Theorem 2.4) an isomorphism between $\mathcal{A}(\mathcal{B}(A))$ and A. That is, $\mathcal{A}\mathcal{B}$ is naturally isomorphic to the identity functor on R-IGA. On the other hand, for a reduced pair $\langle B, \delta\rangle$, $\mathcal{B}(\mathcal{A}(B)) = B\varphi$ and $\bar{\delta} = \varphi\delta$. Thus $\mathcal{B}\mathcal{A}$ is naturally isomorphic to the identity functor on $\langle\mathcal{B}\downarrow R\rangle$.

We conclude by considering some special cases. In particular, the collection of Boolean R-pairs $\langle B, \delta_0\rangle$ where $e\delta_0 = 0$ if $e \neq 0$ and $0\delta_0 = R$ are the objects of a full subcategory of $\langle\mathcal{B}\downarrow R\rangle$. Since this subcategory is isomorphic to the category of Boolean rings, we denote it by Borng. If R has only the trivial idempotents then $\langle B, \delta_0\rangle$ is reduced. The following corollary is a generalization of a result of McCrea for the special case in which R is the ring of integers.

COROLLARY 4.8. *If R is a ring with only the trivial idempotents then the category of R-torsion free idempotent generated R-algebras is equivalent to the category of Boolean rings.*

COROLLARY 4.9. *If R is a field, R-IGA is equivalent to Borng.*

Thus for fields F_1 and F_2, F_1-IGA and F_2-IGA are equivalent. In particular, for prime integers p, we obtain the result of Stringall [10] that the categories of p-rings are equivalent.

References

[1] G. M. Bergman, *Boolean rings of projection maps*, J. London Math. Soc. 4 (1972), pp. 593–599.

[2] S. MacLane, *Categories for the Working Mathematician*, Graduate Texts in Mathematics 5, N. Y., 1971.

[3] K. D. Magill, Jr., *The semigroup of endomorphisms of a Boolean ring*, J. Aust. Math. Soc. 11 (1970), pp. 411–416.

[4] C. J. Maxson, *On semigroups of Boolean ring endomorphisms*, Semigroup Forum 4 (1972), pp. 78–82.

[5] M. K. McCrea, *Commutative idempotent generated rings as generalized Boolean rings*, Ph. D. Dissertation, University of California, 1972.

[6] R. S. Pierce, *Modules over commutative regular rings*, Mem. Amer. Math. Soc. 70, Amer. Math. Soc., Providence, R. I., 1967.

[7] B. M. Schein, *Ordered sets, semilattices, distributive lattices and Boolean algebras with homomorphic endomorphism semigroups*, Fund. Math. 58 (1970), pp. 31–50.

[8] D. B. Smith, Jr., and J. Luh, *On semigroups of endomorphisms of generalized Boolean rings*, preprint.

[9] R. P. Stanley, *Structure of incidence algebras and their automorphism groups*, Bull. Amer. Math. Soc. 76 (1970), pp. 1236–1239.

[10] R. W. Stringall, *The categories of p-rings are equivalent*, Proc. Amer. Math. Soc. 29 (1971), pp. 229–235.

DEPARTMENT OF MATHEMATICS
TEXAS A&M UNIVERSITY
College Station, Texas

Accepté par la Rédaction le 17. 9. 1974

3-dimensional AR's which do not contain 2-dimensional ANR's

by

S. Singh* (Altoona, Penn.)

Abstract. There exists an upper semicontinuous decomposition G of 3-dimensional cell B^3 such that the decomposition space B^3/G is a 3-dimensional AR which does not contain any 2-dimensional ANR.

1. Introduction and terminology. By an AR (ANR) we understand a compact metric absolute retract (compact absolute neighborhood retract). One may consult [9] for additional information on AR's (ANR's) and related terminology.

If G is an upper semicontinuous decomposition of a topological space X we denote the associated decomposition space by X/G and by $p: X \to X/G$ the canonical projection, unless otherwise stated. For more information on upper semicontinuous decompositions see [21]. A survey of results on upper semicontinuous decompositions can be found in [2] and [21].

Let n denote a positive integer. By E^n we shall always mean an n-dimensional Euclidean space, by B^n the closed ball of unit radius, and by S^{n-1} the boundary sphere of B^n. By a disc we shall always understand a space homeomorphic to B^2. All maps will be continuous.

A family (collection, sequence) C of subsets of metric space X is called a *null family* (*collection*, *sequence*) provided that for each $\varepsilon > 0$ at most a finite number of elements of C are of diameter greater than ε.

The purpose of this paper is to provide an affirmative answer to the following question which arises in Bing and Borsuk [8] and Armentrout [4]:

Do there exist 3-dimensional AR's which do not contain 2-dimensional AR's or even 2-dimensional ANR's?

In [8], Bing and Borsuk described an upper semicontinuous decomposition G of B^3 whose nondegenerate elements form a countable null family of arcs such that the decomposition space B^3/G is a 3-dimensional AR which does not contain any disc. They asked whether their 3-dimensional AR B^3/G contained any 2-dimensional AR. Armentrout [4] described an upper semicontinuous decomposition G of B^3 similar to the one described by Bing and Borsuk [8] such that B^3/G is a 3-dimensional AR which does not contain any disc but does contain 2-dimensional AR's.

* The author wishes to thank Professor Steve Armentrout of The Pennsylvania State University for many helpful suggestions.

Hence, we seek 3-dimensional AR's which do not contain any 2-dimensional AR's, or 3-dimensional AR's which do not contain any 2-dimensional ANR's. Indeed we have the following:

THEOREM. *There exists an upper semicontinuous decomposition G of B^3 whose nondegenerate elements form a countable null family of arcs such that the decomposition space $B^3\setminus G$ is a 3-dimensional AR which does not contain any 2-dimensional ANR. Hence, $B^3\setminus G$ does not contain any 2-dimensional AR.*

The following corollary is immediate:

COROLLARY. *There exists an upper semicontinuous decomposition G of E^3 whose nondegenerate elements form a countable null family of arcs such that E^3/G does not contain any 2-dimensional ANR and hence any 2-dimensional AR.*

We let $\dim X$, $\operatorname{Int} X$, PL, $D(f, g)$, $\varrho(A, B)$ and $\operatorname{Bd} X$ respectively denote dimension of X, interior of X, piecewise linear, distance between two functions and distance between two sets. Most notations used are standard and will be clear from the context.

2. Homological linking. In this section we shall describe homological linking between a simple closed curve and a special circle-like continuum. For our purpose, we consider the theory of shape as described by Borsuk in [11]. We need the following definitions:

DEFINITION 1. A compact metric space of dimension 1 is said to be a *circle-like continuum* if and only if X is shape equivalent to circle S^1.

DEFINITION 2. A subset M of a topological space X *has property* UV^∞ if and only if for each open subset U of X containing M, there is an open subset V of X containing M such that (1) $V\subset U$ and (2) V is contractible in U.

DEFINITION 3. Let X be a 1-dimensional continuum in E^3 and G be an upper semicontinuous decomposition of X such that the non-degenerate elements of G form at most a countable null family of arcs. Now X will be called a *special circle-like continuum* if and only if the decomposition space X/G is homeomorphic to the circle S^1.

The following lemma is a result of [17]:

LEMMA 1. *Every special circle-like continuum is shape equivalent to the circle S^1.*

Our immediate goal is to describe certain polyhedral neighborhoods of a circle-like continuum X in E^3. These neighborhoods will be used to define the linking number between X and a simple closed curve. Before we do this rigorously, we state the following lemma for later use:

LEMMA 2. *If M is a 1-dimensional UV^∞ continuum in E^3 and Σ^2 is a 2-sphere in E^3, then $(M\cap\Sigma^2)$ does not separate Σ^2.*

Proof of Lemma 2 is fairly straightforward and hence omitted.

Let $X=\gamma$ be a special circle-like continuum and $\gamma^*=X/G$. It follows from the definition of γ that γ^* is a simple closed curve. Let a^*, $b^*\in\gamma^*$ such that $p^{-1}(a^*)=\{a\}$ and $p^{-1}(b^*)=\{b\}$. Now $\gamma^*=\gamma_1^*\cup\gamma_2^*$, where each γ_i^* is an arc for $i=1,2$, and $\gamma_1^*\cap\gamma_2^*=\{a^*,b^*\}$. It follows that $p^{-1}(\gamma)=p^{-1}(\gamma_1^*)\cup p^{-1}(\gamma_2^*)$, where each $p^{-1}(\gamma_i^*)$ is a UV^∞ continuum in E^3 [5] for each $i=1,2$, and $p^{-1}(\gamma_1^*)\cap p^{-1}(\gamma_2^*)=\{a,b\}$. If π is a plane in E^3 such that π separates a from b in the continuum $\gamma=p^{-1}(\gamma^*)$, then there exists a positive number ε such that the closed balls $B_\varepsilon(a)$, $B_\varepsilon(b)$ in E^3, centered at a, b, respectively, do not meet π. There exist $x_1\neq x_2$ in $[B_\varepsilon(a)\cap\gamma]$ such that $x_1=p^{-1}(x_1^*)$ and $x_2=p^{-1}(x_2^*)$ with $x_1^*, x_2^*\in\gamma^*$ and $d(a,x_i)>\frac{1}{2}\varepsilon$, for $i=1,2$. Similarly, we can find $y_1\neq y_2$ in $[B_\varepsilon(b)\cap\gamma]$ such that $y_1=p^{-1}(y_1^*)$ and $y_2=p^{-1}(y_2^*)$ such that $d(b,y_i)>\frac{1}{2}\varepsilon$, for $i=1,2$, where $y_1^*, y_2^*\in\gamma^*$. Assume we have assigned an orientation to γ^*. By relabelling the points $x_1^*, x_2^*, y_1^*, y_2^*$ if needed, we may assume that the arc γ_3^* between x_1^* and y_1^* and the arc γ_4^* between x_2^* and y_2^* are subarcs of γ_1^* and γ_2^* respectively. Put $\gamma_i=p^{-1}(\gamma_i^*)$ for $i=1,2,3,$ and 4. Now $\gamma_3\subset\gamma_1, \gamma_4\subset\gamma_2$ and $\gamma_3\cap\gamma_4=\emptyset$. Also, each of γ_3 and γ_4 has property UV^∞ in E^3. Hence there are arbitrarily small compact polyhedral neighborhoods of γ_3 and γ_4 which are 3-manifolds with connected boundary. Let H_3 and H_4 be such neighborhoods of γ_3 and γ_4 respectively such that $H_3\cap H_4=\emptyset$. We define

$$P=H_3\cup H_4\cup B_\varepsilon(a)\cup B_\varepsilon(b)\,.$$

It is clear that the set P is a compact neighborhood of the continuum γ in E^3. P will be called a *special neighborhood* of γ in E^3. With the notation and terminology as above, we have shown the following lemma:

LEMMA 3. *Every special circle-like continuum γ in E^3 has a special neighborhood in E^3.*

Let γ be a special circle-like continuum in E^3 and C be a simple closed curve in E^3. In this paragraph, we shall define the notion of linking between γ and C. Intuitively, C links γ if C links some special neighborhood of γ. This will be made precise in the following discussion. Let P be a special neighborhood of a circle-like continuum γ in E^3 and a, b, γ, where a and b are the centers of the balls used to construct P. Let $\varphi\colon S^1\to\gamma^*$ be a homeomorphism. By [5], Lemma 3.2 there exists a map $\varphi_\delta\colon S^1\to E^3$ such that $D(p\circ\varphi_\delta,\varphi)<\delta$, where δ is a positive real number. By simple modifications, we can assume that (1) φ_δ is a homeomorphism, (2) $a, b, \in\varphi_\delta(S^1)$ and (3) $\varphi_\delta(S^1)\subset P$. Pick a sequence of homeomorphism $\varphi_\delta, \varphi_{\delta/2}, \varphi_{\delta/2^2}, \ldots, \varphi_{\delta/2^n}, \ldots$ such that $D(p\circ\varphi_{\delta/2^n},\varphi)<\delta/2^n$ for $n=1,2,3,\ldots$ Also, $\varphi_{\delta/2^n}(S^1)$ contains the set $\{a,b\}$ and $\varphi_{\delta/2^n}(S^1)\subset P$, for $n=1,2,3,\ldots$ Since γ is a circle-like continuum it follows that its fundamental shape group $\underline{\Pi}_1(\gamma,a)$ is isomorphic to Z. The fundamental sequence $\varphi=\{\varphi_{\delta/2^n}\}_{n=0}^{\infty}$ may be assumed to be a generator for the group $\underline{\Pi}_1(\gamma,a)$. Such a fundamental sequence $\varphi=\{\varphi_{\delta/2^n}\}_{n=0}^{\infty}$ will be called a *canonical generator* of $\underline{\Pi}_1(\gamma,a)$. In order to discuss it further, we need the following important lemma:

LEMMA 4. *Let γ be a special circle-like continuum in E^3 and P be a special neighborhood of γ in E^3. Then there exists a simple closed curve ζ in E^3 such that ζ links γ.*

Proof. The special neighborhood P of γ has a decomposition

$$P = B_\varepsilon(a) \cup B_\varepsilon(b) \cup H_1 \cup H_2$$

where H_1 and H_2 are polyhedral 3-manifolds with connected boundaries and $B_\varepsilon(a)$, $B_\varepsilon(b)$ are closed balls of radius ε centered at a and b respectively. There exists a plane Π in E^3 separating a from b. We may assume that Π does not meet $B_\varepsilon(a)$ and $B_\varepsilon(b)$. Also, we may assume that the plane Π and the polyhedron $H_1 \cup H_2$ are in relative general position. It is well-known that $\Pi \cap H_1 = \bigcup_{i=1}^{n_1} D_{1i}$, where each D_{1i} is a punctured disc and $D_{1i} \cap D_{1j} = \emptyset$ for $i \neq j$ and $i, j = 1, 2, \ldots, n_1$. Similarly, we have that $\pi \cap H_2 = \bigcup_{j=1}^{n_2} D_{2j}$, where each D_{2j} is a punctured disc and $D_{2i} \cap D_{2j} = \emptyset$ for $i \neq j$ and $i, j = 1, 2, \ldots, n_2$.

Consider a punctured disc D_0 such that D_0 belongs to the set of disc $\{D_{11}, D_{12}, \ldots, D_{1n_1}, D_{21}, D_{22}, \ldots, D_{2n_2}\}$. Let $C_0, C_1, \ldots, C_k$ denote the boundary curves of D_0. Pick $x_1, x_2, \ldots, x_k$ on C_0 such that $x_1 \neq x_j$. Pick y_1 on C_1 and join x_1 with y_1 by a polygonal line segment l_1 spanning D_0. Pick y_2 on C_2 and a polygonal line segment l_2 spanning D_0 and joining x_2 and y_2 such that $l_1 \cap l_2 = \emptyset$. Assume l_{k-1} is defined. Find a polygonal line segment l_k spanning D_0 and joining x_k with y_k and such that $(\bigcup_{i=1}^{k-1} l_i) \cap l_k = \emptyset$. Let N_i denote a regular neighborhood (in the sense of Whitehead) of l_i in E^3 for $i = 1, 2, \ldots, k$. It is clear that N_i is a PL 3-cell for $i = 1, 2, \ldots, k$. Furthermore, we can choose these 3-cells in such a way that they are mutually disjoint. We may assume that the set of points $x_1, \ldots, x_k$ and $y_1, y_2, \ldots, y_k$ does not meet the 1-dimensional UV^∞ continuum. Consider the 2-sphere $\Sigma_i = \mathrm{Bd}(N_i)$ for some fixed i. Since γ does not separate Σ_i, there exists a polygonal arc m_i joining x_i with y_i and such that (1) $m_i \cap \gamma = \emptyset$ and (2) m_i is contained in Σ_i. There exists a homeomorphism $\varphi\colon E^3 \to E^3$ such that:

(1) φ is a PL-homeomorphism.

(2) For each $i, i = 1, 2, \ldots, k$, $\varphi(x) = x$ for all x outside a small neighborhood M_i of N_i in E^3. We may assume that $M_1, M_2, \ldots$, and M_k are mutually disjoint.

(3) $\varphi(x) \in M_i$, for all $x \in l_i$ and $i = 1, 2, \ldots, k$.

(4) $\varphi(x_i) = x_i$ and $\varphi(y_i) = y_i$ for $i = 1, 2, \ldots, k$.

The map φ adjusts the plane Π such that the plane $\varphi(\Pi)$ has the property that each segment m_i, for $i = 1, 2, \ldots, k$ lie in $\varphi(\Pi)$ and miss the continuum γ. For each $i, i = 1, 2, \ldots, k$, we perform a cut along the segment m_i to obtain a disc D_0^1. Hence we have shown that by appropriately modifying the plane Π, we may assume that each punctured disc belonging to the set $\{D_{11}, \ldots, D_{1n_1}\} \cup \{D_{21}, \ldots, D_{2n_2}\}$ is a disc. We shall use the following theorem of R. L. Moore [16]:

If X and Y are disjoint compact subsets of E^2 which do not separate E^2, then there exists a simple polygon $C \subset (E^2 - (X \cup Y))$ which separates E^2 between X and Y. Moreover, we may assume that all vertices of C have rational coordinates.

Let $X = \bigcup_{i=1}^{n_1} D_{1i}$ and $Y = \bigcup_{i=1}^{n_2} D_{2i}$. Neither X nor Y separate the plane $\Pi_1 = \varphi(\Pi)$. By the theorem mentioned above, there exists a polygonal simple closed curve C in Π_1 such that C separates Π_1 between X and Y. Set $\xi = C$. Let $\Phi = \{\varphi_{\delta/2^n}\}_{n=0}^\infty$ be a canonical generator of $\underline{\Pi}_1(\gamma, a)$. Define a linking number $\lambda(\gamma, \xi)$ between γ and ξ by setting

$$\lambda(\gamma, \xi) = \lim_{n\to\infty} |\lambda(\varphi_{\delta/2^n}, \xi)|$$

where $\lambda(\varphi_{\delta/2^n}, \xi)$ is the linking number as defined in [13], [8]. It is clear that $\lambda(\gamma, \xi)$ is a positive integer which is independent of the choice of the fundamental sequence Φ. Thus γ links ξ.

3. Dyadic Antoine's necklaces and dyadic wreaths.

Dyadic Antoines' necklaces. Let r be a fixed positive integer and T_r be an unknotted polyhedral solid torus in E^3. All tori considered will be solid, unknotted and polyhedral unless otherwise stated. Let $\{T_{r1}, \ldots, T_{rm_{r0}}\}$ denote a chain of linked solid tori in $\mathrm{Int}\,T_r$ circling T_r exactly 2 times such that for each $i = 1, 2, \ldots, m_{r0}$, the diameter of T_{ri} is less than one. For each i, $l < i < m_{r0}$, let $\{T_{ri1}, \ldots, T_{rim_{ri}}\}$ be a chain of linked tori in $\mathrm{Int}\,T_{ri}$ and circling T_{ri} exactly 2 times, with the diameter of each T_{rij} less than $\frac{1}{2}$, where $1 \leqslant j \leqslant m_{ri}$. Proceeding inductively we obtain the following sets:

$$M_{r1} = \bigcup \{T_{ri}\colon 1 \leqslant i \leqslant m_{r0}\},$$
$$M_{r2} = \bigcup \{T_{rij}\colon 1 \leqslant i \leqslant m_{r0}, 1 \leqslant j \leqslant m_{ri}\},$$
$$M_{r3} = \bigcup \{T_{rijk}\colon 1 \leqslant i \leqslant m_{r0}, 1 \leqslant j \leqslant m_{ri}, 1 \leqslant k \leqslant m_{rij}\},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

The set $N_r = \bigcap \{M_{ri}\colon 1 \leqslant i < \infty\}$ will be called a *dyadic Antoine's necklace* circling T_r.

A dyadic wreath substituting for T_r. For each i, $1 \leqslant i \leqslant m_{r0}$, let $\{T_{ri1}, \ldots, T_{rim_{ri}}\}$ be the chain of linked tori in $\mathrm{Int}\,T_{ri}$ which is used in the construction of the set N_r. For each $j, j = 1, 2, \ldots, m_{ri}$, there exists an arc a_{rij} in $\mathrm{Int}\,T_{rij}$ such that a_{rij} contains the set $(N_r \cap T_{rij})$. Construct arcs $b_{ri1}, \ldots, b_{ri(m_{ri}-1)}$ as constructed in [1] such that $(\{a_{rij}\colon 1 \leqslant j \leqslant m_{ri}\}) \cup (\{b_{rik}\colon 1 \leqslant k \leqslant (m_{ri}-1)\})$ is an arc A_{ri}. The arc A_{ri} will be called a *dyadic arc* substituting for T_{ri}. The set $W_r = \{A_{ri}\colon 1 \leqslant i \leqslant m_{r0}\}$ will be called a *dyadic wreath substituting* for T_r and A_{ri}'s will be called *links* of the dyadic wreath W_r.

Remark 1. The above construction of the dyadic Antoine's necklaces and dyadic wreaths can be generalized to *n-adic Antoine's necklaces* and *n-adic wreaths* where $n = 1, 2, 3, \ldots$

4. A sequence of tori.

A sequence $\{A_i\}$ of polyhedral solid tori in E^3 is said to be *A-dense* in E^3 if for each simple closed curve $C \subset E^3$ and an open subset U of E^3, there is an index i such that: (1) $A_i \subset E^3 - C$, (2) the core C_i of A_i is homologically linked with C, and (3) C_i meets U.

For the definition of core and matters related to linking see [11]. We organize the rest of this section in parts (A) to (F) as follows:

(A) There exists in E^3 a countable family F of disjoint polygonal simple closed curves such that for any simple closed curve C in E^3 and any open subset U of E^3, there exists an element P of F such that

(1) P and C are homologically linked, and

(2) P meets U.

It is clear that one can construct an A-dense sequence $\{A_i\}$ of solid polyhedral tori by taking the family F of simple closed curves as the cores of the tori. The above assertions follow from the results of [8] by making appropriate adjustments.

(B) Let B^3 denote a closed unit ball in E^3 with boundary S^2. There exists a countable family of disjoint segments $\{K_i\}$ satisfying the following:

(1) For each i, the end points of K_i lie on S^2.

(2) $\{K_i\}$ is a null sequence.

(3) For each nonempty open subset G of S^2, there is an index j such that both the endpoints of the segment K_j lie in G.

For a proof see [8].

(C) Let $\{K_j\}$ be a countable family of segment as in (B). There exists an A-dense sequence $\{A_i\}$ of solid polyhedral tori contained in $(B^3-S^2)-\bigcup_j K_j$ such that for each j:

(1) The inner radius of A_j is less than $1/j$.

(2) There exists in A_j a dyadic wreath W_j substituting for A_j. Also, $W_k\cap W_j=\emptyset$ for $j\neq k$ and the diameter of each link of W_j is less than $1/j$, for $j=1,2,3,\ldots$

A proof can be constructed by arguments similar to those of [8]. We shall always assume that each A_i is obtained by taking a polygonal simple closed curve C_i as the core of A_i.

(D) Let $\{K_j\}$ be the sequence of disjoint segments as given in (B). For each j, let $\{x_j, x_j'\}$ denote the set of the end points of the segment K_j. Consider the set $B^3-\bigcup_j\{x_j, x_j'\}$. There exists a countable family F of polygonal simple closed curves in $\operatorname{Int}B^3$ such that (1) for any simple closed curve C in B^3 such that either (a) C lies in $\operatorname{Int}B^3$ or (b) $S^2\cap C$ is a polygonal arc with endpoints $\{x_j, x_j'\}$ for some j, and (2) any open subset U of $\operatorname{Int}B^3$, there is an element P of F such that

(1) P homologically links C, and

(2) P meets U.

The assertions of (D) follow from the assertions of (A) by making suitable changes.

(E) There exists a sequence $\{A_i\}$ of solid polyhedral tori such that for each j,

(1) A_j is contained in $(B^3-S^2)-\bigcup_j K_i$.

(2) The inner radius of A_j is less than $1/j$.

(3) For any simple closed curve C such that either $C\subset\operatorname{Int}B^3$ or C meets S^2 in a polygonal arc, there is an i such that the core C_i of A_i is homologically linked with C.

(4) There exists in A_j a dyadic wreath W_j substituting for A_j. Also the diameter of each link of W_j is less than $1/j$, for $j=1,2,3,\ldots$, and $W_k\cap W_l=\emptyset$ for $k\neq l$.

Let us construct a family $\{A_i\}$ of solid, polyhedral and unknotted tori such that each A_i is a tubular neighborhood of some element P of F with P as its core, where F is the family of polygonal simple closed curves as given in (D). Furthermore, for each polygonal simple closed curve P in F there is a solid torus A with P as its core and of arbitrarily small inner radius such that A belongs to the family $\{A_i\}$. The following assertion is easy to prove: Given (1) a simple closed curve C such that (a) C lies in $\operatorname{Int}B^3$, or (b) C meets S^2 in arc whose set of endpoints is $\{x_j, x_j'\}$ for some j, and (2) an open subset U of $\operatorname{Int}B^3$, there is an A_j belonging to the family $\{A_i\}$ of polyhedral solid tori as described above such that (1) the core of A_j is homologically linked with C and (2) $(A_j\cap U)$ contains a polyhedral disc which is meridional in A_j. (For a definition of a meridional disc, see [1].)

(F) The following conventions will be used for the rest of this paper:

(1) The sequence $\{T_i\}$ will always denote the sequence $\{A_i\}$.

(2) For each i, C_i will denote the core of T_i and therefore $\{C_i\}$ will denote the corresponding sequence of the cores of the sequence $\{T_i\}$.

(3) We shall denote the corresponding sequence of dyadic wreaths by $\{W_i\}$ for the sequence $\{T_i\}$.

(4) The sequence $\{K_i\}$ will always stand for a sequence of segments as described in (B) above.

5. Upper semicontinuous decompositions.

A decomposition of B^3. Let $\{K_j\}$ be the countable family of the segments in B^3, $\{T_i\}$ be the countable family of solid polyhedral tori, and W_i be the countable family of the dyadic wreaths, where W_j is a dyadic wreath substituting for T_j for each $j=1,2,\ldots$, as described in Section 4. We define a decomposition G of B^3 as follows: x is a nondegenerate element of G if and only if x is a link of some dyadic wreath W_j, or $x=K_i$ for some i. It is clear from the construction that the nondegenerate elements of G form a countable null collection of arcs. Hence, G is an upper semicontinuous decomposition of B^3 [21]. Recall that $P\colon B^3\to B^3/G$ denotes the canonical projection map.

The following properties of the decomposition space B^3/G are of interest.

PROPOSITION 1. *The decomposition space* B^3/G *is a compact metric space of dimension* 3.

For a proof see [8].

PROPOSITION 2. *The decomposition space* B^3/G *is an* AR.

Proof. Since the dimension of the space B^3/G is finite and for each y in B^3/G the fibre $P^{-1}(y)$ is an AR, it follows from [9] that the space B^3/G is an AR.

A decomposition of E^3. Since E^3 is a homeomorphic to $\operatorname{Int} B^3$, it is enough to describe the elements of a decomposition in $\operatorname{Int} B^3$. Define a decomposition $\bar{G}$ of E^3 as follows: x is a nondegenerate element of $\bar{G}$ if and only if x is a link of some dyadic wreath W_j belonging to the collection $\{W_j\}$. The nondegenerate elements of $\bar{G}$ form a countable null collection of arcs and hence $\bar{G}$ is upper semicontinuous decomposition of E^3 [21]. Here we denote by $q\colon E^3 \to E^3/\bar{G}$, the canonical projection map.

We state the following two propositions without proof:

PROPOSITION 3. *If A is any 2-dimensional continuum in $E^3/\bar{G}$ such that A contains a simple closed curve which is nullhomotopic in A, then A contains countably many distinct points x_i such that $q^{-1}(x_i)$ is a nondegenerate element of G where $i = 1, 2, \ldots$*

Remark 2. Proposition 3 remains true if "$E^3/\bar{G}$" is replaced by the decomposition space "B^3/G".

PROPOSITION 4. *The decomposition space $E^3/\bar{G}$ is a 3-dimensional noncompact metric absolute retract.*

6. 2-dimensional AR's **in** B^3/G. In this section, we shall show that the decomposition space $E^3/\bar{G}$ does not contain any 2-dimensional AR. This will show that the decomposition space B^3/G does not contain any 2-dimensional AR. We need the following lemmas:

LEMMA 5. *Suppose M is an AR in $E^3/\bar{G}$ and U_0 is an open subset of $E^3/\bar{G}$ containing M. Then there is a sequence $\{U_i\}_{i=0}^{\infty}$ of open subsets of $E^3/\bar{G}$ such that* (1) *$M \subset \bigcap_{i=0}^{\infty} U_i$,* (2) *for each i,* (a) *$U_{i+1} \subset U_i$ and* (b) *each loop in U_{i+1} is nullhomotopic in U_i.* (For a proof see [1]).

The following lemma is a consequence of Lemma 9 of [1].

LEMMA 6. *Suppose M is a subset of $E^3/\bar{G}$ which has a sequence $\{U_i\}_{i=0}^{\infty}$ of open subsets of $E^3/\bar{G}$ such that* (1) *$M \subset \bigcap_{i=0}^{\infty} U_i$ and* (2) *for each i* (a) *$U_{i+1} \subset U_i$ and* (b) *each loop in U_{i+1} is nullhomotopic in U_i. Then the sequence $\{V_i\}$ of open subsets of E^3, where $V_i = q^{-1}(U_i)$ for $i = 0, 1, 2, \ldots$ has the properties*: (1) *$q^{-1}(M) \subset \bigcap_{i=0}^{\infty} V_i$,* (2) *for each i,* (a) *$V_{i+1} \subset V_i$ and* (b) *each loop in V_{i+1} is nullhomotopic in V_i.*

LEMMA 7. *If X is a circle-like continuum in $E^3/\bar{G}$ then $q^{-1}(X)$ is a circle-like continuum.*

Proof. The restriction map $q\colon q^{-1}(X) \to X$ a cell-like map in the sense of Sher [17]. By Theorem 11 of [17] it follows that X and $q^{-1}(X)$ are shape equivalent. Since X is shape equivalent to S^1, we have that $q^{-1}(X)$ is shape equivalent to S^1. The dimension of $q^{-1}(X)$ is one [19].

Remark 3. The above lemmas can be appropriately stated for B^3/G.

Let A be a 2-dimensional AR in $E^3/\bar{G}$. Since A contains simple closed curves, $q^{-1}(A)$ contains special circle-like continua. To illustrate the technique of our proof we shall assume that $q^{-1}(A) = A_1$ contains a simple closed curve. The transition to the case when A_1 contains a special circle-like continuum is straightforward. So we begin with the following:

THEOREM 1. *The decomposition space $E^3/\bar{G}$ does not contain any 2-dimensional AR A such that $A_1 = q^{-1}(A)$ contains a simple closed curve.*

Proof. Let U_0 be an open subset $E^3/\bar{G}$ such that (1) $A \subset U_0$ and (2) there is an open subset W_1 of $E^3/\bar{G}$ with $W_1 \cap U_0 = \emptyset$. By Lemma 5, there exists a sequence $\{U_i\}_{i=0}^{\infty}$ of open subsets of $E^3/\bar{G}$ such that (1) $A \subset \bigcap_{i=0}^{\infty} U_i$ and (2) for each i (a) $U_{i+1} \subset U_i$ and (b) each loop in U_{i+1} is nullhomotopic in U_i. Define $V_i = q^{-1}(U_i)$, for $i = 0, 1, 2, \ldots$, and let $W = q^{-1}(W_1)$. The sequence $\{V_i\}_{i=0}^{\infty}$ of saturated open subsets of E^3 has the following properties: (1) $A_1 \subset \bigcap_{i=0}^{\infty} V_i$, (2) for each i (a) $V_{i+1} \subset V_i$ and (b) each loop in V_{i+1} is nullhomotopic in V_i.

By our hypothesis, there is a simple closed curve C contained in A_1. Let T_α be a polyhedral solid torus in E^3 such that (1) the core C_α of T_α is linked with C, (2) T_α belongs to the sequence $\{T_i\}$ used to construct the dyadic wreaths $\{W_i\}$ and (3) there is a polyhedral meridional disc D in T_α such that D is contained in $W = q^{-1}(W_1)$. Since there are solid tori belonging to the sequence $\{T_i\}$ of arbitrarily small inner radius with C_α as their core, we can find a T_α satisfying (3).

Let $\{T_{\alpha 1}, \ldots, T_{\alpha m_\alpha}\}$ be the chain of tori used in the construction of the dyadic Antoine's necklace N_α. Now the simple closed curve C is contained in $V_{m_\alpha+2}$. By compactness of C, there is a positive real number ε such that the set $N_\varepsilon(C) = \{x \in E^3\colon \varrho(x, C) < \varepsilon\}$ is contained in $V_{m_\alpha+2}$. For each $\delta > 0$ such that $\delta < \varepsilon$, there is a polygonal simple closed curve C_δ such that $C_\delta \subset V_{m_\alpha+2}$ and $\varrho(C_\delta, C) < \delta$. For each $\delta > 0$ but sufficiently small there is a polygonal simple closed curve C_δ such that (1) C_δ links T_α, (2) C_δ is contained in $V_{m_\alpha+2}$ with $\varrho(C, C_\delta) < \delta$ and (3) C_δ is homotopic to C in $V_{m_\alpha+2}$. By hypothesis C_δ is nullhomotopic in $V_{m_\alpha+1}$. Since $V_{m_\alpha+1}$ is a PL 3-manifold, there is a polyhedral singular disc D_1 contained in $V_{m_\alpha+1}$ such that D_1 is bounded by C_δ. We shall denote C_δ by C_1 from now on.

Assume that D_1 and T_α are in relative general position. We look at the curves of intersection of D_1 with the 2-manifold $M_\alpha = \bigcup_{i=1}^{m_\alpha} \operatorname{Bd} T_{\alpha i}$. If every curve of intersection of D_1 with M_α is homotopic to zero in M_α then we may build a new polyhedral singular disc by filling in the curves of intersection in M_α. This new polyhedral singular disc will miss the homotopy centerline of T_α. This is a contradiction, since C_1 and T_α are linked. Hence some curve of intersection of D_1 with M_α is not nullhomotopic in M_α.

Our aim is to show that there is a polygonal simple closed curve contained in some $T_{\alpha j}$, where $j = 1, 2, \ldots, m_{\alpha 0}$, such that γ is not nullhomotopic in $T_{\alpha J}$ and γ is contained in $V_{m_\alpha+1}$. If D_1 has a curve of intersection with M which is longitudinal in $T_{\alpha j}$, for some $j, j = 1, 2, \ldots, m_\alpha$, then there is nothing to prove. Without loss of generality we may assume that for $j = 1$, there is a curve of intersection C_2 of D_1 with M_α such that C_2 is a meridional closed curve in $T_{\alpha 1}$. If D_2 is a polyhedral singular subdisc of D_1 such that C_2 bounds D_2, we may assume that every curve of intersection of D_2 with $\operatorname{Bd} T_{\alpha 1}$ is nullhomotopic in $\operatorname{Bd} T_{\alpha 1}$ except C_2.

Let $T^1_{\alpha 1}$ be a polyhedral solid torus in $\operatorname{Int} T_{\alpha 1}$, concentric with $T_{\alpha 1}$, and such that the intersection of the dyadic arc $A_{\alpha 1}$ with $\bigcup_{i=1}^{m_{\alpha 1}} T_{\alpha 1 i}$ lies completely in $T^1_{\alpha 1}$. For each curve of intersection λ of D_2 with $\operatorname{Bd} T_{\alpha 1}$ such that λ is a nullhomotopic in $\operatorname{Bd} T_{\alpha 1}$ replace the subdisc of D_2 bounded by λ by a polyhedral singular disc on $\operatorname{Bd} T_{\alpha 1}$, and push this new polyhedral singular disc slightly into $[(\operatorname{Int} T_{\alpha 1}) - T^1_{\alpha 1}]$. This produces a polyhedral singular D^1_2 with the boundary C_2, $\operatorname{Int} D^1_2 \subset \operatorname{Int} T_{\alpha 1}$, and $(D^1_2 \cap T^1_{\alpha 1}) \subset D_2$.

By the Loop Theorem [20] there is a polyhedral disc D such that the boundary of D is contained in $\operatorname{Bd} T_{\alpha 1}$, $\operatorname{Bd} D$ is not nullhomotopic in $\operatorname{Bd} R_{\alpha 1}$, $\operatorname{Int} D \subset \operatorname{Int} T_{\alpha 1}$, and D lies in a small neighborhood of D_2. We may assume that the set $(D \cap T^1_{\alpha 1})$ is contained in $V_{m_\alpha+1}$ and that D is in relative general position with $\operatorname{Bd} T^1_{\alpha 1}$.

Now D contains a punctured disc D_0 such that $\operatorname{Bd} D_0$ is contained in $\operatorname{Bd} T^1_{\alpha 1}$, $\operatorname{Int} D_0 \subset \operatorname{Int} T^1_{\alpha 1}$, one boundary curve λ_0 of D_0 is not nullhomotopic in $\operatorname{Bd} T^1_{\alpha 1}$, and every other boundary curve is homotopic to zero in $\operatorname{Bd} T^1_{\alpha 1}$. Clearly, D_0 is contained in $V_{m_\alpha+1}$. Now we may construct a polyhedral meridional disc D^1_0 in $T_{\alpha 1}$ as follows: (1) Attach to D_0 an annulus in $(T_{\alpha 1} - \operatorname{Int} T^1_{\alpha 1})$ having λ_0 as its one boundary curve and having as its other a simple closed curve ξ_1 in $\operatorname{Bd} T_{\alpha 1}$ such that ξ_1 is not nullhomotopic in $\operatorname{Bd} T_{\alpha 1}$. (2) Capp every other boundary curve of D_0 with a disc lying, except for its boundary, in $[(\operatorname{Int} T_{\alpha 1}) - T^1_{\alpha 1}]$. Also, we may suppose that $(D^1_0 \cap T^1_{\alpha 1})$ equal D_0.

By Lemma 1 of [1], it follows that there is a subarc $B_{\alpha 1}$ of the dyadic arc $A_{\alpha 1}$ such that the endpoints of $B_{\alpha 1}$ lie on D^1_0 with the interior of $B_{\alpha 1}$ disjoint from D^1_0 and at the endpoints of $B_{\alpha 1}$, $B_{\alpha 1}$ abuts on D^1_0 from opposite sides. Clearly the endpoints of $B_{\alpha 1}$ lie on in D_0. Now $D_0 \cup B_1$ contains a loop γ_1 which is not nullhomotopic in $T_{\alpha 1}$.

Now γ_1 is nullhomotopic in V_{m_α} and hence bounds a polyhedral singular disc in V_{m_α}. By Lemma 2 of [1], there is a polygonal simple closed curve γ_2 in $(T_{\alpha 2} \cap V_{m_\alpha})$ which is not nullhomotopic in $T_{\alpha 2}$. By repeating this argument, we may assume that there is a simple closed curve γ_i in T_i, for each $i = 1, \ldots, m_\alpha$, such that (1) $\gamma_i \subset [T_{\alpha i} \cap V_{(m_\alpha+1)-i}]$ and (2) γ_i is not nullhomotopic in $T_{\alpha i}$. By Lemma 3 of [1], we conclude that there exists a simple closed curve ξ contained in T_α such that ξ is not nullhomotopic in T_α and ξ is contained in V_0. This can be done since each γ_i is contained in V_0 and each γ_i is nullhomotopic in V_0, for $i = 1, 2, \ldots, m_\alpha$.

By construction, T_α contains a meridional disc Λ such that Λ is contained in W. Now ξ meets Λ and hence ξ meets W. Also, ξ is contained V_0. This is a contradiction, since $V_0 \cap W = \emptyset$.

The techniques of Theorem 1 allow us to prove the following more general theorem:

THEOREM 2. *The decomposition space $E^3/\bar{G}$, does not contain any 2-dimensional* AR.

Proof. Let A be a special circle-like continuum in E^3 such that $\gamma \subset q^{-1}(A) = A_1$. There exists a sequence $\{V_i\}$ of open subsets of E^3 such that for each i, (1) $A_1 \subset V_{i+1} \subset V_i$ and (2) each loop in V_{i+1} is nullhomotopic in V_i. Let W be an open subset of E^3 such that $V_0 \cap W = \emptyset$.

Pick a suitable special neighborhood P of γ in E^3. There exists an index α such that the core C_α of torus T_α and γ are linked. There exists a simple closed curve μ belonging to the canonical generator for the group $\Pi_1(\gamma, a)$ such that μ lies in $V_{m_\alpha+2}$. The rest of the proof proceeds as in the proof of Theorem 1.

THEOREM 3. *We have the following*:

(1) *B^3/G does not contain any 2-dimensional* AR.

(2) *$E^3/\bar{G}$ does not contain any 2-dimensional* AR.

Proof. Let A be a 2-dimensional AR in B^3/G. If $P^{-1}(A)$ is contained in the interior of the ball B^3, then there is nothing to prove. Therefore, we may assume that $P^{-1}(A) \cap S^2 \neq \emptyset$. There are two cases.

Case I. If $\dim[P^{-1}(A) \cap S^2] = 2$, then there is a disc D such that $D \subset [P^{-1}(A) \cap S^2]$. There exists an index j such that the segment k_j has its endpoints a_j and b_j in the disc D. Now, there is a polygonal path γ_j between a_j and b_j such that $\gamma_j \subset D$. Now $\gamma_j \cup K_j$ is a simple closed curve which is contained in $P^{-1}(A)$. There exists some core C_α of a torus T_α in $\operatorname{Int} B^3$ such that C_α links $\gamma_j \cup K_j$. The rest of the proof is clear.

Case II. If $\dim[P^{-1}(A) \cap S^2] \leqslant 1$, then $\dim(A \cap S^2/G^*) \leqslant 1$, where G^* is the induced-decomposition on S^2. Now $[A - (A \cap S^2/G^*)]$ must have dimension 2 ([16], p. 32, Cor. 1). By an argument similar to the proof of Theorem 5.9 of [20] on page 118, it can be shown that the space $[A - (A \cap S^2/G^*)]$ contains a simple closed curve. This will show that there exists a special circle-like continuum $\gamma \subset P^{-1}(A)$ such that γ is contained in $\operatorname{Int} B^3$. Again, we are done by (2). This finishes the proof.

7. 2-dimensional ANR's **in** B^3/G. Since there exist 3-dimensional AR's which do not contain 2-dimensional AR's, one may conjecture that there exist 3-dimensional AR's which do not contain any 2-dimensional ANR's. The purpose of this section is to prove that the above conjecture is true. This is a generalization of our previous Theorem 2, however, our technique is essentially the same. In fact, we shall show that there exist 3-dimensional AR's which do not contain 2-dimensional ANR's or proper ANR's of dimension 3.

LEMMA 8. *If A is an* ANR, *then there exists a positive real number η such that each loop γ of diameter less than η is nullhomotopic in A.*

Proof. This is immediate from: (a) A is a compact metric and (B) A is locally contractible.

LEMMA 9. *If A is an* ANR *of dimension n, $n \geqslant 2$, then A contains simple closed curve of arbitrarily small diameters.*

Proof. Without loss of generality, we may assume that A is connected, otherwise we choose a component of A having dimension n. Let η be a positive real number such as described by Lemma 8. By [9], there is a partitioning $P = \{P_1, \ldots, P_k\}$ such that the mesh of P is less than η. Some element P_j of the parition P must have dimension n. Then P_j must contain a simple closed curve ξ; since P_j has diameter less than η it follows that ξ is nullhomotopic in A. This concludes the proof of the lemma.

The following is an important lemma.

LEMMA 10. *Let X be an* AR *and A be an* ANR *such that $A \subset X$. Then there exists a positive real number δ and a sequence $\{U_i\}_{i=0}^{\infty}$ of open neighborhoods of A in X such that for each i,* (1) *$U_{i+1} \subset U_i$ and* (2) *each loop γ of diameter less than δ and lying in U_{i+1} is nullhomotopic in U_i.*

Proof. A proof can be constructed by using (a) Lemma 8 and (b) the technique of Theorem 4 of [24].

We have the following:

THEOREM 4. *The space B^3/G is a 3-dimensional* AR *which does not contain any* ANR *of dimension* 2 *or a proper* ANR *of dimension* 3.

Proof. Let A be an ANR of dimension 2 such that $A \subset B^3/G$. By Lemma 9, there is a simple closed curve γ^* in A such that γ^* is nullhomotopic in A. Now $\gamma = P^{-1}(\gamma^*))$ is a special circle-like continuum in $P^{-1}(A)$. Without loss of generality, we may assume that γ is in $\operatorname{Int} B^3$. Let C_α be the core of a torus T_α such that C_α links γ. Given $\eta > 0$, there is a torus T_{α_0} belonging to the sequence T_i such that (1) C_α is the core of T_{α_0} and (2) if $T_{\alpha_0 1}, \ldots, T_{\alpha_0 m_{\alpha_0}}$ is the chain used in the first stage of construction of the dyadic Antoine's necklace N_{α_0} then the diameters of the set $P(T_{\alpha_0 i})$ is less than η for $i = 1, 2, \ldots, m_{\alpha_0}$. Let δ and the sequence $\{U_i\}_{i=0}^{\infty}$ be as given in Lemma 10. Pick $\eta = \delta$.

Define $V_i = P^{-1}(U_i)$ for $i = 0, 1, 2, \ldots$ By an argument similar to the one in Theorem 1 there exists an integer j such that $T_{\alpha_0 j}$ contains a loop ξ such that (1) ξ is not nullhomotopic in $T_{\alpha_0 j}$ and (2) ξ is contained in $V_{m_{\alpha_0}+2}$. Let $f\colon S^1 \to V_{m_{\alpha_0}+2}$ denote this loop. The diagram

$$\begin{tikzcd} S^1 \arrow[r, "g"] \arrow[dr, "f"'] & U_{m_{\alpha_0}+2} \arrow[d, "p"] \\ & V_{m_{\alpha_0}+2} \end{tikzcd}$$

is commutative. Since g is nullhomotopic in $U_{m_{\alpha_0}+1}$, it follows by [8] that f is nullhomotopic in $V_{m_{\alpha_0}+1}$. Without loss of generality, we may assume that there exists an open set W such that (1) $W \cap U_0 = \emptyset$ and (2) there is a meridional disc D in T_{α_0} such that $D \subset (T_{\alpha_0} \cap W)$. Proceeding as in the proof of Theorem 1, we show that $W \cap V_0 \neq \emptyset$. This is a contradiction. The following corollaries are interesting:

COROLLARY 1. *There exists an upper semicontinuous decomposition G of E^3 such that the decomposition space E^3/G is a noncompact metric absolute retract which does not contain any* ANR *of dimension* 2 *or* 3.

In [18], it is shown that if an AR is embedded in a PL manifold then it has arbitrarily small neighborhoods which are ANR's. The following is a result of [18]:

THEOREM. *Let X be an* AR *in the interior of a* PL *3-manifold M^3. If some neighborhood of X can be embedded in E^3, then $X = \bigcap_{i=1}^{\infty} H_i$, where H_i is a polyhedral cube with handles and $H_{i+1} \subset H_i \subset M^3$.*

One may conjecture that an AR embedded in an AR has arbitrarily small neighborhoods which are ANR's. The following corollary provides a negative answer even in the case of arc.

COROLLARY 2. *No arc in B^3/G has arbitrarily small neighborhoods which are* ANR*'s.*

It follows by [14] that $E^4/\bar{G} \times E^1$ is homeomorphic to E^4. This allows us to exhibit an involution of E^4. Such that the fixed point set $E^3/\bar{G}$. Another example of an involution of E^4 may be found in [7]. We state this result as follows:

COROLLARY 3. *There exists an involution γ of E^4 such that the fixed point set $F(\gamma)$ sas the properties:* (1) *$F(\gamma)$ has dimension* 3, (2) *$F(\gamma)$ does not contain any 3-dimenhional* AR *or* ANR *and* (3) *$F(\gamma)$ does not contain any* AR *or* ANR *of dimension* 2.

COROLLARY 4. *There exists an upper semicontinuous decomposition G^* of S^3 whose nondegenerate elements form a countable null family of arcs such that* (1) *S^3/G^* is an* ANR *of dimension* 3, (2) *S^3/G^* does not contain any* ANR *of dimension* 2 *or* 3, *and* (3) *$S^3/G^* \times S^1 \neq S^3 \times S^1$* [14].

COROLLARY 5. *There exists an* ANR *X of dimension* 4 *such that* (1) *the circle group S^1 acts freely on X and* (2) *the orbit space X/S^1 is an* ANR *of dimension* 3 *which does not contain any* ANR *of dimension* 2 *or any proper* ANR *of dimension* 3.

We say that a compact metric space X which cannot be written as a finite or a countable union of ANR's of arbitrarily small diameters has the *singularity of Mazurkiewicz of type* ANR. This is more general than the usual definition of the singularity of Mazurkiewicz. We have the following:

COROLLARY 6. *The* AR *B^3/G has the singularity of Mazurkiewicz of type* ANR.

Remark 4. In this paper we studied dyadic decompositions. We can similarly define n-adic decompositions for each n as pointed out in Remark 1. The results of this paper hold for n-adic decompositions for $n > 2$. The case $n = 1$ was studied by Bing and Borsuk [8] and Armentrout [4].

References

[1] S. Armentrout, *On the singularity of Mazurkiewicz in absolute neighborhood retracts*, Fund. Math. (1970), pp. 131–145.

[2] S. Armentrout, *Decompositions and retracts*, Topology Conference Emory University, Atlanta, Georgia 1970.
[3] — *Local properties of decomposition spaces*, Proceedings of Conference on monotone mappings and open mappings, State University of New York at Binghampton (1970), pp. 98–109.
[4] — *A Bing–Borsuk retract which contains a 2-dimensional retract* (to appear).
[5] — and T. M. Price, *Decompositions into compact sets with UV properties*, Trans. Amer. Math. Soc. 141 (1969), pp. 433–442.
[6] R. H. Bing, *Partitioning continuous curves*, Bull. Amer. Math. Soc. 58 (1952), pp. 536–556.
[7] — *The cartesian product of a certain non-manifold and a line is E^4*, Ann. of Math. 70 (1959), pp. 399–412.
[8] — and K. Borsuk, *A 3-dimensional absolute retract which does not contain any disk*, Fund. Math. 54 (1964), pp. 159–175.
[9] K. Borsuk, *Theory of Retracts*, Warszawa 1967.
[10] — *On an irreducible 2-dimensional absolute retract*, Fund. Math. 37 (1950), pp. 137–160.
[11] — *Theory of Shape*, Aarhus, Denmark, Aarhus Universitet, Mathematisk Institut, 1970. Lecture Notes Series, No 28.
[12] W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton (1941), pp. 1–165.
[13] S. Lefschetz, *Algebraic Topology*, Amer. Math. Soc. Coll. Publ. Vol. 27 (1942), pp. 1–389.
[14] D. V. Meyer, *More decomposition of E^n which are factors of E^{n+1}*, Fund. Math. 67 (1970), pp. 49–65.
[15] D. R. McMillan, Jr., *A criterion for cellularity in a manifold*, Ann. of Math. 79 (1964), pp. 327–337.
[16] R. L. Moore, *Foundations of Point Set Theory*, (rev. ed.) Amer. Math. Soc. Coll. Publ. Vol. 13 (1962).
[17] R. B. Sher, *Realizing cell-like maps in Euclidean spaces*, Gen. Top. and its Applic. 2 (1972), pp. 75–89.
[18] S. Smale, *A Vietoris mapping theorem for homotopy*, Proc. Amer. Math. Soc. 8 (1957), pp. 604–610.
[19] S. Singh, *A 3-dimensional irreducible compact absolute retract which contains no disc*, Fund. Math. (to appear).
[20] J. Stallings, *On the loop theorem*, Ann. of Math. 72 (1960), pp. 12–19.
[21] G. T. Whyburn, *Analytic Topology*, Amer. Math. Soc. Coll. Publ., Vol. 28 (rev. ed.) Providence, R. I. 1963.

THE PENNSYLVANIA STATE UNIVERSITY
ALTOONA CAMPUS
Altoona, Pennsylvania

Accepté par la Rédaction le 18. 9. 1974

Compactly generated shape theories

by

Thomas J. Sanders (Annapolis, Maryland)

Abstract. For locally compact metric spaces, Borsuk's weak extension of shape to metric spaces and compactly generated shape are equivalent.

1. Introduction. Among the extensions of K. Borsuk's shape theory [1] to non-compact spaces are the ones given by Borsuk for metric spaces [2] and L. Rubin and the author for Hausdorff spaces [9]. Relationships that exist between these two extensions are discussed in [10].

The approach to shape in [9] is through the compact subsets of the Hausdorff space, hence the name "compactly generated shape". A weakened version [3] [4] of Borsuk's approach in [2] is also through the compact subsets of the metric space. In private communication, B. J. Ball posed the question as to whether or not these two approaches are equivalent. We are able to answer affirmative for locally compact metric spaces. The reader is referred to [9], [12] for the development of compactly generated shape. We denote the compactly generated shape category of [12] by $\mathscr{H}$. The full subcategory of $\mathscr{H}$ consisting of locally compact metric spaces is denoted by $\mathscr{H}_0$. We use AR and ANR to denote, respectively, absolute retract and absolute neighborhood retract for general (i.e. possibly not compact) metric spaces.

2. Weak shape. Suppose M and N are AR's and X and Y are closed subsets of M and N, respectively. A *weak sequence* from X to Y in (M, N), $\underline{\varphi} = \{\varphi_k, X, Y\}_{M,N}$, is a sequence of maps $\varphi_k\colon M \to N$ that satisfy the following condition:

(2.1) For every compactum $A \subset X$ there is a compactum $B \subset Y$ such that for every neighborhood V of B (in N) there is a neighborhood U of X (in M) and an integer K such that if $k \geqslant K$ then
$$\varphi_k|_U \simeq \varphi_{k+1}|_U \quad \text{in } V.$$

Note that a fundamental sequence $\underline{\varphi} = \{\varphi_k, X, Y\}_{M,N}$ as defined in [2] is a weak sequence. Intuitively, we have dropped the "external" conditions imposed on a fundamental sequence in [2] and have retained only the "internal" conditions. Compositions and identities may be defined as in [2]. A weak sequence $\underline{\varphi} = \{\varphi_k, X, Y\}_{M,N}$ is an *extension* of $\underline{\psi} = \{\psi_k, X', Y\}_{M,N}$ if $X' \subset X$ and $\varphi_k(x) = \psi_k(x)$ for all $x \in X'$