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The automorphism group of a p-group of maximal class with an abelian maximal subgroup *

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Abstract. In this paper we give a detailed description of the automorphism group of a p-group of maximal class with a maximal subgroup which is abelian.

§ 1. In this paper we will always let G denote a p-group of maximal class of order p^n, $n\geqslant 4$, p an odd prime, and we will let $\mathscr{A}$ be the group of all automorphisms of G. First we note that G has a characteristic cyclic series, that is, there are characteristic subgroups, G_i, $0\leqslant i\leqslant n$, of G with G_i/G_{i+1} cyclic such that

$$G = G_0 \rhd G_1 \rhd \ldots \rhd G_n = E. \tag{1.1.1}$$

This follows from Lemmas 14.2 and 14.4 in [7]. From Durbin and McDonald's result in [3] or [1], $\mathscr{A}$ is supersolvable and its exponent divides $p^t(p-1)$ for some $t>0$. Thus the Sylow p-subgroup P of $\mathscr{A}$ is normal in $\mathscr{A}$, and so it has a p'-complement H.

The characteristic series (1.1.1) may be taken as a composition series, in that case the factors G_i/G_{i+1} have prime order p. Thus any automorphism α of G acts on G_i/G_{i+1} as a power map, i.e. if α is an automorphism of G restricted to G_i/G_{i+1} then $(aG_{i+1})\alpha = \alpha^n G_{i+1}$ for all $aG_{i+1}\in G_i/G_{i+1}$. Consider H' the commutator subgroup of H; clearly H' stabilizes (1.1.1), that is, if $h\in H'$ then h acts trivially on G_i/G_{i+1}, $i = 0, \ldots, n-1$. By Theorem 1 of P. Hall's paper [6] H' is nilpotent, and by Corollary 3.3 of [4], p. 179, it is a p-group. Therefore H' is trivial giving us that H is an abelian p'-group.

LEMMA 1.1. *The automorphism group $\mathscr{A}$ of a p-group G of maximal class is the semidirect product of P by H where P is the normal Sylow p-subgroup of $\mathscr{A}$ and where H is the p'-complement of P. Furthermore H is an abelian p'-subgroup of $\mathscr{A}$ with exponent dividing $p-1$.*

A p'-group H of automorphisms of G may be represented faithfully on the H-module $G/\Phi(G)$ over the field Z_p (integers modulo p). Here $\Phi(G)$ denotes the Frat-

* A p-group of order p^n is of maximal class if it has class $n-1$. N. Blackburn studies these groups in detail in his paper [2]; most of his results are presented in Huppert [7], pp. 361–377.

tini subgroup of G. By Maschke's Theorem the H-module $G/\Phi(G)$ can be written as a direct sum of irreducible H-modules. These irreducible H-modules have dimension one since the elements of H acts on $G/\Phi(G)$ as power maps. Thus we have

(1.2.1) $$G/\Phi(G) = L_1/\Phi(G) \oplus L_2/\Phi(G)$$

where $L_i/\Phi(G)$, $i = 1, 2$, are irreducible H-modules. Let $C_i = C_H(L_i/\Phi(G))$, the centralizer of $L_i/\Phi(G)$ in H, $i = 1, 2$. Each H/C_i is faithfully and irreducibly represented on $L_i/\Phi(G)$, $i = 1, 2$, and so H/C_i is cyclic of order dividing $p-1$ (see above). Now

$$C_1 \cap C_2 = E$$

since H is faithfully represented on $G/\Phi(G)$. Therefore H is embeddable in the direct sum of

(1.2.2) $$H/C_1 \oplus H/C_2 \cong C_{p-1} \oplus C_{p-1}$$

where C_{p-1} is the cyclic group of order $p-1$. Thus we have the following result.

THEOREM 1.2. *A p'-group of automorphisms of a p-group of maximal class G is embeddable in $C_{p-1} \oplus C_{p-1}$ (the direct sum of two cyclic groups of order $p-1$).*

Remark 1.3. Both Lemma 1.1 and Theorem 1.2 could be stated in a more general setting; namely, that G is a p-group with a characteristic cyclic series, like (1.1.1).

§ 2. By Lemmas 14.2 and 14.4 in Huppert [7] we have that the G_i of the composition series (1.1.1) is the ith term of the lower central series of G for $i = 2, \ldots, n$, that is

$$G_2 = G' = [G, G] \quad \text{(the commutator subgroup of } G\text{)},$$
$$G_i = [G_{i-1}, G], \quad i = 3, \ldots, n,$$

and we have that G_1 is centralizer of G_2 modulo G_4 in G, that is

$$G_1 = C_G(G_2/G_4).$$

We now assume that (for Sections 2 and 3)

(2.1.2) i) G_1 is abelian,
ii) G has exponent p.

We also assume that $p > 3$ and $n \geqslant 5$ to avoid some special cases. These assumptions put several severe restrictions on G. For example, G is a regular p-group and so, by Lemma 14.21 of [7], $n \leqslant p$. Clearly

$$C_G(G_i/G_{i+2}) = G_1, \quad i = 2, \ldots, n-2,$$

so G is a "keine Ausnahmegruppe" in Huppert's terminology, see [7].

We define the simple left normed commutator $[x_1, \ldots, x_n]$ on $\{x_1, \ldots, x_n\}$ of weight n by

(2.1.3) $$[x_1, \ldots, x_n] \underset{\mathrm{df}}{=} [[x_1, \ldots, x_{n-1}], x_n]$$

(x_1, is a simple left normed commutator of weight one). We list some standard commutator identities for later use:

(2.1.4) $$[xy, z] = [x, z]^y [y, z], \quad [x, yz] = [x, z][x, y]^z$$
$$[x, y]^z = [x, y][x, y, z].$$

By Lemma 14.8 in [7] we see that there are generators s and s_1 of G such that

(2.1.5) $$G = \langle G_1, s\rangle \quad \text{and} \quad G_1 = \langle G_2, s_1\rangle$$

and if

(2.1.6) $$s_i = [s_{i-1}, s], \quad i = 2, \ldots, n-1 \ (n \geqslant 4)$$

then

(2.1.6)′ $$G_i = \langle G_{i+1}, s_i\rangle, \quad i = 1, \ldots, n-1.$$

Now clearly by the definition above

(2.1.7) $$s_i = [s_1, \overbrace{s, \ldots s}^{i-1}],$$

Clearly any element of g of G can be written in the following form

(2.1.8) $$g = s^k s_1^{k_1} s_2^{k_2} \ldots s_{n-1}^{k_{n-1}}$$

where $0 \leqslant k < p$ and $0 \leqslant k_i < p$ for $i = 1, \ldots, n-1$. This expression, (2.1.8), for g is unique; to see this suppose that we also have

$$g = s^l s_1^{l_1} \ldots s_{n-1}^{l_{n-1}}.$$

First we see that $k = e$ by factoring out G_1. So now we have

$$s_1^{k_1} \ldots s_{n-1}^{k_{n-1}} = s_1^{l_1} \ldots s_{n-1}^{l_{n-1}}$$

but $\{s_1, \ldots s_{n-1}\}$ is a basis for the vector space G_1 over $\boldsymbol{Z}_p$ (the integers modulo p); therefore $k_i = l_i$, $i = 1, \ldots, n-1$.

Let u be a primitive $p-1$ root of unity modulo p, i.e. $u^{p-1} \equiv 1 \bmod p$, but $u^t \not\equiv 1 \bmod p$ for $0 < t < p-1$. Now we define a mapping α on G by setting

(2.1.9) $$(s)\alpha = s, \quad (s_1)\alpha = s_1^u \quad \text{and} \quad (s_i)\alpha = [s_{i-1}\alpha, s], \quad i = 2, \ldots, n-1.$$

and so if g is expressed as in (2.1.8)

(2.1.9)′ $$g\alpha = ((s)\alpha)^k ((s_1)\alpha)^{k_1} \ldots ((s_{n-1})\alpha)^{k_{n-1}}.$$

Now by the uniqueness of the expression (2.1.8) for each g in G, this defines a mapping on G. The mapping α is a homomorphism of G; the calculation used to verify this basically depends on the following

(2.1.10) $$([s_i, s]\alpha = [(s_i)\alpha, (s)\alpha], \quad i = 1, \ldots, n-1.$$

It is clear that α is one to one, and thus α is an automorphism of G. By its definition the order of α is $p-1$.

Now if v is also a primitive $p-1$ root of unity modulo p, then we can define a mapping β on G (as above) by setting

(2.1.11) $\quad (s)\beta = s^v, \quad (s_1)\beta = s_1, \quad (s_i)\beta = [(s_{i-1})\beta, (s)\beta], \quad i = 2, \dots, n-1$

and so if g is expressed as in (2.1.8)

(2.1.11)′ $\quad (g)\beta = ((s)\beta)^k((s_1)\beta)^{k_1} \dots ((s_{n-1})\beta)^{k_{n-1}}$.

The mapping β is a homomorphism of G because

(2.1.12) $\quad ([s_i, s])\beta = [(s_i)\beta, (s)\beta], \quad i = 1, \dots, n-1$.

Thus β is an automorphism of G of order $p-1$. Let K be the group generated by α and β; this group is abelian since

$$s^{\alpha\beta} = s^\beta = s^{\beta\alpha} \quad \text{and} \quad s_1^{\beta\alpha} = s_1^\alpha = s_1^{\alpha\beta}.$$

$$K \cong C_{p-1} \oplus C_{p-1}$$

and we have the following Theorem.

THEOREM 2.1. *If G is a p-group of maximal class of exponent p with a maximal subgroup (a subgroup of index p) which is abelian, then the p'-complement H of the normal Sylow p-subgroup P of the group $\mathscr{A}$ of automorphisms of G is isomorphic to $C_{p-1} \oplus C_{p-1}$.*

§ 3. Now we study the normal Sylow p-subgroup P of the group of automorphisms $\mathscr{A}$. We have of course the following normal chain of subgroups of $\mathscr{A}$:

(3.1.1) $\quad E \lhd \mathscr{I}_1 \lhd \mathscr{I} \trianglelefteq \mathscr{F} \trianglelefteq P$

where $\mathscr{F}$ is the group of p-automorphisms fixing the Frattini factor $G/\Phi(G)$, see [5], $\mathscr{I}$ the group of inner automorphisms of G, and $\mathscr{I}_1$ is the subgroup of inner automorphisms of G induced by elements of the characteristic subgroup G_1 of G. Clearly

(3.1.2) $\quad \mathscr{I} \cong G/Z(G)$

a p-group of maximal class of order p^{n-1} by Lemma 14.2 in [7], and

(3.1.3) $\quad \mathscr{I}_1 \cong G_1/Z(G)$

a maximal abelian subgroup of $\mathscr{I}$.

We now study the group $\mathscr{F}$ of automorphisms. If $\alpha \in \mathscr{F}$ then

$$(x)\alpha = xf_x$$

for any $x \in G$ where $f_x \in \Phi(G)$. We define maps η_i, $i = 1, \dots, n-2$ by

(3.1.4) $\quad s^{\eta_i} = s, \quad s_1^{\eta_i} = s_1[s_i, s] \quad \text{and} \quad s_j^{\eta_i} = [(s_{j-1})^{\eta_i}, s], \quad j = 2, \dots, n-1$

for $i = 1, \dots, n-2$ (η_1 is just the inner automorphism $\bar{s}$, i.e., the inner automorphism induced on G by s). By the uniqueness of (2.1.8) we can extend η_i to G by setting

(3.1.5) $\quad g^{\eta_i} = s^k(s_1^{\eta_i})^{k_i} \dots (s_{n-1}^{\eta_i})^{k_{n-1}}$

for $i = 1, \dots, n-2$. The calculation which verifies that η_i is a homomorphism rests on the fact that

$$([x, s])^{\eta_i} = [(x)^{\eta_i}, s]$$

for x any power of $s_j, j = 1, \dots, n-2$. The homomorphism η_i maps the generating set $\{s, s_1\}$ of G onto the generating set $\{s, s_1[s_i, s]\}$, and so η_i is an automorphism of G for $i = 1, \dots, n-2$. The η_i commute with each other

$$s_1^{\eta_i\eta_j} = s_1[s_i, s][s_j, s][s_{i+j}, s] = s_1^{\eta_j\eta_i}$$

for $1 \leqslant i, j \leqslant n-2$. Also if $l > n-2$

$$s_1^{(\eta_i^l)} = s_1[s_i, s]^{\binom{l}{1}}[s_i, s, s]^{\binom{l}{2}} \dots [\overbrace{s_i, s, \dots, s}^{n-2}]^{\binom{l}{n-2}}$$

where $\binom{l}{i}$ is the binomial coefficient. Thus, since p divides $\binom{p}{i}$ for $i = 1, \dots, p-1$, we have that

$$s_1^{(\eta_i^p)} = s_1$$

giving us that each η_i has exponent p. The order of $\langle \eta_i | \, i = 1, \dots, n-2 \rangle$ is thus p^{n-2}, and the order of $\mathscr{I}_1$ is p^{n-2}. Hence the order of $\langle \mathscr{I}, \eta_i | \, i = 2, \dots, n-2 \rangle$ is $p^{2(n-2)}$, and so

(3.1.6) $\quad \mathscr{F} = \langle \mathscr{I}, \eta_i | \, i = 2, \dots, n-2 \rangle$

since the order of $\mathscr{F}$ is at most $p^{2(n-2)}$, see [5], p. 178.

By Gaschutz's results, there is still another p-automorphism, ([7] and in particular, Proposition 19.1, in [7]). For us this automorphism ϱ is given by

(3.1.7) $\quad s^\varrho = ss_1, \quad s_1^\varrho = s_1 \quad \text{and} \quad s_i^\varrho = s_i, \quad i = 1, \dots, n-1.$

As before one uses (2.1.8) to extend ϱ to a mapping on G. Once again it is necessary to check that the map ϱ is a homomorphism, this follows from the fact that

$$([s_i, s])^\varrho = [s_i^\varrho, s^\varrho] = [s_i, ss_1] = [s_i, s]$$

for $i = 1, \dots, n-2$. The map ϱ takes the generating set $\{s, s_1\}$ onto the generating set $\{ss_1, s_1\}$, thus ϱ is an automorphism of G. The automorphism ϱ fixes G_1 but does not fix $G/\Phi(G)$. Clearly the order of the automorphism ϱ is p by the definition of the map ϱ. Thus the order of $\langle \mathscr{F}, \varrho \rangle$ is p^{2n-3} which is the maximal order which the Sylow p-subgroup P can have, see [5], p. 178. So

(3.1.8) $\quad P = \langle \mathscr{I}_1, \eta_i, \varrho | \, i = 1, \dots, n-2 \rangle$.

The following formulas hold for ϱ and η_i

(3.1.9) $\quad \varrho^{-1}\eta_i\varrho = \eta_i\bar{s}_i, \quad \text{i.e. } [\eta_i, \varrho] = \bar{s}_i$

for $i = 1, \dots, n-2$ and

(3.1.9)′ $\quad \varrho^{-1}\bar{s}_i\varrho = \bar{s}_i, \quad [s_i, \varrho] = \varrho$.

for $i = 1, \ldots, n-2$. Thus $P/\mathscr{I}_1$ is abelian, in fact

$$\Phi(P) = P' = \mathscr{I}_1 . \tag{3.1.10}$$

This means that P is metabelian of class $n-2$. The Sylow p-subgroup has

$$\{\varrho, \eta_i | \; i = 1, \ldots, n-2\} \tag{3.1.11}$$

as a generating set. We summarize some of the above results of this section in the following theorem:

THEOREM 3.1. *If G is a p-group of maximal class of exponent p with a maximal subgroup which is abelian, then the normal Sylow p-subgroup P is metabelian of class $n-2$ and of order p^{2n-3}. The commutator subgroup P' of P is the subgroup of inner automorphisms $\mathscr{I}_1$ induced on G by the maximal subgroup of G which is abelian* (see (3.1.10)).

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Selection theorems for partitions of Polish spaces

by

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Abstract. In this paper we evaluate the (Borel or projective) class of selectors for partitions of Polish spaces into disjoint closed sets. In particular, we improve upon the results pertaining to α^- partitions which have been obtained recently by Kuratowski and Maitra.

1. Introduction. The problem of the existence of "topologically pleasant" selectors for partitions of a Polish space into disjoint, non-empty, closed sets, where the partitions themselves are "topologically pleasant", has been considered by several authors. We mention here the articles of Mazurkiewicz [8], Bourbaki [2], and Kuratowski and Maitra [7].

In this paper we shall be mainly concerned with the evaluation of the (Borel or projective) class of selectors. The first such result known to us was proved by Mazurkiewicz ([8] and [5], p. 389). He showed that any partition of a closed subset of the space of irrationals which is induced by a continuous function defined on it to a separable metric space admits a coanalytic selector. In the same spirit, Bourbaki proved that any upper semi-continuous partition of a Polish space into closed sets admits a G_δ selector ([2], Chap. 9, Ex. 9(a), p. 262). Kuratowski and Maitra [7] extended Bourbaki's result by showing that any α^+ or α^- partition of a Polish space into closed sets admits a selector of multiplicative class $(\alpha+1)$ (for definitions, see Section 2).

We shall establish in this paper some general results on the existence of selectors, from which it will follow that the results of Kuratowski and Maitra for α^- partitions can be improved at all levels $\alpha>0$. Indeed, if $\alpha>0$, we prove that any α^- partition of a Polish space admits a selector of multiplicative class α, and, moreover that, in general, a selector of lower class does not exist.

Our method of defining a selector is as follows. We first define a suitable linear order on each Polish space such that every non-empty closed set has a first element. We achieve this by using a result of Arhangel'skiĭ [1], which states that every Polish space is a continuous open image of the space of irrationals. Using such a continuous open function, we transfer the lexicographic order on the space of irrationals to the given Polish space. The selector is now taken to be the set of all first elements of members of the given partition. Our results on the existence of tractable linear orders