $i\colon (S_1 \cup \ldots \cup S_{n-1}, a) \to (S_1 \cup \ldots \cup S_n, a)$ the inclusions, and by $\alpha\colon (S_n, a) \to (S_1 \cup \ldots \cup S_n, a)$ a continuous map such that if we denote by $[\alpha] \in \pi_l(S_1 \cup \ldots \cup S_n, a)$, where $\dim S_n = l$, the element of the homotopy group generated by α, then $[\alpha] = [if_{n-1} p_{n-1}^n i_n] \circ [i_n]$.

We define

$$f_n(x) = \begin{cases} f_{n-1}(x) & \text{if} \quad x \in S_1 \cup \ldots \cup S_{n-1}, \\ \alpha(x) & \text{if} \quad x \in S_n. \end{cases}$$

Since

$$[q^n_{n-1} f_n i_n] = [q^n_{n-1}\alpha] = [q^n_{n-1} i f_{n-1} p^n_{n-1} i_n] \circ [q^n_{n-1} i_n] = [f_{n-1} p^n_{n-1} i_n],$$

there is a homotopy $\gamma_n\colon S_n \times \langle 0,1\rangle \to (S_1 \cup \ldots \cup S_{n-1})$ joining the maps $f_{n-1} p^n_{n-1}|_{(S_n,a)}$ and $q^n_{n-1} f_n|_{(S_n,a)}$ in $(S_1 \cup \ldots \cup S_{n-1}, a)$.

The homotopy $\varphi_n\colon (S_1 \cup \ldots \cup S_n)\times\langle 0,1\rangle \to S_1 \cup \ldots \cup S_{n-1}$ defined by the formula

$$\varphi_n(x,t) = \begin{cases} f_{n-1}(x) & \text{for every } x \in S_1 \cup \ldots \cup S_{n-1},\ 0 \leqslant t \leqslant 1, \\ \gamma_n(x,t) & \text{for every } x \in S_n \text{ and } 0 \leqslant t \leqslant 1 \end{cases}$$

joins the maps $f_{n-1} p^n_{n-1}$ and $q^n_{n-1} f_n$ in $(S_1 \cup \ldots \cup S_{n-1}, a)$.

Let $g_1(x) = x$ for every $x \in S_1$.

It is clear that $f_1 g_1(x) = x$ and $g_1(x) = x$ for every $x \in S_1$.

We define the maps $\lambda_1, \mu_1\colon S_1 \times \langle 0,1\rangle \to S_1$ by the formulas

$$\lambda_1(x,t) = \mu_1(x,t) = x \quad \text{for every } x \in S_1 \text{ and } 0 \leqslant t \leqslant 1.$$

Let us assume that the following continuous maps are defined:

$$g_j\colon (S_1 \cup \ldots \cup S_j, a) \to (S_1 \cup \ldots \cup S_j, a),$$
$$\lambda_j, \mu_j\colon (S_1 \cup \ldots \cup S_j)\times\langle 0,1\rangle \to S_1 \cup \ldots \cup S_j$$

or $j = 1, \ldots, n-1$, satisfying the conditions $\lambda_j(x,0) = f_j g_j(x)$, $\lambda_j(x,1) = x$, $\mu_j(x,0) = g_j f_j(x)$, $\mu_j(x,1) = x$ for every $x \in S_1 \cup \ldots \cup S_j$ and $\lambda_j(a,t) = \mu_j(a,t) = a$ for every $0 \leqslant t \leqslant 1$.

Let $\beta\colon (S_n, a) \to (S_1 \cup \ldots \cup S_n, a)$ be a continuous map such that

$$[\beta] = [i p^n_{n-1} i_n]^{-1} \circ [i_n].$$

We define the map $g_n\colon (S_1 \cup \ldots \cup S_n, a) \to (S_1 \cup \ldots \cup S_n, a)$ by the formula

$$g_n(x) = \begin{cases} g_{n-1}(x) & \text{if } x \in S_1 \cup \ldots \cup S_{n-1}, \\ \beta(x) & \text{if } x \in S_n. \end{cases}$$

It is easy to verify that $[p^n_{n-1} g_n i_n] = [g_{n-1} q^n_{n-1} i_n]$. Then there is a homotopy $\hat{\gamma}_n\colon S_n \times \langle 0,1\rangle \to S_1 \cup \ldots \cup S_{n-1}$ joining the maps $p^n_{n-1} g_n|_{(S_n,a)}$ and $g_{n-1} q^n_{n-1}|_{(S_n,a)}$ in $(S_1 \cup \ldots \cup S_{n-1}, a)$.

The map $\psi_n\colon (S_1 \cup \ldots \cup S_n)\times\langle 0,1\rangle \to S_1 \cup \ldots \cup S_{n-1}$ defined by the formula

$$\psi_n(x,t) = \begin{cases} g_{n-1}(x) & \text{if } x \in S_1 \cup \ldots \cup S_{n-1},\ 0 \leqslant t \leqslant 1, \\ \hat{\gamma}_n(x,t) & \text{if } x \in S_n \text{ and } 0 \leqslant t \leqslant 1 \end{cases}$$

is a homotopy joining the maps $p^n_{n-1} g_n$ and $g_{n-1} q^n_{n-1}$ in $(S_1 \cup \ldots \cup S_{n-1}, a)$.

Since $[f_n g_n i_n] = [i_n]$ and $[g_n f_n i_n] = [i_n]$, there exist homotopies $\hat{\lambda}_n\colon S_n \times \langle 0,1\rangle \to S_1 \cup \ldots \cup S_n$ and $\hat{\mu}_n\colon S_n \times \langle 0,1\rangle \to S_1 \cup \ldots \cup S_n$ such that $\hat{\lambda}_n(x,0) = f_n g_n(x)$, $\hat{\mu}_n(x,0) = g_n f_n(x)$, $\hat{\lambda}_n(x,1) = x$, $\hat{\mu}_n(x,1) = x$ for every $x \in S_n$ and $\hat{\lambda}_n(a,t) = \hat{\mu}_n(a,t) = a$ for every $0 \leqslant t \leqslant 1$.

The maps $\lambda_n, \mu_n\colon (S_1 \cup \ldots \cup S_n)\times\langle 0,1\rangle \to S_1 \cup \ldots \cup S_n$ we define by the formulas

$$\lambda_n(x,t) = \begin{cases} \lambda_{n-1}(x,t) & \text{if } x \in S_1 \cup \ldots \cup S_{n-1},\ 0 \leqslant t \leqslant 1, \\ \hat{\lambda}_n(x,t) & \text{if } x \in S_n \text{ and } 0 \leqslant t \leqslant 1, \end{cases}$$

$$\mu_n(x,t) = \begin{cases} \mu_{n-1}(x,t) & \text{if } x \in S_1 \cup \ldots \cup S_{n-1},\ 0 \leqslant t \leqslant 1, \\ \hat{\mu}_n(x,t) & \text{if } x \in S_n \text{ and } 0 \leqslant t \leqslant 1. \end{cases}$$

It is easy to verify that λ_n and μ_n satisfy the required conditions.

It follows from conditions $(1)_n$-$(4)_n$, for $n = 1, 2, \ldots$, that $\underline{f} = ((f_n), \mathrm{id}_N)$ and $\underline{g} = ((g_n), \mathrm{id}_N)$ are maps of inverse sequences ([9], p. 41–42) and the conditions $\underline{fg} \simeq \underline{\mathrm{id}}_{(Y,a)}$ and $\underline{gf} \simeq \underline{\mathrm{id}}_{(X,a)}$ are satisfied.

Then $\mathrm{Sh}(X, a) = \mathrm{Sh}(Y, \hat{a})$ ([10], p. 62 and [9], pp. 41–44).

7. Fundamental sequences of usual n-dimensional spherical bouquets, with $n > 1$. In this part we shall classify the fundamental sequences from usual n-dimensional spherical bouquets to usual n-dimensional spherical bouquets, where $n > 1$.

(7.1) THEOREM. *If (X, a) and (Y, b) are usual finite n-dimensional spherical bouquets, $n > 1$, and $f, g\colon (X, a) \to (Y, b)$ are continuous maps, then f and g are homotopic if and only if they induce the same homomorphism $f_* = g_*\colon H_n(X) \to H_n(Y)$ of homology groups with integer coefficients.*

Proof. If $f \simeq g$, then $f_* = g_*$.

Let us assume that $f_* = g_*$.

Let $\{S_1, \ldots, S_k\}$ and $\{P_1, \ldots, P_m\}$ be the families of leaves of (X, a) and (Y, b), respectively, and let $\alpha_i\colon (S_i, a) \to (S_1 \cup \ldots \cup S_k, a)$ be the inclusions.

The homomorphism $\varphi_i\colon \pi_n(Y, b) \to H_n(Y)$ defined by the formula

$$\varphi_i([\beta]) = \beta_*(e_i) \quad \text{for every } [\beta] \in \pi_n(Y, b),$$

where e_i is the generator of the group $H_n(S_i)$, is an isomorphism for every i ([7], p. 208).

Since

$$\varphi_i([f\alpha_i]) = f_*(\alpha_i)_*(e_i) = g_*(\alpha_i)_*(e_i) = \varphi_i([g\alpha_i]),$$

for every $i = 1, \ldots, k$ there is a homotopy

$$\chi_i\colon S_i \times \langle 0,1\rangle \to Y$$

joining the maps $f\alpha_i$ and $g\alpha_i$ in (Y, b).

The map $\chi\colon X \times \langle 0,1\rangle \to Y$ defined by the formula

$$\lambda(x,t) = \lambda_i(x,t) \quad \text{for every } x \in S_i,\ 0 \leqslant t \leqslant 1,\ i = 1, \ldots, k$$

is a homotopy joining the maps f and g in (Y, b).

(7.2) THEOREM. *Let (X, a) and (Y, b) be two usual l-dimensional spherical bouquets $l > 1$. The fundamental sequences*

$$\underline{f} = \{f_n, (X, a), (Y, b)\} \quad \textit{and} \quad \underline{g} = \{g_n, (X, a), (Y, b)\}$$

are homotopic if and only if they induce the same homomorphism $\underline{f}_* = \underline{g}_*\colon H_l(X) \to H_l(Y)$ *of Vietoris homology groups with integer coefficients.*

Proof. It is known ([1], p. 242) that if $\underline{f} \simeq \underline{g}$ then $\underline{f}_* = \underline{g}_*$.

Let us assume that $\underline{f}_* = \underline{g}_*$. We shall show that $\underline{f} \simeq \underline{g}$.

Let U be any fixed neighborhood of Y in the Hilbert cube Q. There exists an $\varepsilon > 0$ such that $K(Y, \varepsilon) \subset U$ and we can find, by (4.3), a finite sequence $Y_1, \ldots, Y_k$ of leaves of (Y, b) and a retraction $\hat{r}\colon Y \to Y_1 \cup \ldots \cup Y_k$ satisfying the condition: $\varrho(\hat{r}(y), y) < \varepsilon$ for every $y \in Y$.

There are: a closed neighborhood W of Y in Q and a continuous extension $r\colon W \to Y_1 \cup \ldots \cup Y_k$ of $\hat{r}$ such that $\varrho(r(z), z) < \varepsilon$ for every $z \in W$.

Let V_0 be a neighborhood of X in Q and n_1 a natural number such that $f_n(V) \subset W$ and $g_n(V) \subset W$ for every $n \geqslant n_1$.

There is a $\delta > 0$ such that $K(X, \delta) \subset V$. We can find, by (4.3), a finite sequence $X_1, \ldots, X_m$ of leaves of (X, a) and a retraction $\hat{p}\colon X \to X_1 \cup \ldots \cup X_m$ such that $\varrho(\hat{p}(x), x) < \delta$ for every $x \in X$.

Let V_0 be a neighborhood of X in Q such that there is a continuous extension $p\colon V_0 \to X_1 \cup \ldots \cup X_m$ of $\hat{p}$ satisfying the condition: $\varrho(p(x), x) < \delta$ for every $x \in V_0$.

It is clear that

$$(1) \qquad p \simeq \mathrm{id}_{(V_0, a)} \quad \text{in } (V, a)$$

and

$$(2) \qquad r \simeq \mathrm{id}_{(W, b)} \quad \text{in } (U, b).$$

Let $r_1\colon Q \to Q$ be a continuous extension of r and $r_n = r_1$ for $n = 1, 2, \ldots$ The sequence $\underline{r} = \{r_n, (Y, b), (Y_1 \cup \ldots \cup Y_k, b)\}$ is a fundamental sequence. Let $\underline{i} = \{i_n, (X_1 \cup \ldots \cup X_m, a), (X, a)\}$ be a fundamental sequence such that $i_n(x) = x$ for every $x \in Q$ and $n = 1, 2, \ldots$

Since $\underline{f}_* = \underline{g}_*$, then $(\underline{rfi})_* = (\underline{rgi})_*$.

The set $Y_1 \cup \ldots \cup Y_k$ is an ANR-set, thus the fundamental sequences $\underline{rfi}$ and $\underline{rgi}$ are generated by maps ([1], p. 228). Let us denote them by $f\colon (X_1 \cup \ldots \cup X_m, a) \to (Y_1 \cup \ldots \cup Y_k, b)$ and $g\colon (X_1 \cup \ldots \cup X_m, a) \to (Y_1 \cup \ldots \cup Y_k, b)$, respectively.

Since $f_* = (\underline{rfi})_* = (\underline{rgi})_* = g_*$ ([1], p. 242) it follows from Theorem (7.2) that $f \simeq g$ and then $\underline{rfi} \simeq \underline{rgi}$.

Thus there is an $n_0 \geqslant n_1$ such that

$$(3) \qquad r_1 f_n|_{(X_1 \cup \ldots \cup X_m, a)} \simeq r_1 g_n|_{(X_1 \cup \ldots \cup X_m, a)} \quad \text{in } (Y_1 \cup \ldots \cup Y_k, b)$$

for every $n \geqslant n_0$.

Let $n \geqslant n_0$ be any positive integer.

Since relations (1)-(3) are satisfied, we have

$$f_n|_{(V_0, a)} = f_n \mathrm{id}_{(V_0, a)} \simeq f_n p|_{(V_0, a)} \simeq r_1 f_n p|_{(V_0, a)} \quad \text{in } (U, b)$$

and

$$g_n|_{(V_0, a)} = g_n \mathrm{id}_{(V_0, a)} \simeq g_n p|_{(V_0, n)} \simeq r_1 g_n p|_{(V_0, a)} \quad \text{in } (U, b).$$

These relations and the relation

$$r_1 f_n p|_{(V_0, a)} \simeq r_1 g_n p|_{(V_0, a)} \quad \text{in } (U, b)$$

give the required relation

$$f_n|_{(V_0, a)} \simeq g_n|_{(V_0, a)} \quad \text{in } (U, b).$$

This proves that $\underline{f} \simeq \underline{g}$.

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Accepté par la Rédaction le 4. 11. 1974