Higher Tall axioms

by

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Abstract. Tall has considered a weakened version of Martin's axiom which applies only to sets of forcing conditions which are constructible from some real number. We show the consistency (relative to the existence of a weakly compact cardinal) of a natural modification of Tall's axiom which applies to $\varkappa$ closed sets of forcing conditions with the $\varkappa^+$ chain condition.

Solovay and Tennenbaum [6] proved the relative consistency with the negation of the continuum hypothesis of an axiom which has since come to be known as Martin's axiom. A modification of this axiom was later introduced by Tall [9].

Both of these axioms are strengthenings of the Rasiowa–Sikorski lemma for countable chain condition Boolean algebras. It is natural to look for modification of these axioms which will apply to other algebras. In this paper, we show the consistency (relative to a weakly compact cardinal) of such a generalization of Tall's axiom.

§ 1. **Preliminaries.** We will generally follow the notation of Jech [2]. In particular, if $\mathfrak{P} = \langle P, \leqslant_p \rangle$ is a poset then the initial segments of $\mathfrak{P}$ generate a topology on P. The regular open sets of this topological space form a complete Boolean algebra (cBa), $\mathscr{B} = \langle B, \leqslant_{\mathscr{B}} \rangle = \mathrm{RO}(\mathfrak{P})$. There is a canonical embedding of $\mathfrak{P}$ as a dense subset of $\langle B - \{0\}, \leqslant_{\mathscr{B}} \rangle$.

If M is a standard transitive model of set theory and $\mathscr{B} \in M$ is a cBa in M then M , is the Scott-Solovay Boolean valued model of set theory. $[\![\varphi]\!] = [\![\varphi]\!]_{\mathscr{B}} \in \mathscr{B}$ is the truth value of the statement φ in $M^{(\mathscr{B})}$.

There is a canonical embedding of M into $M^{(\mathscr{B})}$ where the image in $M^{(\mathscr{B})}$ of $m \in M$ is $\check{m}$. If G is an M-generic ultrafilter on $\mathscr{B}$ then we may regard the elements of $M^{(\mathscr{B})}$ as names for the elements of $M[G]$; let i_G be the denotation function so that $m \in M^{(\mathscr{B})}$ is a name for $i_G(m) \in M[G]$. For $m \in M$, $i_G(\check{m}) = m$.

Two elements of a poset are called *incompatible* if they have no common predecessor. Two elements of a Boolean algebra are called *incompatible* if their product is 0. (The ambiguity which arises from the fact that a Boolean algebra is a particular kind of poset should cause no serious confusion.)

A poset or a Boolean algebra is said to satisfy the $\varkappa$-*chain condition* ($\varkappa$cc) provided that every set of pairwise incompatible elements has cardinality less than $\varkappa$. It is easy to see that a poset $\mathfrak{P}$ is $\varkappa$cc iff $\mathscr{B} = \mathrm{RO}(\mathfrak{P})$ is $\varkappa$cc.

If every decreasing sequence in $\mathfrak{P}$ of length less than $\varkappa$ has a lower bound in $\mathfrak{P}$ then $\mathfrak{P}$ is said to be *$\varkappa$-closed*. If $\mathscr{B}$ is a cBa and there is a $\varkappa$-closed poset $\mathfrak{P}$ dense in $\langle \mathscr{B}-\{0\}, \leqslant\rangle$ then we say that $\mathscr{B}$ is *$\varkappa$-closed*.

The following facts are well-known:

1.0. Proposition. *If $\mathscr{B} = \langle B, \leqslant_{\mathscr{B}}\rangle \in M$ is a* cBa, *$\mathfrak{P} = \langle P, \leqslant_p\rangle$ is a substructure of $\mathscr{B}$ such that P is dense in*

$$\langle B-\{0\}, \leqslant_{\mathscr{B}}\rangle \quad \text{and} \quad G = \{b \in B|\ (\exists p \in P \cap G)[p \leqslant b]\},$$

then the following are equivalent:

a. *G is an M-generic ultrafilter on $\mathscr{B}$,*

b. *$G \cap P$ is $\mathfrak{P}$-generic over M.*

Proof. [7, pp. 30-34].

1.1. Proposition. *Suppose $\mathfrak{P} \in M$ is $\varkappa$cc and G is $\mathfrak{P}$-generic over M. If $\varkappa < \lambda$ is a cardinal in M then λ is a cardinal in $M[G]$. If $\varkappa$ is a regular cardinal in M then $\varkappa$ is a regular cardinal in $M[G]$.*

Proof. [2, p. 65].

1.2. Proposition. *Suppose $\mathfrak{P} \in M$ is $\varkappa$-closed. If G is $\mathfrak{P}$-generic over M and $\lambda < \varkappa$ then $(\mathfrak{P}(\lambda)$ in $M) = (\mathfrak{P}(\lambda)$ in $M[G])$.*

Proof. [4, p. 372].

When X is a set we write $[X]^2$ for the set of all unordered pairs of distinct elements of X. A subset X of $\varkappa$ is said to be *homogeneous for a function* $f\colon [\varkappa]^2 \to \lambda$ if $f''[X]^2$ is a singleton. An uncountable cardinal $\varkappa$ is said to be *weakly compact* if whenever $\lambda < \varkappa$ and $f\colon [\varkappa]^2 \to \lambda$, then there is an $X \subseteq \varkappa$ of power $\varkappa$ which is homogeneous for f. A weakly compact cardinal is strongly inaccessible [5, Theorem 4.5].

§ 2. Some generalizations of the Rasiowa–Sikorski lemma.

2.0. Theorem (Rasiowa–Sikorski). *If $\mathfrak{P}$ is a $\varkappa$-closed poset and $\overline{\overline{F}} \leqslant \varkappa$ then there is a G which is $\mathfrak{P}$-generic over F.*

Proof. [7, p. 29].

The most natural way to try to strengthen the Rasiowa–Sikorski theorem is to allow F to be larger. However, to avoid collapsing cardinals it is necessary to restrict $\mathfrak{P}$ in some way.

Definition. The $\varkappa$-Martin axiom ($\varkappa$-MA) is the *statement* that if $\mathfrak{P}$ is a $\varkappa$-closed and $\varkappa^+$cc poset and $\overline{\overline{F}} < 2^{\varkappa}$ then there is a G which is $\mathfrak{P}$-generic over F.

The $\varkappa$-Martin axiom is easily seen to be a consequence of $2^{\varkappa} = \varkappa^+$. Solovay and Tennenbaum constructed a model of ZFC+CH+$\aleph_0$−MA. Such a model satisfies the Souslin hypothesis and has many other interesting properties [3], [8]. Whether ZFC+$\varkappa$-MA+$2^{\varkappa} > \varkappa^+$ is consistent for an uncountable $\varkappa$ remains an open question.

Tall has suggested a weakened form of $\varkappa$-MA which we will call the $\varkappa$-Tall axiom.

Definition. $\varkappa$-TA is the *statement* that if $\mathfrak{P}$ is a $\varkappa$-closed $\varkappa^+$cc poset which is constructible from a bounded subset of $\varkappa$ and $\overline{\overline{P}}, \overline{\overline{F}} < 2^{\varkappa}$ then there is a G which is $\mathfrak{P}$-generic over F.

Of course the Solovay–Tennenbaum model satisfies the $\aleph_0$-TA; however, $\aleph_0$-MA $\to 2^{\aleph_0} = 2^{\aleph_1}$. Tall has constructed [9] a model of ZFC+$\aleph_0$TA+$2^{\aleph_1} > 2^{\aleph_0} > \aleph_1$. Moreover, he shows that the restriction of $\overline{\overline{P}} < 2^{\aleph_0}$ is no real restriction in the $\aleph_0$-TA.

We show in this paper the consistency, relative to the existence of a weakly compact cardinal, of $\varkappa$-TA for more or less arbitrary $\varkappa$.

§ 3. Sums of partially ordered sets.

Suppose that ν is an ordinal and when $\alpha < \nu$ then $\mathfrak{P}_\alpha = \langle P_\alpha, \leqslant_{\mathfrak{P}_\alpha}\rangle$ is a poset. If $\varkappa$ is an infinite cardinal, define a new poset $\mathfrak{P} = \sum_{\alpha<\nu}^{\varkappa} \mathfrak{P}_\alpha = \langle P, \leqslant_{\mathfrak{P}}\rangle$ where $P = \{p|\ p$ is a function $\Lambda\ \mathrm{dom}\,p \subseteq \nu \wedge \overline{\overline{p}} < \varkappa \wedge (\forall \alpha \in \mathrm{dom}\,p)[p(\alpha) \in P_\alpha]\}$. If $p, q \subset P$ then $p \leqslant q$ means $\mathrm{dom}\,p \supseteq \mathrm{dom}\,q$ and $(\forall \alpha \in \mathrm{dom}\,q)\, p(\alpha) \leqslant_{\mathfrak{P}_\alpha} q(\alpha)$.

By $\mathfrak{P}_0 \times \mathfrak{P}_1$ is meant $\sum_{\alpha<2}^{(\aleph_0)} \mathfrak{P}_\alpha$; we use ordered pair notation for the elements of this structure.

3.0. Product Theorem. *Let $\mathfrak{P}_1$ and $\mathfrak{P}_2$ be posets in a standard transitive model M of set theory. If G_1 is $\mathfrak{P}_1$-generic over M and G_2 is $\mathfrak{P}_2$-generic over $M[G_1]$ then $G_1 \times G_2$ is $(\mathfrak{P}_1 \times \mathfrak{P}_2)$-generic over M and $M[G_1 \times G_2] = M[G_1][G_2]$. Every set which is $\mathfrak{P}_1 \times \mathfrak{P}_2$-generic is obtained in this way.*

Proof. [4, p. 367].

3.1. Corollary. *If $\mathfrak{P} = \sum_{\alpha<\nu}^{(\varkappa)} \mathfrak{P}_\alpha$ and G is $\mathfrak{P}$-generic over M then*

$$G^\alpha = \{p \upharpoonright \alpha|\ p \in G\} \quad \text{and} \quad G_\alpha = \{p(\alpha)|\ p \in G\}$$

are respectively $\sum_{\beta<\alpha}^{(\varkappa)} \mathfrak{P}_\beta$-generic over M and $\mathfrak{P}_\alpha$-generic over $M[G^\alpha]$.

3.2. Theorem. *Suppose λ is weakly compact and $\varkappa < \lambda$. If $\mathfrak{P}_\alpha$ is a λcc poset whenever $\alpha < \nu$ then $\mathfrak{P} = \sum_{\alpha<\nu}^{(\varkappa)} \mathfrak{P}_\alpha$ is λcc.*

Proof. Suppose $\langle p_\alpha|\ \alpha < \lambda\rangle$ is a pairwise incompatible enumeration of elements of $\mathfrak{P}$. Let $\langle \xi_\gamma|\ \gamma < \lambda\rangle$ be a one-to-one enumeration of a set of ordinals containing $\bigcup_{\alpha<\lambda} \mathrm{dom}\,p_\alpha$. Define $d_\alpha = \{\gamma|\ \xi_\gamma \in \mathrm{dom}\,p_\alpha\}$. We may assume that all of the d_α have the same order type $\eta < \varkappa$. Define d_α^β to be the βth element of d_α when $\beta < \eta$.

We can choose a $\theta < \eta$ and again reduce $\langle p_\alpha|\ \alpha < \lambda\rangle$ to one of its subsequences so that for $\alpha < \beta < \lambda$:

1. if $\gamma < \theta$ then $d_\alpha^\gamma = d_\beta^\gamma$ (call this common value d^γ),
2. if $\theta \leqslant \gamma \leqslant \delta < \lambda$ then $d_\alpha^\gamma < d_\beta^\delta$.

In order to do this, define θ to be smallest so that $\{d_\alpha^\theta|\ \alpha < \lambda\}$ is not bounded below λ, or set $\theta = \eta$ if there is no such ordinal. It is now easy to reduce to a subsequence satisfying 1, and subsequently to a subsequence also satisfying 2.

Notice that $\operatorname{dom}(p_\alpha) \cap \operatorname{dom}(p_\beta) = \{d^\gamma|\ \gamma<\theta\}$. Thus when $\alpha<\beta<\lambda$, there is a $\gamma<\theta$ such that $p_\alpha(d^\gamma)$ and $p_\beta(d^\gamma)$ are incompatible. Let $f(\{\alpha, \beta\})$ be the least such γ. Since λ is weakly compact we may assume that f assumes a constant value ε. But then $\{p_\alpha(d^\varepsilon)|\ \alpha<\lambda\}$ is a pairwise incompatible subset of $\mathfrak{P}_{d^\varepsilon}$, which is impossible.

Let us say that a topological space satisfies the $\varkappa$-chain condition ($\varkappa$cc) if every set of pairwise disjoint open sets has power less than $\varkappa$. The proof of Theorem 3.2 may be used to show the following.

3.3. Theorem. *Let $\varkappa$ be weakly compact, and when $\alpha<\nu$ let X_α be a $\varkappa$cc topological space. Then $X = \prod_{\alpha<\gamma} X_\alpha$ is a $\varkappa$cc topological space.*

§ 4. A model of $\varkappa$-TA. Let M be a countable standard transitive model of ZFC in which λ is a weakly compact cardinal and $\varkappa$, ν are regular cardinals with $2^\varkappa \leqslant \nu$ and $\varkappa<\lambda<\nu$. Let $\langle \mathfrak{P}_{2\alpha}|\ \alpha<\nu\rangle$ be an enumeration in M of all λcc, $\varkappa$-closed posets of power less than ν which are constructible from a bounded subset of $\varkappa$. Let $\{\mathfrak{P}_{2\alpha+1}|\ \alpha<\nu\}$ be the set of $\varkappa$-closed λcc posets for collapsing η to $\varkappa$ where $\varkappa\leqslant\eta<\lambda$. We may assume that each $\mathfrak{P}_\alpha$ appears unboundedly often in the enumeration $\langle \mathfrak{P}_\alpha|\ \alpha<\nu\rangle$.

For $\gamma\leqslant\nu$, define $\mathfrak{P}^\gamma = \sum_{\alpha<\gamma}^{(\varkappa)} \mathfrak{P}_\alpha$. Let G^ν be $\mathfrak{P}^\nu$-generic over M and define (as in Theorem 3.1),

$$G^\gamma = G^\nu \cap P^\gamma, \quad G_\gamma = \{p(\gamma)|\ p \in G\}.$$

4.0. Theorem. $M[G^\nu]$ *is a model of* ZFC$+\varkappa$-TA$+(2^\varkappa = \nu)+(\varkappa^+ = \lambda)+$ $+(2^\lambda = (2^\lambda)^M+\nu)$.

Proof. By Theorem 3.2, $\mathfrak{P}^\nu$ is a λcc poset and clearly $\mathfrak{P}^\nu$ is $\varkappa$-closed. Thus $\varkappa$ is a cardinal in $M[G^\nu]$ and $\varkappa^+ = \lambda$. Since each $G_{2\gamma+1}$ introduces a new subset of $\varkappa$, the remaining cardinal equalities are easily seen.

Suppose that in $M[G^\nu]$, $\mathfrak{P}$ is a $\varkappa$-closed λcc poset which is constructible from some $a\subseteq\eta<\varkappa$. Since $a \in M$, $\mathfrak{P} \in M$. Any decreasing sequence from $\mathfrak{P}$ of length less than $\varkappa$ which is in $M[G^\nu]$ is also in M, so $\mathfrak{P}$ is $\varkappa$-closed in M. Any subset of P of power λ in M must also be of power λ in $M[G^\nu]$, so $\mathfrak{P}$ is λcc in M.

If in addition to the assumptions of the preceeding paragraph, F is a set of subsets of P of power less than ν in $M[G^\nu]$, then for some $\gamma<\nu$, $F\in M[G^\gamma]$. For some $\delta>\gamma$, $\mathfrak{P}$ and $\mathfrak{P}_\delta$ are isomorphic. By Theorem 3.1, $G_\delta \in M[G^\nu]$ is (within isomorphism) $\mathfrak{P}$-generic over F.

Remark. We really have a somewhat stronger result. Suppose $N\subseteq M$ is such that if $a \in M$ is a bounded subset of $\varkappa$ then ($\nu^{\underset{\smile}{\varkappa}}$ in $N[a]) = \nu$. Then $\langle \mathfrak{P}_{2\alpha}|\ \alpha<\nu\rangle$ could be taken as an enumeration of all posets relatively constructible from N and a bounded subset of $\varkappa$. We would then have an N-relativization of the $\varkappa$-TA.

The author conjectures that this construction for $M = N$ will, possibly with some additional large cardinal assumptions, give a model of $\varkappa$-MA. We recall the following lemma of Solovay and Tennenbaum [6] which says that two stage forcing is no better than one stage forcing.

Lemma. *If $\mathscr{B}$ is $\varkappa$-closed, λcc cBa in M and $\mathfrak{D}$ is a $\check{\varkappa}$-closed λcc cBa in $M^{(\mathscr{B})}$ then there is a $\varkappa$-closed, λcc cBa $\mathfrak{G} \in M$ such that if H is $\mathfrak{G}$-generic over M then ($\exists F$, $\mathscr{B}$-generic over M) ($\exists G \in M[F, H]$) (G is $i_F(\mathfrak{D})$-generic over $M[F]$).*

If the existential quantifier on F could be made universal by a suitable change in the definition of $\mathfrak{G}$ this would say that essentially no new $\varkappa$-closed λcc posets are introduced by $\varkappa$-closed λcc forcing. This would allow Theorem 4.0 to be modified for $\varkappa$-MA.

It is natural to ask whether the restriction on the cardinality of the poset in $\varkappa$-TA is necessary. Tall [9] presents an argument of *W.* Boos to show that for $\varkappa = \omega$, this restriction is not essential. In some cases, Boos' argument generalizes to larger cardinals. The Skolem-Lowenheim Theorem for infinitary languages will be useful.

4.1. Lemma. *Let $\mathfrak{B}$ be an infinite structure of type μ and let λ, $\varkappa$ be infinite cardinals such that $\mu\leqslant\lambda = \lambda^{\underset{\smile}{\varkappa}}<\overline{\overline{|\mathfrak{B}|}}$. Then there is a structure $\mathfrak{A}\underset{\varkappa\varkappa}{\prec}\mathfrak{B}$ such that $\overline{\overline{|\mathfrak{A}|}} = \lambda$.*

Proof. See [1].

4.2. Theorem. *Assume* ZFC$+\varkappa$-TA *and* $(\forall\alpha)[\aleph_\alpha<2^\varkappa\to(\exists\beta\geqslant\alpha)\aleph_\beta^{\underset{\smile}{\varkappa}} = \aleph_\beta<2^\varkappa]$. *Then if r is a bounded subset of $\varkappa$, $\mathfrak{P} \in L[r]$ is a $\varkappa$-closed $\varkappa^+$cc poset and $\overline{\overline{F}}<2^\varkappa$, there is a G which is $\mathfrak{P}$-generic over F.*

Proof. We may assume $\overline{\overline{F}} = \mu$ where $\mu^{\underset{\smile}{\varkappa}} = \mu<2^\varkappa$. For some η, $\mathfrak{P} \in L_\eta[r]$. Consider the structure

$$\mathfrak{A} = \langle L_\eta[r], \in, r, P, \leqslant; \mathfrak{D}; \alpha\rangle_{\mathfrak{D}\in F;\ \alpha<\varkappa}.$$

By Lemma 4.1 there is a $\mathscr{B}\subseteq L_\eta[r]$, $\overline{\overline{\mathscr{B}}} = \mu$ and $\mathfrak{B}\underset{\varkappa\varkappa}{\prec}\mathfrak{A}$ where

$$\mathfrak{B} = \langle B, \in, r, P^{\mathfrak{B}}, \leqslant_{\mathfrak{P}}^{\mathfrak{B}}; \mathfrak{D} \cap P^{\mathfrak{B}}; \alpha\rangle_{\mathfrak{D}\in F;\ \alpha<\varkappa}.$$

Let i be the collapsing isomorphism onto

$$\mathfrak{G} = \langle L_\xi(r), \in, r, P^{\mathfrak{C}}, \leqslant_{\mathfrak{P}}^{\mathfrak{C}}; \mathfrak{D}^{\mathfrak{C}}; \alpha\rangle_{\mathfrak{D}\in F;\ \alpha<\varkappa}.$$

Since $\mathfrak{P}^{\mathfrak{C}} = \langle P^{\mathfrak{C}}, \leqslant_{\mathfrak{P}}^{\mathfrak{C}}\rangle \in L[r]$ is $\varkappa$-closed and $\varkappa^+$cc (because $\mathfrak{P}^{\mathfrak{B}}\subseteq\mathfrak{P}^{\mathfrak{A}}$ and compatibility is first order), there is a G which is $\mathfrak{P}^{\mathfrak{C}}$-generic over $\{\mathfrak{D}^{\mathfrak{C}}|\ \mathfrak{D} \in F\}$. It is now easy to see that $i^{-1}(G)$ is a filterbase for an F-generic filter on $\mathfrak{P}$.

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