Some quantitative properties of shapes

by

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Abstract. Let A be a compactum lying in a space $M \in \mathrm{AR}$ and B a compactum lying in a space $N \in \mathrm{AR}$. If U is a neighborhood of A in M and V is a neighborhood of B in N, then one introduces in a natural way the relation of U-domination ($A \underset{U}{\leqslant} B$ in M, N), the relation of (U, V)-affinity of A with B ($A \underset{(U,V)}{\leftrightarrow} B$ in M, N) and the relation of (U, V)-equivalence of A with B ($A \underset{(U,V)}{=} B$ in M, N). Those relations allow us to define some shape invariant relations: the quasi-domination $A \overset{q}{\leqslant} B$, the quasi-affinity $A \overset{q}{\leftrightarrow} B$ and the quasi-equivalence $A \overset{q}{=} B$, weaker than the relations $\mathrm{Sh}(A) \leqslant \mathrm{Sh}(B)$, $\mathrm{Sh}(A) \leqslant \mathrm{Sh}(B)$ and $\mathrm{Sh}(B) \leqslant \mathrm{Sh}(A)$, and $\mathrm{Sh}(A) = \mathrm{Sh}(B)$ respectively. If $A \overset{q}{\leqslant} B$, then $p_n(A) \leqslant p_n(B)$ and if B is movable, then A is also movable.

§ 1. **Introduction.** In the theory of shape one considers a classification of compacta based on their global topological properties. In this way one obtains the notion of shape, which has a qualitative character. Though in some cases we can say that the shape of one compactum A is less than the shape of another compactum B, we are not able to estimate how much the shape of A differs from the shape of B.

In the present paper we introduce for a given neighborhood U of a compactum A (lying in a space $M \in \mathrm{AR}$) a relation of the U-domination of A by another compactum B (lying in a space $N \in \mathrm{AR}$) and we define also, for given neighborhoods U of A and V of B, relations of (U, V)-affinity and of (U, V)-equivalence. These relations allow us to consider shapes from the quantitative point of view. They allow us also to introduce the concepts of quasi-domination, quasi-affinity and quasi-equivalence, which are weaker than the relations of fundamental domination and of fundamental equivalence considered in the theory of shape.

We assume as known the basic notions and the most elementary results of the theory of shape. The reader can find them in [1].

§ 2. U-**domination.** Let A, B be two compacta lying in spaces $M, N \in \mathrm{AR}$, respectively, and let V be a neighborhood of B in N. Two fundamental sequences

$$\underline{f} = \{f_k, A, B\}_{M,N}, \quad \underline{f}' = \{f'_k, A, B\}_{M,N}$$

are said to be *V-homotopic* (notation: $\underline{f} \underset{V}{\simeq} \underline{f}'$) if the following condition is satisfied:

(2.1) *There exists a neighborhood U_0 of A in M such that $f_k/U_0 \simeq f'_k/U_0$ in V for almost all k.*

It is clear that

(2.2) *If $\underline{f} \simeq \underline{f}'$, then $\underline{f} \underset{V}{\simeq} \underline{f}'$ for every neighborhood V of B in N, and conversely.*

(2.3) *If $\underline{f} \underset{V}{\simeq} \underline{f}'$ and $\underline{f}' \underset{V}{\simeq} \underline{f}''$ then $\underline{f} \underset{V}{\simeq} \underline{f}''$.*

(2.4) *If $V \subset V'$, then $\underline{f} \underset{V}{\simeq} \underline{f}'$ implies $\underline{f} \underset{V'}{\simeq} \underline{f}'$.*

(2.5) Remark. If V is an open neighborhood of B in N, then condition (2.1) is equivalent to the following one:

(2.1)' $$f_k/A \simeq f'_k/A \quad \text{in } V \quad \text{for almost all } k.$$

Proof. It is clear that (2.1) implies (2.1)'. On the other hand, if (2.1)' is satisfied, then there is an index k_0 such that $f_k/A \simeq f'_k/A$ in V for every $k \geqslant k_0$. Since $N \in \mathrm{AR}$, there exists for every $k \geqslant k_0$ a neighborhood U_k of A in M such that

(2.6) $$f_k/U_k \simeq f'_k/U_k \quad \text{in } V \quad \text{for} \quad k \geqslant k_0.$$

But $\underline{f}$ and $\underline{f}'$ are fundamental sequences, consequently there exists a neighborhood U' of A in M and an index $k_1 \geqslant k_0$ such that

(2.7) $$\begin{matrix} f_k/U' \simeq f_{k_1}/U' & \text{in } V \\ f'_k/U' \simeq f'_{k_1}/U' & \text{in } V \end{matrix} \quad \text{for every } k \geqslant k_1.$$

Consider a neighborhood $U_0 \subset U' \cap U_k$ of A in M. It follows by (2.6) and (2.7) that

$$f_k/U_0 \simeq f'_k/U_0 \quad \text{in } V \quad \text{for every } k \geqslant k_1,$$

hence condition (2.1) is satisfied.

Let U be a neighborhood of A in M. We say that A is *U-dominated by B in M, N* (notation: $A \underset{U}{\leqslant} B$ in M, N) if there exist two fundamental sequences

(2.8) $$\underline{f} = \{f_k, A, B\}_{M,N}, \quad \hat{\underline{f}} = \{\hat{f}_k, B, A\}_{N,M}$$

such that $\hat{\underline{f}}\underline{f}$ is U-homotopic to the fundamental identity sequence $\underline{i}_{A,M}$ for A in M.

It is clear that:

(2.9) *If $U \subset U'$, then $A \underset{U}{\leqslant} B$ in M, N implies $A \underset{U'}{\leqslant} B$ in M, N.*

(2.10) *If M, N are two AR-spaces containing A, then $A \underset{U}{\leqslant} A$ in M, N for every neighborhood U of A in M.*

(2.11) *If $\mathrm{Sh}(A) \leqslant \mathrm{Sh}(B)$, then $A \underset{U}{\leqslant} B$ in M, N for every neighborhood U of A in M.*

(2.12) *If $A \neq \emptyset$ is contractible in U, then $A \underset{U}{\leqslant} B$ in M, N for every compactum $B \neq \emptyset$.*

In fact, in this case for every two fundamental sequences (2.8) the relation $\hat{\underline{f}}\underline{f} \underset{U}{\simeq} \underline{i}_{A,M}$ holds true.

§ 3. (U, V)-affinity and (U, V)-equivalence. Let U be a neighborhood of A in M and V be a neighborhood of B in N, where A, B are compacta lying in spaces $M, N \in \mathrm{AR}$, respectively. Then we say shortly that (U, V) is a neighborhood of (A, B) in (M, N). The compacta A, B are said to be *(U, V)-affinite in M, N* if the relations $A \underset{U}{\leqslant} B$ in M, N and $B \underset{V}{\leqslant} A$ in N, M both hold true. Then we write $A \underset{(U,V)}{\leftrightarrow} B$ in M, N.

It is clear that

(3.1) *If $U \subset U'$, $V \subset V'$, then $A \underset{(U,V)}{\leftrightarrow} B$ in M, N implies that $A \underset{(U',V')}{\leftrightarrow} B$ in M, N.*

(3.2) *If $A \underset{(U,V)}{\leftrightarrow} B$ in M, N, then $B \underset{(V,U)}{\leftrightarrow} A$ in N, M.*

Two compacta A, B are said to be *(U, V)-equivalent in M, N* if there exist two fundamental sequences (2.8) such that the following conditions are satisfied:

$$\hat{\underline{f}}\underline{f} \simeq \underline{i}_{A,M} \quad \text{and} \quad \underline{f}\hat{\underline{f}} \simeq \underline{i}_{B,N}.$$

Then we write $A \underset{(U,V)}{\simeq} B$ in M, N.

Let us observe that

(3.3) *$A \underset{(U,V)}{\simeq} B$ in M, N implies $A \underset{(U,V)}{\leftrightarrow} B$ in M, N.*

(3.4) *If $\mathrm{Sh}(A) = \mathrm{Sh}(B)$, then $A \underset{(U,V)}{\simeq} B$ in M, N for every neighborhood (U, V) of (A, B) in (M, N).*

(3.5) *If $U \subset U'$, $V \subset V'$, then $A \underset{(U,V)}{\simeq} B$ in M, N implies $A \underset{(U',V')}{\simeq} B$ in M, N.*

(3.6) *If $A \neq \emptyset$ is contractible in U and $B \neq \emptyset$ is contractible in V, then $A \underset{(U,V)}{\simeq} B$ in M, N.*

§ 4. Some examples. In order to illustrate the sense of the notions introduced in § 3, consider the following examples:

(4.1) Example. Let A, B be two non-empty continua lying in the plane E^2. Assume that $E^2 \setminus B$ is connected and $E^2 \setminus A$ is not. Setting $M = N = E^2$, consider two fundamental sequences (2.8). If U is a neighborhood of A in E^2 containing all bounded components of $E^2 \setminus A$, then A is contractible in U and we infer by (2.12) that $A \underset{U}{\leqslant} B$ in E^2, E^2. Moreover, B is contractible in every neighborhood V of B in E^2. Hence $B \underset{V}{\leqslant} A$ in E^2, E^2 and consequently $A \underset{(U,V)}{\leftrightarrow} B$ in E^2, E^2.

If, however, U is a neighborhood of A in E^2 such that at least one bounded component of E^2 A is not contained in U, then there exists in A a 1-dimensional true cycle γ which is not homologous to zero in U. Consider two fundamental sequences $\underline{f}$, $\hat{\underline{f}}$ given by (2.8). If we observe that every 1-dimensional true cycle in B si null-homologous in B, we easily infer that the fundamental sequence $\hat{\underline{f}}\underline{f}$ assigns to γ a true cycle γ' homologous to zero in U; hence $\gamma \sim \gamma'$ in U and consequently the relation $\underline{\hat{f}}\underline{f} \simeq \underline{i}_{A,U}$ fails. Thus in this case A is not U-dominated by B.

(4.2) Example. Let A be a circle and B a dyadic solenoid, both lying in the space $M = N = E^3$. Let (U, V) be a neighborhood of (A, B) in (E^3, E^3) such that A is not contractible in U and B is not contractible in V.

Consider two fundamental sequences $\underline{f}, \underline{\hat{f}}$ given by (2.8). One can easily see that $\underline{f}$ assigns to each 1-dimensional true cycle $\underline{\gamma}$ in A with integers as coefficients a 1-dimensional true cycle in B homologous to zero in V. Since there are in A one-dimensional true cycles with integers as coefficients which are not homologous to zero in U, we infer — as in Example (4.1) — that the relation $\underline{\hat{f}}\underline{f} \simeq \underline{i}_{A,M}$ fails. Hence A is not U-dominated by B in E^3, E^3.

On the other hand, B is not V-dominated by A in E^3, E^3 because there exists in B a 1-dimensional true cycle $\hat{\gamma}$, with rational coefficients, which is not homologous to zero in V. The fundamental sequence $\underline{\hat{f}}$ assigns to $\hat{\gamma}$ a 1-dimensional true cycle γ in A, and $\underline{f}$ assigns to γ a 1-dimensional true cycle in B which is homologous to zero in V. It follows that $\underline{f}\underline{\hat{f}}$ is not homotopic in V to the fundamental identity sequence $\underline{i}_{B,N}$ and consequently B is not V-dominated by A in E^3, E^3.

(4.3) Example. Let C_0 denote the circle in E^2 given by the equation $x^2+y^2 = 1$, and let C_n denote the circle given by the equation $x^2+y^2 = n/(n+1)$ for $n = 1, 2, \ldots$ Setting

$$A = C_0 \cup \bigcup_{n=1}^{\infty} C_n,$$

we get a compactum in E^2. If we add to A the point $a = (0, 0)$, then we get another compactum B. It is clear that A is a retract of B and B is homeomorphic to a retract of A. It follows that

$$\mathrm{Sh}(A) \leqslant \mathrm{Sh}(B) \quad \text{and} \quad \mathrm{Sh}(B) \leqslant \mathrm{Sh}(A)$$

and we infer by (2.7) that

(4.4) $\quad A \underset{(U,V)}{\leftrightarrow} B$ in E^2, E^2 for every neighborhood (U, V) of (A, B) in (E^2, E^2).

Now let us consider an open neighborhood U of A (in E^2) and an open neighborhood W of the point a in E^2 such that $U \cap W = \emptyset$. It is clear that none of the circles C_m is contractible in U and that $V = U \cup W$ is an open neighborhood of B in E^2.

Suppose that $A \underset{(U,V)}{\simeq} B$ in E^2, E^2. Then there exist two fundamental sequences

$$\underline{f} = \{f_k, A, B\}_{E^2,E^2}, \qquad \underline{\hat{f}} = \{\hat{f}_k, B, A\}_{E^2,E^2}$$

such that $\underline{\hat{f}}\underline{f} \underset{U}{\simeq} \underline{i}_{A,E^2}$ and $\underline{f}\underline{\hat{f}} \underset{V}{\simeq} \underline{i}_{B,E^2}$.

Since $\underline{\hat{f}}$ is a fundamental sequence, there exists an open disk $D \subset W$ containing the point a and such that

(4.5) $$\hat{f}_k(D) \subset U \quad \text{for almost all } k.$$

Moreover, the relation $\underline{f}\underline{\hat{f}} \simeq \underline{i}_{B,E^2}$ in V implies that there is an index n_0 such that

(4.6) $$f_k(C_{n_0}) \subset D \quad \text{for almost all } k.$$

It follows by (4.5) and (4.6) that for almost all k the restriction $\hat{f}_k f_k / C_{n_0}$ is null-homotopic in U. But this is impossible, because the relation $\underline{\hat{f}}\underline{f} \underset{U}{\simeq} \underline{i}_{B,E^2}$ implies that $\hat{f}_k f_k / C_{n_0} \simeq i/C_{n_0}$ in U and C_{n_0} is not contractible in U. Thus A and B are not (U, V)-equivalent in E^2, E^2.

It follows by Example (4.3) and by (3.3) that:

(4.7) *The relation of (U, V)-affinity is less restrictive than the relation of (U, V)-equivalence.*

§ 5. Role of spaces M, N. Let us prove the following

(5.1) Theorem. *Let A, B, A', B' be compacta lying in spaces $M, N, M', N' \in \mathrm{AR}$, respectively, and let A be homeomorphic to A', and B homeomorphic to B'. Then for every neighborhood U' of A' in M' there exists a neighborhood U of A in M such that $A \underset{U}{\leqslant} B$ in M, N implies $A' \underset{U'}{\leqslant} B'$ in M', N'.*

Proof. Let $h_1\colon A \xrightarrow[\text{onto}]{} A'$ and $h_2\colon B \xrightarrow[\text{onto}]{} B'$ be homeomorphisms. Since $M, M', N, N' \in \mathrm{AR}$, there exist maps

$$\alpha\colon M \to M', \quad \hat{\alpha}\colon M' \to M, \quad \beta\colon N \to N', \quad \hat{\beta}\colon N' \to N$$

such that

$$\alpha(x) = h_1(x) \quad \text{for every } x \in A, \qquad \hat{\alpha}(x') = h_1^{-1}(x') \quad \text{for every } x' \in A',$$
$$\beta(y) = h_2(y) \quad \text{for every } y \in B, \qquad \hat{\beta}(y') = h_2^{-1}(y') \quad \text{for every } y' \in B'.$$

It follows that there is a neighborhood U of A in M such that

(5.2) $$\alpha(U) \subset U'.$$

Let us show that $A \underset{U}{\leqslant} B$ in M, N implies $A' \underset{U'}{\leqslant} B'$ in M', N'.

Consider two fundamental sequences

$$\underline{f} = \{f_k, A, B\}_{M,N} \quad \text{and} \quad \underline{\hat{f}} = \{\hat{f}_k, B, A\}_{N,M},$$

satisfying the condition

(5.3) $$\underline{\hat{f}}\underline{f} \underset{U}{\simeq} \underline{i}_{A,M}.$$

Setting

$$g_k = \beta f_k \hat{\alpha}, \quad \hat{g}_k = \alpha \hat{f}_k \hat{\beta} \quad \text{for every } k = 1, 2, \ldots,$$

we get two fundamental sequences:

$$\underline{g} = \{g_k, A', B'\}_{M',N'}, \quad \underline{\hat{g}} = \{\hat{g}_k, B', A'\}_{N',M'}.$$

It follows by (5.3) that there is a neighborhood $U_0 \subset U$ of A in M such that

(5.4) $$\hat{f}_k f_k / U_0 \simeq i/U_0 \quad \text{in } U \quad \text{for almost all } k.$$

Now let us select a neighborhood V_1 of B in N such that

(5.5) $$\hat f_k(V_1)\subset U_0 \quad \text{for almost all } k,$$

and let $V_2\subset V_1$ be a neighborhood of B in N such that

(5.6) $$\hat\beta\beta/V_2\simeq i/V_2 \quad \text{in } V_1.$$

Moreover, there exists a neighborhood $U_2\subset U_0$ of A in M such that

(5.7) $$f_k(U_2)\subset V_2 \quad \text{for almost all } k,$$

and there is a neighborhood $U_0'\subset U'$ of A' in M' such that

(5.8) $$\hat\alpha(U_0')\subset U_2$$

and that

(5.9) $$\alpha\hat\alpha/U_0'\simeq i/U_0' \quad \text{in } U'.$$

One infers by (5.8), (5.7) and (5.6) that

$$\hat\beta\beta f_k\hat\alpha/U_0'\simeq f_k\hat\alpha/U_0' \quad \text{in } V_1 \quad \text{for almost all } k.$$

By virtue of (5.5) we infer that

$$\hat f_k\hat\beta\beta f_k\hat\alpha/U_0'\simeq \hat f_k f_k\hat\alpha/U_0' \quad \text{in } U_0\subset U \quad \text{for almost all } k.$$

Using (5.8) and the inclusion $U_2\subset U_0$, we infer by (5.4) that

$$\hat f_k f_k\hat\alpha/U_0'\simeq\hat\alpha/U_0' \quad \text{in } U \quad \text{for almost all } k.$$

The last relation, combined with relation (5.3), gives

$$\hat g_k g_k/U_0' = \alpha\hat f_k\hat\beta\beta f_k\hat\alpha/U_0'\simeq\alpha\hat\alpha/U_0' \quad \text{in } U' \quad \text{for almost all } k.$$

It remains to apply (5.9) in order to obtain the homotopy

(5.10) $$\hat g_k g_k/U_0'\simeq i/U_0' \quad \text{in } U' \quad \text{for almost all } k.$$

Hence $A' \underset{U'}{\leqslant} B'$ in M', N' and Theorem (5.1) is proved.

(5.11) COROLLARY. *If A, B, A', B' are compacta lying in AR-spaces M, N, M', N', respectively, and if A is homeomorphic to A' and B is homeomorphic to B' then for every neighborhood (U', V') of (A', B') there is a neighborhood (U, V) of (A, B) in (M, N) such that $A \underset{(U,V)}{\leftrightarrow} B$ in M, N implies $A' \underset{(U',V')}{\leftrightarrow} B'$ in M', N'.*

(5.12) THEOREM. *If A, B, A', B' are compacta lying in AR-spaces M, N, M', N', respectively, where A is homeomorphic to A' and B — to B', then for every neighborhood (U', V') of (A', B') in (M', N') there is a neighborhood (U, V) of (A, B) in (M, N) such that $A \underset{(U,V)}{\simeq} B$ in M, N implies $A' \underset{(U',V')}{\simeq} B'$ in M', N'.*

Proof. Preserving the notation used in the proof of Theorem (5.1), consider a neighborhood V of B in N such that

$$\beta(V)\subset V'.$$

If $A \underset{(U,V)}{\simeq} B$ in M, N, then the fundamental sequences $\underline{f}$ and $\underline{\hat f}$ may be selected so that not only condition (5.3), but also the condition

$$\underline{f}\,\underline{\hat f} \underset{V}{\simeq} \underline{i}_{B,N}$$

is satisfied.

In the proof of Theorem (5.1) it is shown that there exists a neighborhood U_0' of A' in M' satisfying (5.10). By the same argument one shows that there is a neighborhood V_0' of B' in N' such that

(5.13) $$g_k\hat g_k/V_0'\simeq i/V_0' \quad \text{in } V' \quad \text{for almost all } k.$$

It follows by (5.10) and (5.13) that $A' \underset{(U',V')}{\simeq} B'$ in M', N' and the proof of Theorem (5.12) is concluded.

§ 6. Quasi domination, quasi-affinity and quasi-equivalence. Let A, B be two compacta lying in spaces $M, N\in$ AR, respectively. We say that A is *quasi-dominated by B (in M, N)* if $A \underset{U}{\leqslant} B$ in M, N for every neighborhood U of A in M. Observe that Theorem (5.1) implies that this relation is topological and the choice of spaces M, N is immaterial. In fact, let A', B' be homeomorphic to A and B, respectively, and contained in spaces $M', N'\in$ AR, respectively. Assume that A' is quasi-dominated by B' in M', N'. If U is any neighborhood of A in M, then a neighborhood U' of A' in M' may be selected so that $A' \underset{U'}{\leqslant} B'$ in M', N' implies $A \underset{U}{\leqslant} B$ in M, N. Hence A is quasi-dominated by B (in M, N). It follows that the words "in M, N" are superfluous and we can say shortly that A is *quasi-dominated by B* (notation: $A \overset{q}{\leqslant} B$). Moreover, we see that the relation $A \overset{q}{\leqslant} B$ is topological.

If $A \underset{(U,V)}{\leftrightarrow} B$ in M, N for every neighborhood (U, V) of (A, B) in (M, N), then we say that A and B are quasi-affinite (in M, N). Using Corollary (5.11) we see that the words "in M, N" are superfluous. Thus we say shortly that A and B are *quasi-affinite* (notation: $A \overset{q}{\leftrightarrow} B$). Moreover, we see that this relation is topological.

If $A \underset{(U,V)}{\simeq} B$ in M, N for every neighborhood (U, V) of (A, B) in (M, N), then we say that A and B are quasi-equivalent (in M, N). One infers by Theorem (5.12) that the words "in M, N" are superfluous. Thus we say shortly that A and B are *quasi-equivalent* (notation: $A \overset{q}{\simeq} B$). Theorem (5.12) implies also that this relation is topological.

Using (2.11), we infer that

(6.1) $$\mathrm{Sh}(A)\leqslant\mathrm{Sh}(B) \quad \text{implies that} \quad A \overset{q}{\leqslant} B,$$

and using (3.4), we infer that

(6.2) $$\mathrm{Sh}(A)=\mathrm{Sh}(B) \quad \text{implies} \quad A \overset{q}{\simeq} B.$$

Now let us prove a theorem which implies in particular that statements converse to (6.1) and to (6.2) are not true.

(6.3) THEOREM. *All 0-dimensional infinite compacta are quasi-equivalent.*

Proof. Let A, B be two 0-dimensional infinite compacta. Since quasi-equivalence is a topological relation, we can assume that A and B are subsets of the 1-dimensional Euclidean space E^1. It is clear that for every neighborhood $(\hat{U}, \hat{V})$ of (A, B) in (E^1, E^1) there exist neighborhoods $U \subset \hat{U}$ and $V \subset \hat{V}$ of A and of B, respectively, in E^1 of the following form:

U is the union of n intervals $I_\nu = \langle x_\nu, x'_\nu \rangle$, where $x_\nu < x'_\nu$ for $\nu = 1, 2, \ldots, n$ and $x'_\nu < x_{\nu+1}$ for $\nu = 1, 2, \ldots, n-1$, such that there exists a point $a_\nu \in A \cap I_\nu$ for $\nu = 1, 2, \ldots, n$.

V is the union of n intervals $J_\nu = \langle y_\nu, y'_\nu \rangle$, where $y_\nu < y'_\nu$ for $\nu = 1, 2, \ldots, n$ and $y'_\nu < y_{\nu+1}$ for $\nu = 1, 2, \ldots, n-1$, such that there exists a point $b_\nu \in B \cap J_\nu$ for every $\nu = 1, 2, \ldots, n$.

Setting

$$f(x) = b_\nu \quad \text{for every point } x \in A \cap I_\nu, \qquad \nu = 1, 2, \ldots, n,$$
$$\hat{f}(y) = a_\nu \quad \text{for every point } y \in B \cap J_\nu, \qquad \nu = 1, 2, \ldots, n,$$

we get two maps $f\colon A \to B$ and $\hat{f}\colon B \to A$ such that $\hat{f}f(x) = a_\nu$ for every point $x \in A \cap I_\nu$ and $f\hat{f}(y) = b_\nu$ for every point $y \in B \cap J_\nu$.

It readily follows that $f\hat{f} \simeq i/A$ in U and $\hat{f}f \simeq i/B$ in V. One infers that $\underline{f} = \{f_k, A, B\}_{E^1, E^1}$ and $\underline{\hat{f}} = \{\hat{f}_k, B, A\}_{E^1, E^1}$ are fundamental sequences generated by f and $\hat{f}$, respectively, then

$$\underline{\hat{f}}\,\underline{f} \underset{U}{\simeq} \underline{i}_{A,E^1} \quad \text{and} \quad \underline{f}\,\underline{\hat{f}} \underset{V}{\simeq} \underline{i}_{B,E^1}.$$

Hence $A \underset{(U,V)}{\simeq} B$ in E^1, E^1 and consequently also (by (3.5)) $A \underset{(\hat{U},\hat{V})}{\simeq} B$ in E^1, E^1. Hence $A \overset{q}{\simeq} B$ and the proof of Theorem (6.3) is finished.

(6.4) Corollary. *There exist compacta A, B such that* $\mathrm{Sh}(A) < \mathrm{Sh}(B)$ *and* $A \overset{q}{\simeq} B$.

It is so, for instance, if A is a countable compactum and B is the Cantor discontinuum.

§ 7. Transitivity. Let us prove the following

(7.1) Theorem. *Let A, B be compacta lying in spaces $M, N \in$ AR, respectively, and let U be a neighborhood of A in M such that $A \underset{U}{\leqslant} B$ in M, N. Then there is a neighborhood V of B in N such that for every compactum C lying in a space $P \in$ AR the relation $B \underset{V}{\leqslant} C$ in N, P implies the relation $A \underset{U}{\leqslant} C$ in M, P.*

Proof. $A \underset{U}{\leqslant} B$ in M, N means that there exist two fundamental sequences,

$$\underline{f} = \{f_k, A, B\}_{M,N}, \quad \underline{\hat{f}} = \{\hat{f}_k, B, A\}_{N,M},$$

and a neighborhood U_0 of A in M such that

$$\hat{f}_k f_k / U_0 \simeq i/U_0 \quad \text{in } U \quad \text{for almost all } k. \tag{7.2}$$

Moreover, there exists a neighborhood V of B in N such that

$$\hat{f}_k(V) \subset U \quad \text{for almost all } k. \tag{7.3}$$

If $B \underset{V}{\leqslant} C$ in N, P, then there are fundamental sequences

$$\underline{g} = \{g_k, B, C\}_{N,P}, \quad \underline{\hat{g}} = \{\hat{g}_k, C, B\}_{P,N}$$

and a neighborhood V_0 of B in N such that

$$\hat{g}_k g_k / V_0 \simeq i/V_0 \quad \text{in } V \quad \text{for almost all } k. \tag{7.4}$$

Now let us select a neighborhood $U'_0 \subset U_0$ of A in M such that

$$f_k(U'_0) \subset V_0 \quad \text{for almost all } k. \tag{7.5}$$

Setting

$$h_k = g_k f_k, \quad \hat{h}_k = \hat{f}_k \hat{g}_k \quad \text{for} \quad k = 1, 2, \ldots,$$

consider the fundamental sequences

$$\underline{h} = \underline{g}\,\underline{f} = \{h_k, A, C\}_{M,P}, \quad \underline{\hat{h}} = \underline{\hat{f}}\,\underline{\hat{g}} = \{\hat{h}_k, C, A\}_{P,M}.$$

It follows by (7.4) and (7.5) that

$$\hat{g}_k g_k f_k / U'_0 \simeq f_k / U'_0 \quad \text{in } V \quad \text{for almost all } k.$$

Using (7.3), (7.2) and the inclusion $U'_0 \subset U_0$, one obtains

$$\hat{h}_k h_k / U'_0 = \hat{f}_k \hat{g}_k g_k f_k / U'_0 \simeq \hat{f}_k f_k / U'_0 \simeq i/U'_0 \quad \text{in } U \quad \text{for almost all } k,$$

and consequently

$$\underline{\hat{h}}\,\underline{h} \simeq \underline{i}_{A,M} \quad \text{in } U. \tag{7.6}$$

Thus we have shown that $A \underset{U}{\leqslant} C$ in M, P and the proof of Theorem (7.1) is concluded.

(7.7) Corollary. *If A, B, C are compacta lying in AR-spaces M, N, P, respectively, and if U is a neighborhood of A in M and W is a neighborhood of C in P such that for every neighborhood V of B in N*

$$A \underset{(U,V)}{\leftrightarrow} B \quad \text{in } M, N \quad \text{and} \quad B \underset{(V,W)}{\leftrightarrow} C \quad \text{in } N, P,$$

then $A \underset{(U,W)}{\leftrightarrow} C$ in M, P.

In fact, $A \underset{(U,V)}{\leftrightarrow} B$ in M, N means that

$$A \underset{U}{\leqslant} B \quad \text{in } M, N, \tag{7.8}$$

$$B \underset{V}{\leqslant} A \quad \text{in } N, M. \tag{7.9}$$

$B \underset{(V,W)}{\leftrightarrow} C$ in N, P means that

$$B \underset{V}{\leqslant} C \quad \text{in } N, P, \tag{7.10}$$

$$C \underset{W}{\leqslant} B \quad \text{in } P, N. \tag{7.11}$$

Using Remark (2.5), we infer by Theorem (7.1) that if the neighborhood V of B in N is sufficiently small, then (7.8) and (7.10) imply that $A \underset{U}{\leqslant} C$ in M, P and (7.9), (7.11) imply that $C \underset{W}{\leqslant} A$ in P, M. Hence $A \underset{(U,W)}{\leftrightarrow} C$ in M, P.

(7.12) COROLLARY. *The relations of quasi-domination and of quasi-affinity are transitive.*

In fact, if $A \overset{q}{\leqslant} B$ and $B \overset{q}{\leqslant} C$, then for any spaces $M, N, P \in \mathrm{AR}$ containing A, B, C, respectively, and for every neighborhood U of A in M and V of B in N the relations $A \underset{U}{\leqslant} B$ in M, N and $B \underset{V}{\leqslant} C$ in N, P hold true. By Theorem (7.1) the neighborhood V can be selected so that these relations imply $A \underset{U}{\leqslant} C$ in M, P. Hence $A \overset{q}{\leqslant} C$.

Using Corollary (7.7), we show in the same way that $A \overset{q}{\leftrightarrow} B$ and $B \overset{q}{\leftrightarrow} C$ imply $A \overset{q}{\leftrightarrow} C$.

(7.13) PROBLEM. *Is the relation of quasi-equivalence transitive?*

§ 8. Relations of quasi-domination, quasi-affinity and quasi-equivalence as shape invariants. Let us prove the following

(8.1) THEOREM. *If* $\mathrm{Sh}(A) = \mathrm{Sh}(A')$ *and* $\mathrm{Sh}(B) = \mathrm{Sh}(B')$, *then* $(A \overset{q}{\leqslant} B) \Rightarrow (A' \overset{q}{\leqslant} B')$, $(A \overset{q}{\leftrightarrow} B) \Rightarrow (A' \overset{q}{\leftrightarrow} B')$ *and* $(A \overset{q}{\simeq} B) \Rightarrow (A' \overset{q}{\simeq} B')$.

Proof. The first two implications are simple consequences of (2.11), (3.4) and of Corollary (7.12). In order to prove the third implication, it suffices to establish the following proposition:

(8.2) If $A \overset{q}{\simeq} B$ and if $\mathrm{Sh}(B) = \mathrm{Sh}(C)$, then $A \overset{q}{\simeq} C$.

Let us assume that the compacta A, B, C lie in spaces $M, N, P \in \mathrm{AR}$ respectively. It suffices to show that if U is an open neighborhood of A in M and W is an open neighborhood of C in P, then $A \underset{(U,W)}{\simeq} C$ in M, P.

Since $\mathrm{Sh}(B) = \mathrm{Sh}(C)$, there exist fundamental sequences

$$\underline{g} = \{g_k, B, C\}_{N,P} \quad \text{and} \quad \underline{\hat{g}} = \{\hat{g}_k, C, B\}_{P,N}$$

such that $\underline{\hat{g}}\,\underline{g} \simeq \underline{i}_{B,N}$ and $\underline{g}\,\underline{\hat{g}} \simeq \underline{i}_{C,P}$. Then there is a neighborborhood $W_0 \subset W$ of C in P such that

(8.3) $$g_k \hat{g}_k / W_0 \simeq i / W_0 \quad \text{in } W \quad \text{for almost all } k.$$

Moreover, there exists a neighborhood V of B in N such that

(8.4) $$g_k(V) \subset W_0 \quad \text{for almost all } k.$$

The hypothesis $A \overset{q}{\simeq} B$ implies that $A \underset{(U,V)}{\leftrightarrow} B$, i.e., there exist fundamental sequences

$$\underline{f} = \{f_k, A, B\}_{M,N} \quad \text{and} \quad \underline{\hat{f}} = \{\hat{f}_k, B, A\}_{N,M}$$

and a neighborhood $(U', V') \subset (U, V)$ of (A, B) in (M, N) such that

(8.5) $$\hat{f}_k f_k / U' \simeq i / U' \quad \text{in } U \quad \text{for almost all } k,$$

and

(8.6) $$f_k \hat{f}_k / V' \simeq i / V' \quad \text{in } V \quad \text{for almost all } k.$$

Setting $h_k = g_k f_k$, $\hat{h}_k = \hat{f}_k \hat{g}_k$ for $k = 1, 2, \ldots$, we get two fundamental sequences

$$\underline{h} = \{h_k, A, C\}_{M,P} = \underline{g}\,\underline{f} \quad \text{and} \quad \underline{\hat{h}} = \{\hat{h}_k, C, A\}_{P,M} = \underline{\hat{f}}\,\underline{\hat{g}}.$$

Since $\underline{\hat{f}}\colon B \to A$, there is a neighborhood V_1 of B in N such that

(8.7) $$\hat{f}_k(V_1) \subset U \quad \text{for almost all } k.$$

Since $\underline{\hat{g}}\,\underline{g} \simeq \underline{i}_{B,N}$, there is a neighborhood V_2 of B in N such that

(8.8) $$\hat{g}_k g_k / V_2 \simeq i / V_2 \quad \text{in } V_1 \quad \text{for almost all } k.$$

Moreover, $f_k(A) \subset V_2$ for almost all k, and we infer by (8.8) that

$$\hat{g}_k g_k f_k / A \simeq f_k / A \quad \text{in } V_1 \quad \text{for almost all } k.$$

Using (8.5) and (8.7), one infers that

(8.9) $$\hat{h}_k h_k / A = \hat{f}_k \hat{g}_k g_k f_k / A \simeq \hat{f}_k f_k / A \simeq i / A \quad \text{in } U \quad \text{for almost all } k.$$

Moreover, $\hat{g}_k(C) \subset V'$ for almost all k. It follows by (8.6) that

$$f_k \hat{f}_k \hat{g}_k / C \simeq \hat{g}_k / C \quad \text{in } V \quad \text{for almost all } k.$$

Using (8.4), we infer that $g_k f_k \hat{f}_k \hat{g}_k / C \simeq g_k \hat{g}_k / C$ in W_0 for almost all k. Since $W_0 \subset W$, we infer by (8.3) that

(8.10) $$h_k \hat{h}_k / C = g_k f_k \hat{f}_k \hat{g}_k / C \simeq i / C \quad \text{in } W \quad \text{for almost all } k.$$

Relations (8.9) and (8.10) imply that $A \underset{(U,V)}{\simeq} C$ in M, P and thus the proof of Proposition (8.2), and hence also of Theorem (8.1), is finished.

§ 9. Case of ANR-spaces. In the case of ANR-spaces the relations of quasi-domination, of quasi-affinity and of quasi-equivalence reduce to the well-known relations for shapes. In order to show this, let us first prove the following

(9.1) LEMMA. *Let A, B be compact ANR's lying in $M, N \in \mathrm{AR}$, respectively. Let U be a closed neighborhood of A in M such that A is a retract of U. Then for every fundamental sequence $\underline{f} = \{f_k, A, B\}_{M,N}$ there exists a fundamental sequence $\underline{f}' = \{f_k', A, B\}_{M,N}$ homotopic to $\underline{f}$ and such that $f_k'(U) \subset B$ for $k = 1, 2, \ldots$*

Proof. Since $B \in \mathrm{ANR}$, we can assume that $\underline{f}$ is generated by a map $f\colon A \to B$. Let $r\colon U \to A$ be a retraction. Since U is closed in M and $N \in \mathrm{AR}$, there is a map $f'\colon M \to N$ such that $f'(x) = fr(x)$ for every point $x \in U$. Setting $f_k' = f'$ for every $k = 1, 2, \ldots$, we can easily see that $\underline{f} = \{f_k', A, B\}_{M,N}$ is a fundamental sequence satisfying the required conditions.

(9.2) THEOREM. *If A, B are two compact ANR-spaces lying in $M, N \in \mathrm{AR}$, respectively, then there exists a neighborhood U of A (in M) such that $A \underset{U}{\leqslant} B$ in M, N*

implies $\mathrm{Sh}(A) \leqslant \mathrm{Sh}(B)$, *and there exists a neighborhood* (U, V) *of* (A, B) *in* (M, N) *such that* $A \underset{(U,V)}{\simeq} B$ *in* M, N *implies* $\mathrm{Sh}(A) = \mathrm{Sh}(B)$.

Proof. Since A, B are compact ANR-sets, there exist a closed neighborhood U of A in M and a closed neighborhood V of B in N such that A is a retract of U and B is a retract of V.

Let $\underline{f} = \{f_k, A, B\}_{M,N}$, $\underline{\hat{f}} = \{\hat{f}_k, B, A\}_{N,M}$ be fundamental sequences satisfying the condition

$$(9.3) \qquad \underline{\hat{f}}\,\underline{f} \underset{U}{\simeq} \underline{i}_{A,M}.$$

Using Lemma (9.1), we may assume that

$$(9.4) \qquad f_k(U) \subset B \quad \text{and} \quad \hat{f}_k(V) \subset A \quad \text{for every } k = 1, 2, \ldots$$

We infer by (9.3) that there is an index k_0 such that

$$(9.5) \qquad \hat{f}_{k_0} f_{k_0}/A \simeq i/A \quad \text{in } U.$$

Setting $\varphi(x) = f_{k_0}(x)$ for $x \in A$ and $\psi(y) = \hat{f}_{k_0}(y)$ for $y \in B$, we get two maps,

$$\varphi\colon A \to B \quad \text{and} \quad \psi\colon B \to A,$$

such that

$$\psi\varphi = \hat{f}_{k_0} f_{k_0}/A \simeq i/A \quad \text{in } U.$$

Hence A is homotopically dominated by B. Thus $\mathrm{Sh}(A) \leqslant \mathrm{Sh}(B)$.

If we assume also that the relation $\underline{f}\,\underline{\hat{f}} \underset{V}{\simeq} \underline{i}_{B,N}$ holds true, then the index k_0 can be selected so that not only condition (9.5) is satisfied, but also the condition

$$f_{k_0} \hat{f}_{k_0}/B \simeq i/B \quad \text{in } V.$$

Then we show, by an analogous argument, that the maps φ, ψ satisfy the two conditions $\psi\varphi \simeq i/A$ and $\varphi\psi \simeq i/B$. Hence A and B are homotopically equivalent and consequently $\mathrm{Sh}(A) = \mathrm{Sh}(B)$.

§ 10. Betti numbers. We use here the Vietoris homology theory based on the notion of the true cycle (compare [1], p. 35). In the sequel we consider only true cycles with rational coefficients. By the *n-th Betti number* $p_n(A)$ of a compactum A we understand the maximal number of n-dimensional true cycles of A which are homologically independent in A, i.e., a linear combination of those true cycles with integer coefficients is null-homologous in A only if all coefficients vanish.

First let us prove the following

(10.1) Lemma. *Let A be a compactum lying in a space M. If $\underline{\gamma}_1, \underline{\gamma}_2, \ldots, \underline{\gamma}_m$ are n-dimensional true cycles in A which are homologically dependent in every neighborhood U of A in M, then they are homologically dependent in A.*

Proof (by induction). Let $m = 1$ and let

$$\underline{\gamma}_1 = \underline{\gamma} = (\gamma_1, \gamma_2, \ldots)$$

be an n-dimensional true cycle in A. If the system consisting of $\underline{\gamma}$ is homologically dependent in every neighborhood U of A in M, then $\underline{\gamma} \sim 0$ in U and we infer that for every $\varepsilon > 0$ the cycle γ_k is for almost all k the boundary of an ε-chain lying in U. We infer that there exists a sequence of indices $j_1 < j_2 < \ldots$ such that for $j_\nu \leqslant i < j_{\nu+1}$ there is in A a $1/\nu$-chain $\varkappa_i$ such that $\gamma_i = \partial\varkappa_i$. For $i = 1, 2, \ldots, j_1 - 1$ we define $\varkappa_i$ as an arbitrary chain in A satisfying the condition $\partial\varkappa_i = \gamma_i$. It is clear that $\underline{\varkappa} = (\varkappa_1, \varkappa_2, \ldots)$ is an infinite chain in A with $\partial\underline{\varkappa} = \underline{\gamma}$. Hence $\underline{\gamma} \sim 0$ in A and the proof in the case of $m = 1$ is finished.

Assume now that the statement is true for a natural number m and consider the system of n-dimensional true cycles in A:

$$(10.2) \qquad \underline{\gamma}_1, \underline{\gamma}_2, \ldots, \underline{\gamma}_m, \underline{\gamma}_{m+1}.$$

Let $U_1 \supset U_2 \supset \ldots$ be a sequence of neighborhoods of A in M shrinking to A. If system (10.2) is homologically dependent in every neighborhood U of A in M, then for every $r = 1, 2, \ldots$ there exist integers $l_1^{(r)}, l_2^{(r)}, \ldots, l_{m+1}^{(r)}$, not all vanishing and such that the true cycle

$$\sum_{i=1}^{m+1} l_i^{(r)} \cdot \underline{\gamma}_i$$

is homologous to zero in U_r. If $l_{m+1}^{(r)} = 0$ for almost all r, then we infer that the system $\underline{\gamma}_1, \underline{\gamma}_2, \ldots, \underline{\gamma}_m$ is homologically dependent in every neighborhood U of A in M; hence the hypothesis of induction implies that it is homologically dependent in A. Then also the system (10.2) is homologically dependent in A.

Thus we can assume that there exists a sequence of indices $r_1 < r_2 < \ldots$ such that $l_{m+1}^{(r_\nu)} \neq 0$ for $\nu = 1, 2, \ldots$ Setting

$$w_i^{(\nu)} = -\frac{l_i^{(r_\nu)}}{l_{m+1}^{(r_\nu)}},$$

we get rational numbers such that

$$\underline{\gamma}_{m+1} \sim w_1^{(\nu)} \cdot \underline{\gamma}_1 + \ldots + w_m^{(\nu)} \cdot \underline{\gamma}_m \quad \text{in } U_{r_\nu} \quad \text{for} \quad \nu = 1, 2, \ldots$$

If there exists a system of rational numbers $(w_1, w_2, \ldots, w_m)$ such that $(w_1^{(\nu)}, \ldots, w_m^{(\nu)}) = (w_1, \ldots, w_m)$ for an infinite number of indices ν, then we infer that $w_1 \cdot \underline{\gamma}_1 + \ldots$ $\ldots + w_m \cdot \underline{\gamma}_m - \underline{\gamma}_{m+1}$ is a true cycle homologous to 0 in every neighborhood U of A (in M); hence it is also homologous to zero in A. Then system (10.2) is homologically dependent in A.

However, if such a system of rational numbers $(w_1, w_2, \ldots, w_m)$ does not exist, then for every $\nu = 1, 2, \ldots$ there exists an index $\alpha(\nu) > \nu$ such that

$$(w_1^{(\nu)}, w_2^{(\nu)}, \ldots, w_m^{(\nu)}) \neq (w_1^{(\alpha(\nu))}, w_2^{(\alpha(\nu))}, \ldots, w_m^{(\nu(\alpha))}).$$

It follows that

$$(w_1^{(\nu)} - w_1^{(\alpha(\nu))}) \cdot \underline{\gamma}_1 + (w_2^{(\nu)} - w_2^{(\alpha(\nu))}) \cdot \underline{\gamma}_2 + \ldots + (w_m^{(\nu)} - w_m^{(\alpha(\nu))}) \cdot \underline{\gamma}_m \sim 0 \quad \text{in } U_\nu,$$

and consequently, by the hypothesis of induction, the system $(\underline{\gamma}_1, \underline{\gamma}_2, \ldots, \underline{\gamma}_m)$ is homologically dependent in A. Hence also the system (10.2) is homologically dependent in A and the proof of Lemma (10.1) is finished.

(10.3) THEOREM. *If $A \overset{q}{\leqslant} B$, then $p_n(A) \leqslant p_n(B)$, and if $A \overset{q}{\leftrightarrow} B$, then $p_n(A) = p_n(B)$.*

Proof. In order to prove that $A \overset{q}{\leqslant} B$ implies $p_n(A) \leqslant p_n(B)$, we can assume that $p_n(B) = m$ is finite. Then there exists in B a system $\underline{\gamma}'_1, \underline{\gamma}'_2, \ldots, \underline{\gamma}'_m$ of n-dimensional true cycles such that each n-dimensional true cycle $\underline{\gamma}'$ in B is homologous in B to a linear combination

$$w_1 \cdot \underline{\gamma}'_1 + w_2 \cdot \underline{\gamma}'_2 + \ldots + w_m \cdot \underline{\gamma}'_m$$

with rational coefficients $w_1, w_2, \ldots, w_m$.

Let $\underline{\gamma}$ be an n-dimensional true cycle in A and let M be an AR-space containing A, and let N be an AR-space containing B. If $A \overset{q}{\leqslant} B$, then for every neighborhood U of A in M there are two fundamental sequences

$$\underline{f} = \{f_k, A, B\}_{M,N}, \quad \underline{\hat{f}} = \{\hat{f}_k, B, A\}_{N,M}$$

such that

$$\underline{\hat{f}}\underline{f} \underset{U}{\simeq} \underline{i}_{A,M}. \tag{10.4}$$

The fundamental sequence $\underline{f}$ assigns to $\underline{\gamma}$ an n-dimensional true cycle $\underline{\gamma}'$ in B. Then there exist rational numbers $w_1, w_2, \ldots, w_m$ such that

$$\underline{\gamma}' \sim \sum_{i=1}^{m} w_i \cdot \underline{\gamma}'_i.$$

It follows by (10.4) that $\underline{\hat{f}}$ assigns to $\underline{\gamma}'$ a true cycle in A which is homologous in U to the true cycle $\underline{\gamma}$. If $\underline{\gamma}_i$ is a true cycle assigned by $\underline{\hat{f}}$ to $\underline{\gamma}'_i$, we infer that

$$\underline{\gamma} \sim \sum_{i=1}^{m} w_i \cdot \underline{\gamma}_i \quad \text{in } U.$$

Thus we have shown that for every neighborhood U of A in M there exists in A a system of m true n-dimensional cycles $\underline{\gamma}_1, \underline{\gamma}_2, \ldots, \underline{\gamma}_m$ such that every n-dimensional true cycle $\underline{\gamma}$ in A is homologous in U to a linear combination of that system. It follows that every system consisting of $m+1$ true n-dimensional cycles in A is homologically dependent in every neighborhood U of A. We infer by Lemma (10.1) that a such system is homologically dependent in A; hence $p_n(A) \leqslant m$.

If $A \overset{q}{\leftrightarrow} B$ then $A \overset{q}{\leqslant} B$ and also $B \overset{q}{\leqslant} A$; hence $p_n(A) \leqslant p_n(B)$ and also $p_n(B) \leqslant p_n(A)$, and thus $p_n(A) = p_n(B)$.

(10.5) PROBLEM. *Is it true that $A \overset{q}{\simeq} B$ implies that every homology group $H_n(A, \mathfrak{A})$ of A over an arbitrary Abelian group $\mathfrak{A}$ is isomorphic with the group $H_n(B, \mathfrak{A})$?*

§ 11. Movability. Let us prove the following

(11.1) THEOREM. *If $A \overset{q}{\leqslant} B$ and if B is movable, then A is movable.*

Proof. Assume that $A \subset M$, $B \subset N$, where $M, N \in \mathrm{AR}$. Let U be a neighborhood of A in M. Then there exist two fundamental sequences,

$$\underline{f} = \{f_k, A, B\}_{M,N} \quad \text{and} \quad \underline{\hat{f}} = \{\hat{f}_k, B, A\}_{N,M},$$

such that

$$\underline{f}\underline{\hat{f}} \underset{U}{\simeq} \underline{i}_{A,M}. \tag{11.2}$$

It follows by (11.2) that there exists a neighborhood $W \subset U$ of A in M such that

$$\hat{f}_k f_k / W \simeq i / W \quad \text{in } U \quad \text{for almost all } k. \tag{11.3}$$

Moreover, there exists a neighborhood V of B in N such that

$$\hat{f}_k(V) \subset U \quad \text{for almost all } k. \tag{11.4}$$

Since B is movable, there is a neighborhood $V_0 \subset V$ of B in N such that for every neighborhood V' of B in N there exists a homotopy

$$\psi\colon V_0 \times \langle 0, 1\rangle \to V$$

such that $\psi(y, 0) = y$ and $\psi(y, 1) \in V'$ for every point $y \in V_0$.

Now let us select a neighborhood $U_0 \subset W$ of A in M such that

$$f_k(U_0) \subset V_0 \quad \text{for almost all } k. \tag{11.5}$$

If U' is an arbitrary neighborhood of A in M, then we can select the neighborhood V' of B in N so that

$$\hat{f}_k(V') \subset U' \cap W \quad \text{for almost all } k. \tag{11.6}$$

Consider an index k_0 such that for $k = k_0$ conditions (11.3), (11.4), (11.5) and (11.6) are satisfied. It follows by (11.3) that there is a homotopy

$$\vartheta\colon W \times \langle 0, 1\rangle \to U$$

such that $\vartheta(x, 0) = x$ and $\vartheta(x, 1) = \hat{f}_{k_0} f_{k_0}(x)$ for every point $x \in W$. Setting

$$\varphi(x, t) = \vartheta(x, 2t) \qquad \text{for } x \in U_0 \text{ and for } 0 \leqslant t \leqslant \tfrac{1}{2},$$
$$\varphi(x, t) = \hat{f}_{k_0}\psi(f_{k_0}(x), 2t-1) \qquad \text{for } x \in U_0 \text{ and for } \tfrac{1}{2} \leqslant t \leqslant 1,$$

we get a homotopy

$$\varphi\colon U_0 \times \langle 0, 1\rangle \to U$$

such that $\varphi(x, 0) = x$ and $\varphi(x, 1) \in U'$ for every point $x \in U$. Thus A is movable and Theorem (11.1) is proved.

(11.7) Remark. In the same way one can show that if $A \overset{q}{\leqslant} B$ and if B is movable in dimension n (see [2]) (or, more generally, is A-movable, see [3]), then A is movable in dimension n (or A-movable).

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Characterizations of real functions by continua

by

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Abstract. For a real-valued function f with domain an open interval, definitions for the functional concepts of Darboux at a point and connected at a point are examined and two-sided conditions in these definitions are reduced to one-sided conditions. The existence and characterizations of two new subclasses of Darboux functions are obtained and several examples are given to indicate that none of the four classes mentioned above are equivalent.

1. Introduction. This paper deals with properties of real functions which can be characterized by the types of continua which they intersect. A function f is said to be a *Darboux function* if $f(C)$ is connected whenever C is a connected subset of the domain of f. Equivalently, a real function f is a *Darboux function* if every horizontal interval which meets $f(+)$ and $f(-)$ meets f. A function which has a connected graph is called a *connected function*. In a paper published in 1965, [3], Bruckner and Ceder define what it means for a function to be Darboux at a point and in a paper published in 1971, [4], Garrett, Nelms, and Kellum define what it means for a function to be connected at a point. The main theorems in this paper reduce these definitions and still retain the results of [3] and [4]. Also, we exhibit two new classes of real functions each of which are subclasses of the class of Darboux functions and each of which contain the class of connected functions. The author wishes to express his appreciation to Harvey Rosen for many helpful ideas.

2. Notation. If M is a subset of the plane E then $(M)_X$ denotes the X-projection of M and M_K denotes those points of M which have X-projection in K where K is a subset of the X-axis. We denote the vertical line through the point $(x, 0)$ by l_x. If f is a real function with domain a subset of the real line R then $f(+)$ denotes the subset of E consisting of all ordered pairs (x, y) where x is in $(f)_X$ and $y>f(x)$. We define $f(-)$ similarly. A continuum is a closed connected subset of E. A horizontal segment is a bounded open connected subset of a horizontal line and a horizontal interval is the closure of a horizontal segment. Unless otherwise stated, all functions considered are real functions with domain an open connected subset of R. No distinction will be made between a function and its graph.