# On uniform spaces with linearly ordered bases II ($\omega_\mu$-metric spaces)

by

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**Abstract.** The object of the paper are uniform spaces with linearly ordered bases, i.e. $\omega_\mu$-metric spaces. Necessary and sufficient conditions for the $\omega_\mu$-metrizability of a topological space are given, generalizing famous metrization theorems of Nagata, Bing, Frink, Morita, and others. Moreover, a large number of necessary and sufficient conditions for the metrizability of a $\omega_\mu$-metrizable space are presented, yielding new metrization theorems as well.

§ 1. **Introduction.** Although this paper is self-contained, it can be considered as a continuation of a paper [15] by the same authors. It is mainly devoted to presenting necessary and sufficient conditions for topological spaces to have a uniform structure with a linearly ordered base, i.e. to be $\omega_\mu$-metric in the sense of R. Sikorski [18] (§ 4). It also provides a study of several special properties of $\omega_\mu$ metric spaces and — as corollaries — a large variety of metrization theorems (§ 3), i.e. necessary and sufficient metrizability conditions for several classes of topological spaces, involving (locally) compact spaces, first countable spaces and others.

§ 2. **$\omega_\mu$-metric spaces.** Let $\omega_\mu$ denote the $\mu$th infinite initial ordinal number. A linearly ordered abelian group $(G, <)$ is said to have *character* $\omega_\mu$, if there exists a strictly decreasing $\omega_\mu$-sequence converging to 0 in the order topology. A *$\omega_\mu$-metric* on a set $X$ is a function $\varrho$ from $X\times X$ to $(G, <)$ such that

(i) $\varrho(x, y)>0$, $\varrho(x, y) = 0$ iff $x = y$,

(ii) $\varrho(x, y) = \varrho(y, x)$,

(iii) $\varrho(x, y)\leqslant\varrho(x, z)+\varrho(z, y)$ $\forall x, y, z\in X$.

$(X, \varrho)$ is then called a *$\omega_\mu$-metric space.* As it is well known, the class of $\omega_0$-metric spaces coincides with the class of metric spaces (this will also be apparent from the results in § 3) even when $G$ is not $\boldsymbol{R}$. Since an $\omega_\mu$-metric space is first countable iff $\mu = 0$, $\omega_\mu$-metric spaces are very useful in studying spaces of "high cardinality".

Researchers who have studied $\omega_\mu$-metric spaces include F. Hausdorff [5], pp. 285, 286, L. W. Cohen and C. Goffman [1], R. Sikorski [18] (who was the first

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to devote extensive work to this field), F. W. Stevenson and W. J. Thron [20], Wang Shu-Tang [23], P. Nyikos and H. C. Reichel [15], A. Hayes [6] and D. Harris [4].

In [20] the authors have shown that a separated uniform space $(X, \mathfrak{U})$ is $\omega_\mu$-metrizable iff $(X, \mathfrak{U})$ has a linearly ordered base and $\aleph_\mu$ is the least power of such a base. Thus, for example, by a theorem in [6], every $\omega_\mu$-metric space is paracompact.

In order to present another characterization, let us recall the so-called natural topology on the spaces $A^B$, as defined by A. K. Steiner and E. F. Steiner [19]: Let $A$ be a set and $B$ a well-ordered set. For each $x \in A^B$ and each $\alpha \in B$ define

$$x(\alpha) = \{y \in A^B \mid y_\beta = x_\beta \text{ for all } \beta \leqslant \alpha\}.$$

Then the *natural topology* $\mathfrak{N}$ on $A^B$ is defined to be the topology generated by the base $\{x(\alpha) \mid x \in A^B, \alpha \in B\}$. $(A^B, \mathfrak{N})$ is a normal, totally disconnected, $T_2$-space, the topology of which coincides with the product topology iff $B$ has order type $\leqslant \omega_0$ or (trivially) $A$ has at most one element. These spaces are generalizations of the well-known Baire's zero dimensional sequence spaces $\Omega^N$ (see, e.g. [12]).

The topology $\mathfrak{N}$ can be induced by a uniformity $\mathfrak{U}$ with a linearly ordered base

$$\mathfrak{B} = \{\mathfrak{B}_\alpha \mid \alpha \in B\}, \quad \mathfrak{B}_\alpha = \{(x, y) \mid x_\beta = y_\beta \text{ for } \beta \leqslant \alpha\}.$$

Thus all spaces $(A^B, \mathfrak{N})$ are $\omega_\mu$-metrizable. As a partial converse, we have shown the following [15]:

THEOREM 1. *A non-metrizable space is $\omega_\mu$-metrizable if and only if $X$ is homeomorphic to a subspace of a suitable space $(A^B, \mathfrak{N})$, $B = \omega_\mu$, $\mu > 0$.*

Since it is easy to see that $x(\alpha) \cap y(\beta) \neq \emptyset$ implies $x(\alpha) \subset y(\beta)$ or $x(\alpha) \supset y(\beta)$, Theorem 1 shows that every non-metrizable $\omega_\mu$-metric space, i.e. $\mu > 0$, is *non-archimedean*. In other words, such a space has a base $\mathfrak{M}$ with the property that either two basis sets are disjoint or one contains the other [16].

**§ 3. Metrization-theorems.** As an aid to determining for which $\omega_\mu$ a "linearly uniformizable" space is $\omega_\mu$-metrizable, we will use the spaces $D_\mu^*$. Each $D_\mu^*$ can be characterized topologically either as $\omega_\mu + 1$ with the limit ordinals removed or as $\omega_\mu + 1$ with all points except $\omega_\mu$ isolated and the sets $(\alpha, \omega_\mu] = \{\beta: \alpha < \beta \leqslant \omega_\mu\}$ as a local base, as $\alpha$ ranges over the ordinats smaller than $\omega_\mu$. Another characterization, used by D. Harris [4] is that they are point-sets of cardinality $\aleph_\mu$ and a distinguished point $\omega_\mu$ such that all points other than $\omega_\mu$ are isolated and a set is an (open) neighborhood of $\omega_\mu$ if, and only if, it contains $\omega_\mu$ and its complement is of cardinality $< \aleph_\mu$.

LEMMA 1. *Let $X$ be a $\omega_\mu$-metrizable space. For each non-isolated point $x$ of $X$ there is a closed embedding $f$ of $D_\mu^*$ in $X$ such that $f(\omega_\mu) = x$. For no other regular cardinal $\aleph_\lambda$ is $D_\lambda^*$ embeddable in $X$.*

Proof. Clearly, every non-isolated point $x$ has a totally ordered local base $\{V_\alpha \mid \alpha < \omega_\mu\}$, such that $V_\alpha \subsetneq V_\beta$ when $\alpha > \beta$. For each $\alpha < \omega_\mu$ let $f(\alpha)$ be any point in $V_{\alpha+1} \setminus V_{\alpha+2}$, and let $f(\omega_\mu) = x$. Since every intersection of $< \omega_\mu$ open subsets of $X$ is open, it is trivial to verify that $f$ is a closed map, and continuity is also clear.



On the other hand, if $f: D_\lambda^* \to X$ is an embedding, then $f(\omega_\lambda)$ is a non-isolated point of $X$. But if $\omega_\lambda < \omega_\mu$, then $f(D_\lambda^*)$ is a discrete subspace of $X$, which yields a contradiction. Hence $\omega_\lambda > \omega_\mu$. But $x = f(\omega_\mu)$ is an intersection of $\aleph_\mu$ relatively open subsets $V_\alpha$ of $f(D_\lambda^*)$, so that $f^{-1}(x) = f^{-1}(\bigcap V_\alpha) = \bigcap f^{-1}(V_\alpha)$ contains infinitely many points of $D_\lambda^*$.

With the aid of Lemma 1 we can prove a multitude of "metrization theorems". The following theorem collects some, but by no means all, of the properties which, in the presence of $\omega_\mu$-metrizability, imply metrizability. (The reference numbers indicate papers with detailed studies of the properties in question):

THEOREM 2. *Let $X$ be $\omega_\mu$-metrizable for some $\mu$. If $X$ is not discrete, then $X$ is metrizable if and only if some non-isolated point $X$ is a $G_\delta$. Moreover, $X$ is metrizable if it is any of the following*:

1.a) *First countable*; 1.b) *perfectly normal*; 1.c) *semistratifiable* [13]; 1.d) *a $\sigma$-space* [13]; 1.e) *stratifiable*; 1.f) *quasi-metrizable* [13]; 1.g) *a $\sigma^\#$-space* [13].

2.a) *A $k$-space* (*"Kelley-space"*); 2.b) *sequential*; 2.c) *Fréchet*; 2.d) *locally compact*; 2.e) *compact*.

3.a) *A $q$-space* [7]; 3.b) a *$w\theta$-space* [7].

4.a) *A $\Sigma^*$-space* [17]; 4.b) *a $\Sigma$-space* [13]; 4.c) *a $w\Delta$-space* [7]; 4.d) *quasi-complete*; 4.e) *a $p$-space* [13]; 4.f) *an $M$-space* [13]; 4.g) *countably compact*; 4.h) *pseudocompact*; 4.i) *Čech-complete.*

5.a) *Not strongly zerodimensional* [2]; 5.b) *not totally disconnected*; 5.c) *locally connected*; 5.d) *connected.*

6.a) *A ccc-space* (*i.e. every collection of disjoint open sets is countable*); 6.b) *separable*; 6.c) *hereditarily Lindelöf.*

Remark. *Each of these results* determines a *metrization theorem* like the following corollary. This family of results may be more interesting than Theorem 2 itself.

COROLLARY 3.1. *A first-countable space $X$ is metrizable if and only if its topology can be induced by a separated uniformity $\mathfrak{U}$ with a linearly ordered base.*

(Herein "first-countable" can be replaced by any other property listed in Theorem 2).

Thus we get for example:

COROLLARY 3.2. *A compact space $X$ is metrizable if and only if its (in fact unique) uniform structure (i.e. the system of all open neighbourhoods of the diagonal in $X \times X$) has a totally ordered base.*

Moreover, since metric spaces automatically share the properties labeled by 1; 2a, b, c; 3a, b and 4a, b, c, e, f, we can formulate corollaries like the following:

COROLLARY 3.3. *A $\omega_\mu$-metric space $X$ is a Kelley-space if and only if $\mu = 0$.*

(Herein "Kelley-space" can be replaced by any of the above mentioned properties.)

Remark 1. Theorem 2 can also be used for constructing uniform spaces which do not have compatible uniform structures with linearly ordered bases. As an example take the following

Remark 2. Let $X$ be any non-metrizable topological space and $\alpha X$ be any compactification of $X$. Then the (unique) uniform structure $\mathfrak{U}$ of $\alpha X$ has no linearly ordered base.

Proof of Theorem 2. If some non-isolated point $x \in X$ is a $G_\delta$, it follows by Lemma 1 that $\mu = 0$, hence $X$ is metrizable. The results headed by "1" follow from the fact that, if $X$ is not discrete, then any of the listed conditions implies that $X$ has a nonisolated $G_\delta$-point, thus $X$ is forced to be metrizable. In the other cases, all properties labeled by a certain number imply the first property labeled by the same number (sometimes with the addition of paracompactness, which every $\omega_\mu$-metrizable space has, § 2). Thus we only have to prove the results 2.a), 3.a), 4.a), 5.a) and 6.a):

To prove 2.a), we note that every closed subspace of a $k$-space is a $k$-space. Now in $D_\mu^*$, $\mu \neq 0$, every compact subset is finite. Hence $D_\mu^* \setminus \{\omega_\mu\}$ has a closed intersection with every compact subset, but is not closed. So $D_\mu^*$ is *not* a $k$-space for $\mu \neq 0$.

To prove 3.a) recall that in a $q$-space ([7]), a sequence each point $x_n$ of which belongs to $g(n, x)$ (an open set containing $x$) is required to have a cluster point. But if $X$ is not metrizable, every countable subset of $X$ is discrete.

To prove 5.a) we need only note that every zero-set of a $\omega_\mu$-metrizable space, $\mu \neq 0$, is open.

6.a) follows from the observation that any non-isolated point in a $\omega_\mu$-metrizable space has a totally ordered local base $\{V_\alpha \mid \alpha < \omega_\mu\}$ such that $\overline{V_{\alpha+1}} \subsetneq V_\alpha$ for all $\alpha$. Thus, if $\mu \neq 0$, $\{(V_\alpha \setminus V_{\alpha+1}) / \alpha < \omega_\mu\}$ is a non-countable collection of disjoint open sets.

It remains only to show 4.a):

A $\Sigma^*$-space is a space having a cover $\mathfrak{K}$ of countably compact subsets, and an outer network $\{\mathfrak{H}_n \mid n \in N\}$ for $\mathfrak{K}$ such that each $\mathfrak{H}_n$ is hereditarily closure-preserving [17]. That is, given any $\mathfrak{H}' \subset \mathfrak{H}_n$, $\mathfrak{H}' = \{H_\gamma \mid \gamma \in \Gamma\}$, and any choice of subsets $A_\gamma \subset H_\gamma$, the closure of $\bigcup_{\gamma \in \Gamma} A_\gamma$ is the union of the closures of the $A_\gamma$.

In a $\omega_\mu$-metrizable space, $\mu \neq 0$, every countably compact subset is finite. Hence $\mathfrak{K}$ would have to be a cover of $X$ by finite sets. Since a closed subspace of a $\Sigma^*$-space is a $\Sigma^*$-space, we may assume $X = D_\mu^*$.

Let $K \in \mathfrak{K}$. By definition, it is required that for every open set $U$ containing $K$, there exists $n$ and $H \in \mathfrak{H}_n$ such that $K \subset H \subset U$. It is trivial to obtain a subcollection $\mathfrak{K}'$ of $\mathfrak{K}$, such that $|\mathfrak{K}'| = \aleph_\mu$, and such that for each $K \in \mathfrak{K}'$ there exists a point $p_K \neq \omega_\mu$ of $K$ which is in no other member of $\mathfrak{K}'$. Well-order these points $p_K$ in a transfinite sequence $\{p_\alpha \mid \alpha < \omega_\mu\}$. For each $\alpha$ there exists $H_\alpha \in \mathfrak{H}_n$ for some $n$ such that $p_\alpha \in H_\alpha$, $p_\alpha \notin H_\beta$ for all $\beta < \alpha$.



Since $\aleph_\mu$ is uncountable and regular, there exists $\mathfrak{H}_n$ containing $\aleph_\mu$ sets of the form $H_\alpha$. Now let $A_\alpha = \{p_\alpha\}$ for these $\alpha$. Clearly, $\omega_\mu$ is in the closure of $\bigcup A_\alpha$, but is not in the closure of any $A_\alpha$. This completes the proof.

The result that $D_1^*$ is not a $\Sigma^*$-space is due to A. Okuyama [17]. His proof that $D_1^*$ is a $\Sigma^\#$-space obviously generalizes to all $D_\mu^*$.

**§ 4. $\omega_\mu$-metrizability theorems.** Unlike the previous section, this one begins with an arbitrary space $X$ and establishes several necessary and sufficient conditions for its $\omega_\mu$-metrizability, most of which generalize classical metrization theorems though completely different methods are needed. (As pointed out in § 2, all these theorems yield necessary and sufficient conditions for $X$ to have an uniform structure with a linearly ordered base, as follows by the theorem of T. W. Stevenson and W. J. Thron [20]). Our principal such theorem will be based on the "generalized metrization theorem" of J. Nagata [12]:

THEOREM. *A $T_1$-space $X$ is metrizable if and only if for each point $p$ of $X$, there exist two sequences $\mathfrak{U} = \{U_n(p) \mid n = 1, 2, \ldots\}$ and $\mathfrak{B} = \{V_n(p) \mid n = 1, 2, \ldots\}$ of neighborhoods of $p$ such that*

(i) *$\{U_n(p) \mid n = 1, 2, \ldots\}$ is a local base at $p$,*
(ii) *$q \notin U_n(p)$ implies $V_n(q) \cap V_n(p) = \emptyset$, and*
(iii) *$q \in V_n(p)$ implies $V_n(q) \subset U_n(p)$.*

The generalization will come by letting $\mathfrak{U}$ be $\{U_\tau(p) \mid \tau < \omega_\mu\}$, letting $\mathfrak{B}$ be $\{V_\tau(p) \mid \tau < \omega_\mu\}$, and adding condition (iv): $\bigcap_{\tau \leqslant \gamma} U_\tau(p)$ and $\bigcap_{\tau \leqslant \gamma} V_\tau(p)$ is a neighborhood of $p$ for all $\gamma < \omega_\mu$. This condition clearly yields the same class of spaces as (iv'): every intersection of fewer than $\aleph_\mu$ open subsets of $X$ is open or (iv'') $U_\tau(p) \subset U_\sigma(p)$ whenever $\tau > \sigma$.

Spaces satisfying our generalization of (i), (ii) and (iv'') were introduced by J. Vaughan [22] under the name of *Nagata spaces over $\alpha$* ($\alpha = \omega_\mu$ for some $\omega$). Vaughan also introduced the class of spaces stratifiable over $\alpha$ and showed that every Nagata space over $\alpha$ is stratifiable over $\alpha$. In [14], P. Nyikos showed that every space stratifiable over $\alpha$ which is also suborderable — that is, embeddable as a subspace of a totally ordered set with the order-topology — is $\omega_\mu$-metrizable for $\omega_\mu = \alpha$. Conversely, every $\omega_\mu$-metrizable space is stratifiable (and also Nagata) over $\omega_\mu$, and if $\mu \neq 0$ it is suborderable as well.

To simplify the proof of our generalization of Nagata's theorem, we introduce the following definition:

DEFINITION. A set $\{\mathfrak{U}_\gamma \mid \gamma \in \Gamma\}$ of covers of a set $X$ is *locally ultra-starring for $X$* if for each $x \in X$ and each neighborhood $U$ of $x$, there exists a set $V$ such that $x \in V \subset U$ and a $\gamma \in \Gamma$ such that $\mathrm{St}(V, \mathfrak{U}_\gamma) = V$.

As usual, the expression $\mathrm{St}(A, \mathfrak{B})$, where $\mathfrak{B}$ is a collection of sets, denotes the union: $\bigcup \{V \in \mathfrak{B} \mid A \cap V \neq \emptyset\}$.

In the definition, it is not required that $V$ be an open set. However, if each $\mathfrak{U}_\gamma$ is an open cover, or merely a family of sets whose interiors cover $X$, or a locally finite closed cover, then any $V$ for which $\mathrm{St}(V, \mathfrak{U}_\gamma) = V$ will actually be clopen.

Remark. In this doctoral thesis, P. Nyikos proved that a $T_0$ space $X$ is non-archimedeanly metrizable if and only if there is a countable system of open covers which is locally ultra-starring for $X$. One aspect of the following theorem is an analogue of that result.

THEOREM 3. *Let $X$ be a $T_1$-space and $\omega_\mu$ a regular initial ordinal. The following are equivalent:*

(A) *$X$ is $\omega_\mu$-metrizable.*

(B) *There are two systems $\mathfrak{U} = \{U_\tau(p)|\ \tau<\omega_\mu\}$ and $\mathfrak{V} = \{V_\tau(p)|\ \tau<\omega_\mu\}$ satisfying:*

(i) *$\mathfrak{U}$ is a local base at $p$,*

(ii) *$q \notin U_\tau(p)$ implies $V_\tau(q) \cap V_\tau(p) = \emptyset$,*

(iii) *$q \in V_\tau(p)$ implies $V_\tau(q) \subset U_\tau(p)$,*

(iv) *$\bigcap_{\tau\leqslant\gamma} U_\tau(p)$ and $\bigcap_{\tau\leqslant\gamma} V_\tau(p)$ are neighborhoods of $p$ for all $\gamma \geqslant \omega_\mu$.*

(C) *$X$ is metrizable if $\mu = 0$, and if $\mu \neq 0$, then there exists a system $\mathfrak{V} = \{V_\tau(p)|\ \tau<\omega_\mu\}$ of neighborhoods of each $p \in X$ such that $V_\tau(p) \subset V_\sigma(p)$ whenever $\sigma<\tau$ and such that $\{\mathfrak{V}_\tau|\ \tau<\omega_\mu\}$, $\mathfrak{V}_\tau = \{V_\tau(p)|\ p \in X\}$, is locally ultra-starring for $X$.*

Proof. (A)$\Rightarrow$(B): If $\mu = 0$, $X$ is metrizable and we can take the spheres with center $p$ and radius $r = 2^{-n}$ for $U_n(p)$ and $r = 2^{-(n+1)}$ for $V_n(p)$ respectively. If $\mu>0$, we let $\{a_\tau|\ \tau<\omega_\mu\}$ be a $\omega_\mu$-sequence of elements of $(G, <)$ converging to the neutral element, and $a_{\tau+1}+a_{\tau+1} \leqslant a_\tau$ for all $\tau<\omega_\mu$, and let

$$U_\tau(p) = \{q \in X|\ \varrho(p, q)<a_\tau\} \quad \text{and} \quad V_\tau(p) = \{q \in X|\ \varrho(p, q)<a_{\tau+1}\}.$$

(Compare § 2.)

(B)$\Rightarrow$(C): If $\mu = 0$, this is Theorem VI. 2 in the book of Nagata [12]. So let $\mu>0$. We can replace $U_\tau(p)$ by $\bigcap_{\alpha\leqslant\tau} U_\alpha(p)$ and analogously, $V_\tau(p)$ by $\bigcap_{\alpha\leqslant\tau} V_\alpha(p)$, so that we may assume $U_\tau(p) \subset U_\sigma(p)$ and $V_\tau(p) \subset V_\sigma(p)$ whenever $\tau \geqslant \sigma$. For a given $U_\tau(p) =: U_{\tau_1}(p)$ and $V_\tau(p) =: V_{\tau_1}(p)$ we can find a $U_{\tau_2}(p) \in \mathfrak{U}$ such that $U_{\tau_2}(p) \subset V_{\tau_1}(p)$ by property (i). Now take the corresponding $V_{\tau_2}(p) \in \mathfrak{V}$ and repeat the construction. Inductively, for each $\tau = \tau_1$, we obtain sequences $\{U_{\tau_n}(p)|\ n = 1, 2, \ldots\}$ and $\{V_{\tau_n}(p)|\ n = 1, 2, \ldots\}$. Let

$$U^\tau(p) = \bigcap_{n=1}^{\infty} U_{\tau_n}(p) = \bigcap_{n=1}^{\infty} V_{\tau_n}(p),$$

which is a neighborhood of $p$ by property (iv). Now let $\alpha_\tau = \sup\{\tau_n|\ n = 1, \ldots\}$. We shall show that $\mathrm{St}(U^\tau, \mathfrak{V}_{\alpha_\tau}) = U^\tau$. Let $q \in X$. Suppose $V_{\alpha_\tau}(q) \cap U^\tau \neq \emptyset$. Then $V_{\tau_n}(q) \cap V_{\tau_n}(p) \neq \emptyset$ for all $n$, by (ii). Hence $q \in U_{\tau_n}(p)$ for every $n$, implying $V_{\tau_n}(q) \subset U_{\tau_n}(p)$ for all $n$, and so $V_{\alpha_\tau}(q) \subset U^\tau$. Hence the collection of all

$$\mathfrak{V}_\tau = \{V_\tau(p)|\ p \in X\}, \quad \tau<\omega_\mu,$$

is locally ultra-starring for $X$.

(C)$\Rightarrow$(A): Obvious if $\mu = 0$. If $\mu>0$, we define, for each $p \in X$ and $\tau<\omega_\mu$, $V^\tau(p) = \{q|\ q$ is chained to $p$ by $\mathfrak{V}_\tau\}$ (in other words, there exists a finite sequence $V_1, \ldots, V_n$ of member of $\mathfrak{V}_\tau$ such that (a) $p \in V_1$, (b) $V_i \cap V_{i+1} \neq \emptyset$ for $i = 1, \ldots, n-1$, (c) $q \in V_n$).



For each $\tau$, $\{V^\tau(p)|\ p \in X\}$ is a partition of $X$ into (closed and) open sets. (Indeed, if $q \in V^\tau(p)$, then $V_\tau(q) \subset V^\tau(p)$. Hence the sets are open; and moreover $V^\tau(q) \subset V^\tau(p)$; symmetrically, $p \in V^\tau(q)$, and so $V^\tau(p) \subset V^\tau(q)$.) Furthermore, $V^\tau(p)$ refines $V^\sigma(p)$ whenever $\tau>\sigma$. By the local ultra-starring property, the union of all the partitions is a base for the topology on $X$. Indeed, if $\mathrm{St}(V, \mathfrak{V}_\tau) = V$, then for each $p \in V$, $V^\tau(p) \subset V$.

Letting $B_\tau = \{(p, q)|\ V^\tau(p) = V^\tau(q)\}$, it follows that $\{B_\tau|\ \tau<\omega_\mu\}$ is a totally ordered base for a uniformity on $X$. Thus $X$ is $\omega_\mu$-metrizable by the theorem of Stevenson and Thron, mentioned in § 2 ([20]).

But a simple direct proof is also available: let $(G, <)$ an ordered group with a transfinite well-ordered sequence $\{a_\tau|\ \tau<\omega_\mu\}$ converging to the identity element, with $a_\tau<a_\sigma$ whenever $\tau>\sigma$. Now let $\varrho(p, q) = \inf\{\alpha_\tau|\ (x, y) \in B_\tau\}$. It is trivial to verify that $\varrho$ is a (non-archimedean) $\omega_\mu$-metric for $X$. ■

Remark 1. A $\omega_\mu$-metric $\varrho$ is *non-archimedean* if it satisfies the "strong" triangle inequality:

$$\varrho(x, y) \leqslant \max\{\varrho(x, z), \varrho(z, y)\}\ \forall x, y, z \in X.$$

We just have seen (Theorem 3) that *every non-metrizable, $\omega_\mu$-metrizable space* (i.e. $\mu>0$) *has a compatible non-archimedean $\omega_\mu$-metric.*

In *valuation theory* certain $\omega_\mu$-metrics on fields have been studied (mostly without requiring commutativity of $G$); for an account e.g. see the book of Jacobson [8], Chapter V.

Remark 2. Theorem 3 is a dramatic illustration of how the uncountable case is often simpler than the countable case. It is instructive to compare its relatively brief proof with the combined proofs of the Alexandroff–Urysohn metrization theorem and Nagata's general metrization theorem [12, pp. 184–191], keeping in mind that the former is used in proving the latter, and the latter employs the deep theorem that a space is paracompact if every open cover has a $\sigma$-cushioned open refinement. Even the more recent proofs of this theorem [7] are much longer than the proof of Theorem 3 in the uncountable case. With the help of Theorem 3 we can prove several additional $\omega_\mu$-metrization theorems. All of them have their "classical" analogues in metrization theory (i.e. for $\mu = 0$). (Compare e.g. the book of Nagata [12].) For instance:

THEOREM 4. *A $T_1$-space $X$ is $\omega_\mu$-metrizable iff for each $p \in X$ there exists a local base $\{W_\tau(p)|\ \tau<\omega_\mu\}$ such that:*

(i) *$\bigcap_{\tau\leqslant\gamma} W_\tau(p)$ is a neighborhood of $p$ for all $\gamma<\omega_\mu$, and*

(ii) *for every $\tau$ and $p$ there exists $\sigma(\tau, p)$ such that $W_\sigma(p) \cap W_\sigma(q) \neq \emptyset$ implies $W_\sigma(q) \subset W_\tau(p)$.*

Remark 1. Since (i) is satisfied automatically if $\mu = 0$, this theorem is a generalization of a metrization theorem of A. H. Frink [3], [12].

Proof. Theorem 4 is a corollary to Theorem 3. Necessity follows completely analogously, and sufficiency can be shown as follows. Let $\mathfrak{W}_\tau = \{W_\tau(p)|\ p \in X\}$ and let $U_\tau(p) = \mathrm{St}(p, \mathfrak{W}_\tau)$ and $V_\tau(p) = W_{\sigma(\tau,p)}(p)$, $\tau < \omega_\mu$. Then the conditions (i)-(iv) of Theorem 3 are satisfied. Hence $X$ is $\omega_\mu$-metrizable.

Remark 2. Using Theorem 3 for $\mu = 0$, Nagata [12] presented a proof of the famous result of A. M. Stone [21] that the image $S$ of a metrizable space $R$ under a closed continuous mapping is metrizable iff the boundaries of all sets $f^{-1}(y)$, $y \in S$, are compact. Similarly, we can prove a generalization of this theorem, using a lemma which can be proved exactly using the arguments of our paper [15].

Lemma. *Let $X$ be a $\omega_\mu$-metric space and $f\colon X \to Y$ a closed continuous mapping onto $Y$ such that each point $y \in Y$ has a $\omega_\mu$-compact* [18] *preimage. Then $Y$ is $\omega_\mu$-metrizable. (In analogy to the "classical" case we could call such a mapping $\omega_\mu$-perfect.)*

Proposition 4.1. *Let $X$ be a $\omega_\mu$-metric space ($\mu \geqslant 0$) and let $f\colon X \to Y$ be a closed map of $X$ onto $Y$ such that $f^{-1}(y)$ has a $\omega_\mu$-compact boundary for each $y \in Y$. Then $Y$ is a $\omega_\mu$-metric space.*

Proof. If $f^{-1}(y)$ has empty boundary, we let $g(y)$ be any point of $f^{-1}(y)$. Otherwise, we let $g(y)$ be any point of $f^{-1}(y)$. Otherwise we let $g(y)$ be the boundary of $f^{-1}(y)$. Now let $X' \subset X$ be the image of $Y$ under this "multivalued function". The restriction of $f$ to the ($\omega_\mu$-metric) space $X'$ is clearly $\omega_\mu$-perfect and the result follows from the lemma cited above.

Another corollary of Theorem 3 is

Theorem 5. *A $T_1$-space is $\omega_\mu$-metrizable iff there exists a $\omega_\mu$-sequence $\{\mathfrak{U}_\tau|\ \tau < \omega_\mu\}$ of open coverings of $X$ such that*

(i) $\bigcap_{\tau \leqslant \gamma} \mathrm{St}(p, \mathfrak{U}_\tau)$ *is open for all* $\gamma < \omega_\mu$, *and*

(ii) *the stars* $\{\mathrm{St}(\mathrm{St}(p, \mathfrak{U}_\tau), \mathfrak{U}_\tau)|\ \tau < \omega_\mu\}$ *form a local base at* $p$.

Remark 1. Condition (i) is satisfied if for every system

$$\{U_\alpha|\ p \in U_\alpha \in \mathfrak{U}_\alpha,\ \alpha \leqslant \gamma < \omega_\mu\}$$

the intersection of this system is open (1).

Remark 2. For $\mu = 0$, (i) is satisfied automatically, and the theorem coincides with a metrization theorem of K. Morita [11], [12].

Proof. In order to prove necessity for $\mu > 0$, take $\mathfrak{B}^\tau = \{V^\tau(p)|\ p \in X\}$, $\tau < \omega_\mu$, where $V^\tau(p)$ denotes the sets defined in the proof of Theorem 3 (C$\Rightarrow$A). Clearly, $\mathrm{St}(\mathrm{St}(p, \mathfrak{B}^\tau), \mathfrak{B}^\tau) = \mathrm{St}(p, \mathfrak{B}^\tau) = V^\tau(p)$.

Sufficiency: Let $\mathrm{St}(\mathrm{St}(p, \mathfrak{U}_\tau)\mathfrak{U}_\tau) = U_\tau(p)$, and $\mathrm{St}(p, \mathfrak{U}_\tau) = V_\tau(p)$ for every $\tau < \omega_\mu$. And the conditions in (B) of Theorem 3 are obviously satisfied.

We can also prove an analogue of another theorem by Morita [12, p. 192], [11], on locally finite (or even closure-preserving) covers, almost exactly as in Nagata's text. The only differences are that the covers are indexed by ordinals $< \omega_\mu$, and

(1) This is a consequence of the axiom of choice; more exactly, of the general distributivity of $\bigcap$ and $\bigcup$ in set theory.



that, to prove necessity for $\mu > 0$, we note that $\{V^\tau(p)|\ p \in X\}$, being a clopen partition, is a locally finite closed collection for each $\tau$.

R. Sikorski, in his paper [18], defined *$\omega_\mu$-additive spaces*: spaces such that for any system $\{O_\tau|\ \tau \leqslant \gamma\}$, $\gamma < \omega_\mu$, of open sets $O_\tau$, their intersection is open again (2). Clearly, $\omega_\mu$-metric spaces are $\omega_\mu$-additive.

Applying the theorems above to $\omega_\mu$-additive spaces, we may drop condition (iv) in Theorem 3 and conditions (i) in Theorems 4 and 5. In this way we obtain *$\omega_\mu$-metrization theorems for $\omega_\mu$-additive spaces* which are formally analogous to classical metrization theorems. (Obviously, every topological space is $\omega_0$-additive.)

So, for example, we could prove a *$\omega_\mu$-analogue* of the Nagata–Smirnov metrization theorem, using Theorem 3, similarly as in Nagata's text. We do not carry out this possibility, because such a theorem was proved by Wang Shu-Tang [23], who, however, used completely different methods.

R. Sikorski [18] proved another $\omega_\mu$-metrization theorem, analogous to Urysohn's metrization theorem:

*A regular $\omega_\mu$-additive space $X$ is $\omega_\mu$-metrizable if there is a $\omega_\mu$-sequence of open sets $\{O_\tau|\ \tau < \omega_\mu\}$ forming a base for the topology on $X$.*

We also have an analogue of Bing's metrization theorem:

*A regular space is $\omega_\mu$-metrizable iff it is $\omega_\mu$-additive and has a $\omega_\mu$-discrete base (a base which is a union of a $\omega_\mu$-sequence of discrete collections).*

In other words:

Theorem 6. *A regular space $X$ is $\omega_\mu$-metrizable iff $X$ has a base $\mathfrak{B}$ which is a union of a $\omega_\mu$-sequence of discrete collections $\mathfrak{B}_\tau$ ($\tau < \omega_\mu$) such that*

$$\bigcap_{\tau \leqslant \gamma} \{B_\tau(p)|\ p \in B_\tau(p) \in \mathfrak{B}_\tau\}, \quad \textit{where} \quad \gamma < \omega_\mu,$$

*is always open.*

Remark. Obviously, the last condition is satisfied whenever $\mu = 0$ or $X$ is $\omega_\mu$-additive.

Proof. Necessity: we may assume $\mu > 0$. Since all properties are hereditary, we may assume $X$ is homeomorphic with a space $(A^B, \mathfrak{B})$, $B = \omega_\mu$, by Theorem 1. For such a space, the natural base $\mathfrak{U} = \{x(\alpha)|\ x \in A^B, \alpha \in B\}$ is a $\omega_\mu$-sequence of clopen partitions $\{x(\alpha)|\ x \in A^B\}$.

Sufficiency: Any such space satisfies the hypotheses of Wang Shu-Tang's analogue of the Nagata-Smirnov theorem [23].

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(2) It would also be appropriate to say "$\omega_\mu$-multiplicative". Of course, one could also speak of unions of closed sets.

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# A note on tangentially equivalent manifolds


by

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**Abstract.** Let $M_1^m$, $M_2^m$ be two closed 1-connected smooth manifolds which are tangentially equivalent. B. Mazur proved that, for large values of $k$, $k \geqslant m+2$, there exists a diffeomorphism

$$F\colon M_1^m \times D^k \to M_2^m \times D^k.$$

In this note we define an obstruction theory for the existence of such a diffeomorphism in the metastable range, $k \geqslant \frac{1}{2}(m+4)$ and for $m \geqslant 5$.


Recall that two closed oriented smooth manifolds $M_1^m$ and $M_2^m$ are called *tangentially equivalent* iff there exists a smooth homotopy equivalence

$$f\colon M_1^m \to M_2^m,$$

such that $f^*\bar{\tau}(M_2) = \bar{\tau}(M_1)$, where $\bar{\tau}(M_i)$ is the stable tangent bundle of $M_i (i = 1, 2)$. B. Mazur in [6] proved that if $M_1^m$ and $M_2^m$ are two closed simply connected tangentially equivalent manifolds, then for large $k$, $k \geqslant m+2$, there exists a diffeomorphism

$$F\colon M_1^m \times D^k \to M_2^m \times D^k$$

such that the following diagram is commutative up to homotopy:

$$\begin{array}{ccc} M_1^m \times D^k & \xrightarrow{F} & M_2^m \times D^k \\ \downarrow p & & \downarrow p \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

where the vertical maps are projections on the first factor, and $f$ the tangential equivalence. In this note we define an obstruction theory for the existence of the diffeomorphism for values of $k$ in the metastable range, i.e. for $k \geqslant \frac{1}{2}(m+4)$ and for 1-connected manifolds, $m \geqslant 5$.

Let $f\colon M_1^m \to M_2^m$ be a tangential equivalence between the 1-connected closed manifolds $M_1$ and $M_2$. Consider the composition map $i \circ f = f'$

$$M_1^m \xrightarrow{f} M_2^m \xrightarrow{i} M_2^m \times D^k,$$

$i$ is the inclusion map, $f'$ induces an isomorphism between the homotopy groups in all dimensions, hence by Haefliger theorem, for $k \geqslant \frac{1}{2}(m+4)$, [2], $f'$ can be approximated, within its homotopy class, by an imbedding

$$g\colon M_1^m \to M_2^m \times D^k.$$