P-embedding and product spaces

by

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Abstract. Let A be a subset of a topological space X and Y a compact Hausdorff space with weight $\mathfrak{m}$, where $\mathfrak{m}$ is an infinite cardinal number. Our main theorem asserts that A is $P^{\mathfrak{m}}$-embedded in X iff $A \times Y$ is C^*-embedded in $X \times Y$. This theorem settles all the questions posed by R. A. Alò and L. I. Sennott as well as gives an analysis of the theorems obtained recently by M. E. Rudin and by M. Starbird.

§ 1. Throughout this paper by a space we shall mean a topological space and by $\mathfrak{m}$ an infinite cardinal number.

A subspace A of a space X is said to be $P^{\mathfrak{m}}$-*embedded* in X if every locally finite cozero-set cover of A with cardinality $\leqslant \mathfrak{m}$ has a refinement which can be extended to a locally finite cozero-set cover of X. In case A is $P^{\mathfrak{m}}$-embedded in X for every $\mathfrak{m}$, A is said to be P-embedded in X (H. L. Shapiro [11]). For the case $\mathfrak{m} = \aleph_0$ $P^{\aleph_0}$-embedding coincides with C-embedding in the usual sense (T. E. Gantner [3]).

Our main concern in this paper is to describe $P^{\mathfrak{m}}$- or P-embedding in terms of C^*-embedding in product spaces.

As for paracompactness and normality in products the following theorems are obtained by K. Morita [5, Theorems 2.2, 2.4 and 2.7, and 6, Theorem 1.3].

THEOREM 1.1. *For a space X the following statements are equivalent.*

(a) *X is $\mathfrak{m}$-paracompact and normal.*

(b) *$X \times Y$ is normal for every compact Hausdorff space Y of weight $\leqslant \mathfrak{m}$.*

(c) *$X \times I^{\mathfrak{m}}$ is normal.*

(d) *$X \times D^{\mathfrak{m}}$ is normal.*

Here I denotes the unit interval $[0, 1]$ and D the discrete space consisting of exactly two points 0 and 1.

THEOREM 1.2. *Let X be a completely regular Hausdorff space and Y a compact Hausdorff space containing X as its subspace. Then X is paracompact iff $X \times Y$ is normal.*

The following theorems are motivated by the above theorems.

THEOREM 1.3. *For a subspace A of a space X the following statements are equivalent.*

(a) *A is $P^{\mathfrak{m}}$-embedded in X.*

(b) *$A \times Y$ is C^*-embedded in $X \times Y$ for every compact Hausdorff space Y of weight $\leqslant \mathfrak{m}$.*

(c) $A \times I^{\mathfrak{m}}$ *is* C^**-embedded in* $X \times I^{\mathfrak{m}}$.
(d) $A \times D^{\mathfrak{m}}$ *is* C^**-embedded in* $X \times D^{\mathfrak{m}}$.

THEOREM 1.4. *Let A be a subspace of a space X and Y a compact Hausdorff space containing A as its subspace. Then A is P-embedded in X iff $A \times Y$ is C^*-embedded in $X \times Y$.*

As for $P^{\mathfrak{m}}$- or P-embedding in product spaces R. A. Alò and L. I. Sennott [1] have given several interesting results. The essential parts of their results follow readily from our characterizations above. In particular, all of the conjectures or questions which remained unsettled in their paper [1] are proved or solved by our Theorems 1.3 and 1.4.

Furthermore we can establish a more precise result on $P^{\mathfrak{m}}$-embedding.

THEOREM 1.5. *Let A be a subspace of a space X and Y a compact Hausdorff space of weight $\mathfrak{m}$. Then A is $P^{\mathfrak{m}}$-embedded in X iff $A \times Y$ is C^*-embedded in $X \times Y$.*

As was proved essentially by C. H. Dowker [2], a space X is collectionwise (resp. $\mathfrak{m}$-collectionwise) normal iff every closed subset of X is P- (resp. $P^{\mathfrak{m}}$-) embedded in X (for a direct proof, see Theorem 3.3 below). Accordingly as a direct consequence of our Theorem 1.5 we have the following result which has recently been proved by M. E. Rudin [10].

THEOREM 1.6. *If the product space $X \times Y$ of a space X with a compact Hausdorff space Y of weight $\mathfrak{m}$ is normal, then X is $\mathfrak{m}$-collectionwise normal.*

§ 2. Before proving our theorems we shall need some preliminary results. The following is given in [9, Lemma 2.1].

LEMMA 2.1. *A subspace A of a space X is C^*-embedded in X iff every finite cozero-set cover of A has a refinement which can be extended to a normal open cover of X.*

As was proved in [5, Corollary 1.3] an open cover $\{G_\alpha |\ \alpha \in \Omega\}$ of a normal space X is normal iff there exists a normal open cover $\{U_\lambda\}$ of X such that each set U_λ is contained in a union of a finite number of sets of $\{G_\alpha\}$. The following theorem may be compared with this result.

THEOREM 2.2. *Let A be a C^*-embedded subspace of a space X. Then A is $P^{\mathfrak{m}}$-embedded in X iff for every locally finite cozero-set cover $\{H_\alpha |\ \alpha \in \Omega\}$ of A with $\operatorname{card}\Omega \leqslant \mathfrak{m}$ there exists a locally finite cozero-set cover $\{U_\lambda\}$ of X such that each set $U_\lambda \cap A$ is contained in a union of a finite number of sets of $\{H_\alpha\}$.*

Proof. We have only to prove the "if" part. Let $\{G_\alpha |\ \alpha \in \Omega\}$ be a locally finite cozero-set cover of A with $\operatorname{card}\Omega \leqslant \mathfrak{m}$. Then there exist a cozero-set cover $\{H_\alpha |\ \alpha \in \Omega\}$ of A and a continuous map f_α: $A \to I$ for $a \in \Omega$ such that

$$f_\alpha(x) = \begin{cases} 0 & \text{if} \quad x \in H_\alpha , \\ 1 & \text{if} \quad x \in A - G_\alpha . \end{cases}$$

Let g_α: $X \to I$ be an extension of f_α and put

$$\left.\begin{aligned} K_\alpha &= \{x |\ g_\alpha(x) = 0\} \\ L_\alpha &= \{x |\ g_\alpha(x) < 1\} \end{aligned}\right. \quad \text{for } \alpha \in \Omega .$$

Then K_α is a zero-set and L_α a cozero-set of X. On the other hand, for $\{H_\alpha |\ \alpha \in \Omega\}$ there exists $\{U_\lambda |\ \lambda \in \Lambda\}$ with the properties described in the theorem. For each $\lambda \in \Lambda$ let us choose a finite subset Δ_λ of Ω so that

$$U_\lambda \cap A \subset \bigcup \{H_\alpha |\ \alpha \in \Delta_\lambda\} .$$

Then we have (1), (2) and (3) below:

(1) $$K_\alpha \subset L_\alpha \quad \text{for} \quad \alpha \in \Omega .$$

(2) $$H_\alpha \subset K_\alpha \cap A \subset L_\alpha \cap A \subset G_\alpha \quad \text{for} \quad \alpha \in \Omega .$$

(3) $$U_\lambda \cap A \subset \bigcup \{K_\alpha |\ \alpha \in \Delta_\lambda\} \quad \text{for} \quad \lambda \in \Lambda .$$

Finally let us put

$$\mathscr{V} = \{U_\lambda \cap L_\alpha |\ \alpha \in \Delta_\lambda, \lambda \in \Lambda\} \cup \{U_\lambda \cap (X - \bigcup_{\alpha \in \Delta_\lambda} K_\alpha) |\ \lambda \in \Lambda\} .$$

Then by (1) $\mathscr{V}$ is a locally finite cozero-set cover of X and by (2) and (3) $\mathscr{V} \cap A = \{V \cap A |\ V \in \mathscr{V}\}$ refines $\{G_\alpha |\ \alpha \in \Omega\}$. Thus A is $P^{\mathfrak{m}}$-embedded in X, and the proof is completed.

The union of a σ-locally finite collection of cozero-sets is a cozero-set, while the union of a discrete collection of zero-sets is not always a zero-set. Accordingly, the following lemma may be of interest.

LEMMA 2.3 ([1]). *Let $\{K_\alpha |\ \alpha \in \Omega\}$ be a locally finite collection of zero-sets of a space X such that there is a locally finite collection $\{L_\alpha |\ \alpha \in \Omega\}$ of cozero-sets of X with $K_\alpha \subset L_\alpha$ for each $\alpha \in \Omega$. Then $\bigcup \{K_\alpha |\ \alpha \in \Omega\}$ is a zero-set of X.*

Proof. Since K_α is a zero-set and L_α a cozero-set of X with $K_\alpha \subset L_\alpha$, there exists a continuous map f_α: $X \to I$ such that

$$\left.\begin{aligned} K_\alpha &= \{x |\ f_\alpha(x) = 1\} \\ L_\alpha &= \{x |\ f_\alpha(x) > 0\} \end{aligned}\right\} \quad \text{for} \quad \alpha \in \Omega .$$

Let us define a map g: $X \to I$ by

$$g(x) = \sup\{f_\alpha(x) |\ \alpha \in \Omega\} \quad \text{for} \quad x \in X .$$

Then g is continuous, and moreover we have

$$\bigcup \{K_\alpha |\ \alpha \in \Omega\} = \{x |\ g(x) = 1\} .$$

Hence $\bigcup \{K_\alpha |\ \alpha \in \Omega\}$ is a zero-set of X, and this completes the proof.

As another characterization of $P^{\mathfrak{m}}$-embedding we have the following theorem.

THEOREM 2.4. *Let A be a subspace of a space X. Then A is $P^{\mathfrak{m}}$-embedded in X iff A is C-embedded in X and for every discrete collection $\{G_\alpha |\ \alpha \in \Omega\}$ of open sets of A*

[1] For the case that $\operatorname{card}\Omega = \aleph_0$ and $\{G_\alpha\}$ is discrete Lemma 2.3 was observed by T. Ishii in a letter (dated Nov. 20, 1972) to K. Morita.

with $\operatorname{card}\Omega \leqslant \mathfrak{m}$ *and every collection* $\{F_\alpha|\ \alpha\in\Omega\}$ *of closed sets of* A *such that* $\{G_\alpha, A-F_\alpha\}$ *is a normal open cover of* A *for each* $\alpha\in\Omega$, *there exists a locally finite collection* $\{H_\alpha|\ \alpha\in\Omega\}$ *of cozero-sets of* X *such that*

$$F_\alpha \subset H_\alpha \cap A \subset G_\alpha \quad \textit{for each } \alpha\in\Omega .$$

Proof. Suppose that A is $P^{\mathfrak{m}}$-embedded in X. Then A is obviously C-embedded in X. Take $\{G_\alpha|\ \alpha\in\Omega\}$ and $\{F_\alpha|\ \alpha\in\Omega\}$ as is described in the theorem. Then, since $\{G_\alpha, A-F_\alpha\}$ is a normal open cover of A for $\alpha\in\Omega$, there exist a zero-set K_α and a cozero-set L_α of A such that

$$F_\alpha \subset K_\alpha \subset L_\alpha \subset G_\alpha .$$

Let us put

$$\mathscr{U} = \{L_\alpha|\ \alpha\in\Omega\} \cup \{A - \bigcup_{\alpha\in\Omega} K_\alpha\} .$$

Then $\mathscr{U}$ is a locally finite cozero-set cover of A with $\operatorname{card} \leqslant \mathfrak{m}$ since $\bigcup_{\alpha\in\Omega} K_\alpha$ is a zero-set of A by Lemma 2.3, and moreover we have

(1) $$\operatorname{St}(K_\alpha, \mathscr{U}) \subset L_\alpha \quad \text{for each } \alpha\in\Omega .$$

Since A is $P^{\mathfrak{m}}$-embedded in X, there exists a locally finite cozero-set cover $\mathscr{V}$ of X such that $\mathscr{V} \cap A$ refines $\mathscr{U}$. Then we have

(2) $$\text{either} \quad V\cap K_\alpha = \emptyset \quad \text{or} \quad V\cap K_\beta = \emptyset$$

for each set V of $\mathscr{V}$ and $\alpha, \beta\in\Omega$ with $\alpha\neq\beta$; and

(3) $$\operatorname{St}(K_\alpha, \mathscr{V}) \cap A \subset \operatorname{St}(K_\alpha, \mathscr{U}) \quad \text{for} \quad \alpha\in\Omega .$$

Let $H_\alpha = \operatorname{St}(K_\alpha, \mathscr{V})$ for $\alpha\in\Omega$. Then each H_α is a cozero-set of X. Moreover $\{H_\alpha|\ \alpha\in\Omega\}$ is locally finite by (2) and the local finiteness of $\mathscr{V}$, and by (1) and (3) we have

$$F_\alpha \subset K_\alpha \subset H_\alpha \cap A \subset \operatorname{St}(K_\alpha, \mathscr{U}) \subset L_\alpha \subset G_\alpha \quad \text{for} \quad \alpha\in\Omega .$$

Thus, the "only if" part is proved.

Conversely, suppose that $\mathscr{U} = \{U_\alpha|\ \alpha\in\Omega\}$ is a locally finite cozero-set cover of A with $\operatorname{card}\Omega \leqslant \mathfrak{m}$. Then there exist a cozero-set cover $\mathscr{G} = \bigcup \mathscr{G}_n$ and a zero-set cover $\mathscr{F} = \bigcup \mathscr{F}_n$ of A, where $\mathscr{G}_n = \{G_{n\alpha}|\ \alpha\in\Omega_n\}$ and $\mathscr{F}_n = \{F_{n\alpha}|\ \alpha\in\Omega_n\}$ with $\operatorname{card}\Omega_n \leqslant \mathfrak{m}$ for $n = 1, 2, \ldots$ such that

(4) $$\mathscr{G} \text{ refines } \mathscr{U},$$

(5) $$\mathscr{G}_n \text{ is discrete} \quad \text{for} \quad n = 1, 2, \ldots,$$

(6) $$F_{n\alpha} \subset G_{n\alpha} \quad \text{for} \quad \alpha\in\Omega_n,\ n = 1, 2, \ldots$$

Then for every n it is easy to see that $\{G_{n\alpha}|\ \alpha\in\Omega_n\}$ and $\{F_{n\alpha}|\ \alpha\in\Omega_n\}$ satisfy the assumption of the theorem. Hence there exists a locally finite collection $\{H_{n\alpha}|\ \alpha\in\Omega_n\}$ of cozero-sets of X such that

$$F_{n\alpha} \subset H_{n\alpha} \cap A \subset G_{n\alpha} \quad \text{for} \quad \alpha\in\Omega_n,\ n = 1, 2, \ldots$$

Let us put

$$D = \bigcup\{H_{n\alpha}|\ \alpha\in\Omega_n,\ n = 1, 2, \ldots\} .$$

Then D is a cozero-set of X and clearly contains A. Since A is C-embedded in X, by [4, Theorem 1.18], there is a cozero-set E of X such that

$$E\cap A = \emptyset, \quad E\cup D = X .$$

Finally let us put

$$\mathscr{V} = \{E\} \cup \{H_{n\alpha}|\ \alpha\in\Omega_n,\ n = 1, 2, \ldots\} .$$

Then $\mathscr{V}$ is a σ-locally finite cozero-set cover of X, and hence a normal open cover of X such that $\mathscr{V}\cap A$ refines $\mathscr{U}$. Thus, A is $P^{\mathfrak{m}}$-embedded in X. This completes the proof of Theorem 2.4.

The following theorem which was proved in [8, Theorem 2.5] plays an essential role in the present paper.

THEOREM 2.5. *Let* X *be a space and* Y *a compact Hausdorff space. Let* $\mathscr{G} = \{G_\alpha|\ \alpha\in\Omega\}$ *be an open cover of* $X\times Y$. *Then there exists an open cover* $\mathscr{U} = \{U_\lambda|\ \lambda\in\Lambda\}$ *of* X *satisfying conditions* (a), (b) *and* (c) *below*:

(a) $\operatorname{card}\Lambda \leqslant \mathfrak{m}$ *or* $<\aleph_0$ *according as* $\mathfrak{m}\geqslant\aleph_0$ *or* $\mathfrak{m}<\aleph_0$ *where* $\mathfrak{m} = \operatorname{Max}(\operatorname{card}\Omega$, *weight of* $Y)$.

(b) *For a suitable collection* $\{\mathscr{V}_\lambda|\ \lambda\in\Lambda\}$ *of finite open covers of* Y, *the collection* $\{U_\lambda\times V|\ V\in\mathscr{V}_\lambda, \lambda\in\Lambda\}$ *is an open cover of* $X\times Y$ *which refines* $\mathscr{G}$.

(c) $\mathscr{U}$ *is a normal open cover of* X *iff* $\mathscr{G}$ *is a normal open cover of* $X\times Y$.

We need further the following two lemmas, the first of which is given in [8], and the second in [1] (for a proof see [9]).

LEMMA 2.6. *Let* X *be a space and* Y *a normal Hausdorff P-space in the sense of K. Morita* [7]. *If a subset* B *of* Y *is locally compact,* σ*-compact and closed, then* $X\times B$ *is* C*-embedded in* $X\times Y$.

LEMMA 2.7. *Let* A *be a* $P^{\mathfrak{m}}$*-embedded subspace of a space* X. *Then* $A\times Y$ *is* $P^{\mathfrak{m}}$*-embedded in* $X\times Y$ *for any compact Hausdorff space* Y *of weight* $\leqslant\mathfrak{m}$.

In concluding this section we shall prove one more lemma.

LEMMA 2.8. *Let* A *be a subspace of a space* X *and* Y *a non-discrete compact Hausdorff space. If* $A\times Y$ *is* C^**-embedded in* $X\times Y$, *then* A *is* C*-embedded in* X.

Proof. Suppose $A\times Y$ is C^*-embedded in $X\times Y$. First we note that A is C^*-embedded in X. Let $\{U_n|\ n = 1, 2, \ldots\}$ be a countable cozero-set cover of A. Since Y is a non-discrete compact Hausdorff space, Y contains an infinite discrete subset $B = \{y_n|\ n = 1, 2, \ldots\}$. Let us put

$$H_1 = \bigcup\{U_n\times\{y_i|\ i\leqslant n\}|\ n = 1, 2, \ldots\},$$
$$H_2 = \bigcup\{U_n\times(\operatorname{Cl}B - \{y_i|\ i\leqslant n\})|\ n = 1, 2, \ldots\} .$$

Then each of H_1 and H_2 is a cozero-set of $A\times \mathrm{Cl}B$ since each point y_n of B is isolated in $\mathrm{Cl}B$, and it is easy to see that $\{H_1, H_2\}$ covers $A\times \mathrm{Cl}B$. On the other hand, by Lemma 2.6 $A\times \mathrm{Cl}B$ is C-embedded in $A\times Y$, and consequently by the assumption $A\times \mathrm{Cl}B$ is C^*-embedded in $X\times \mathrm{Cl}B$. Then by Lemma 2.1 and Theorem 2.5, there exists a locally finite cozero-set cover $\mathscr{U} = \{U_\lambda |\ \lambda\in\Lambda\}$ of X such that for a suitable collection $\{\mathscr{V}_{-\lambda}|\ \lambda\in\Lambda\}$ of open covers of $\mathrm{Cl}B$ $\{U_\lambda\cap A)\times V|\ V\in\mathscr{V}_\lambda,\ \lambda\in\Lambda\}$ refines $\{H_1, H_2\}$. Suppose that $U_\lambda\cap A\neq\emptyset$. Since B is infinite discrete and Y is compact, $\mathrm{Cl}B-B$ is non-empty. Let V be a set of $\mathscr{V}_\lambda$ with $(\mathrm{Cl}B-B)\cap V\neq\emptyset$. Then first we note that

$$(U_\lambda\cap A)\times V\subset H_2\,. \tag{1}$$

On the other hand, V contains some y_n of B, and so if x is any point of $U_\lambda\cap A$, we have by (1)

$$(x, y_n)\in U_j\times(\mathrm{Cl}B-\{y_i|\ i\leqslant j\})\quad \text{for some } j.$$

Consequently $j\leqslant n$, and $x\in\bigcup\limits_{i\leqslant n} U_i$. Therefore by Theorem 2.2 A is C-embedded in X, and this completes the proof.

§ **3.** Now we proceed to the proof of Theorems 1.3, 1.4 and 1.5.

Proof of Theorem 1.3. (a)→(b). This follows from Lemma 2.7.

(b)→(c). This is obvious.

(c)→(d). Since $D^{\mathfrak{m}}$ is closed in $I^{\mathfrak{m}}$ and $I^{\mathfrak{m}}$ is compact Hausdorff of weight $\mathfrak{m}$, this can be seen easily by Lemma 2.6.

(d)→(a). Our Theorem 1.5 mentioned in the introduction shows this implication. However, the following is a direct proof.

Suppose (d). Let $\{H_\alpha|\ \alpha\in\Omega\}$ be a locally finite cozero-set cover of A with $\mathrm{card}\,\Omega\leqslant\mathfrak{m}$. Here we may assume that $\mathrm{card}\,\Omega=\mathfrak{m}$. For each $\alpha\in\Omega$ let us put $Y_\alpha=D$, and construct the product space $Y=\prod\{Y_\alpha|\ \alpha\in\Omega\}$, which is homeomorphic to $D^{\mathfrak{m}}$; we denote by ϱ_α the projection from Y onto Y_α. Let us put

$$G_0=\bigcup\{H_\alpha\times\varrho_\alpha^{-1}(0)|\ \alpha\in\Omega\}\,,\qquad G_1=\bigcup\{H_\alpha\times\varrho_\alpha^{-1}(1)|\ \alpha\in\Omega\}\,.$$

Then $\{G_0, G_1\}$ is a binary cozero-set cover of $A\times D^{\mathfrak{m}}$ since

$$\{H_\alpha\times\varrho_\alpha^{-1}(i)|\ \alpha\in\Omega,\ i=1,2\}$$

is a locally finite cozero-set cover of $A\times D^{\mathfrak{m}}$. Since $A\times D^{\mathfrak{m}}$ is C^*-embedded in $X\times D^{\mathfrak{m}}$, by Lemma 2.1 and Theorem 2.5 there exists a locally finite cozero-set cover $\mathscr{U}=\{U_\lambda|\ \lambda\in\Lambda\}$ of X such that for a suitable collection $\{\mathscr{V}_\lambda|\ \lambda\in\Lambda\}$ of finite open covers of $D^{\mathfrak{m}}$ the collection $\{(U_\lambda\cap A)\times V|\ V\in\mathscr{V}_\lambda,\ \lambda\in\Lambda\}$ refines $\{G_0, G_1\}$. Here $\mathscr{V}_\lambda$ can be chosen so that for some finite subset $\{\alpha_1, \ldots, \alpha_n\}$ of Ω we have

$$\mathscr{V}_\lambda=\{\bigcap_{i=1}^{n}\varrho_{\alpha_i}^{-1}(k_i)|\ k_i=0 \text{ or } 1 \text{ for } i\leqslant n\}\,.$$

Suppose that

$$(U_\lambda\cap A)\times\bigcap_{i=1}^{n}\varrho_{\alpha_i}^{-1}(k_i)\subset G_0\,.$$

Pick a point y of $\bigcap\limits_{i=1}^{n}\varrho_{\alpha_i}^{-1}(k_i)$ such that $\varrho_\alpha(y)=1$ for $\alpha\notin\{\alpha_1, \ldots, \alpha_n\}$. If $x\notin\bigcup\{H_{\alpha_i}|\ i=1, \ldots, n\}$ then $(x, y)\notin G_0$. Hence we have

$$U_\lambda\cap A\subset\bigcup\{H_{\alpha_i}|\ i=1, \ldots, n\}\,.$$

On the other hand, as is easily seen A is C^*-embedded in X. Therefore by Theorem 2.2 A is $P^{\mathfrak{m}}$-embedded in X. This completes the proof.

COROLLARY 3.1. *A subspace A of a space X is C-embedded in X iff $A\times I$ is C^*-embedded in $X\times I$.*

The equivalence of (a) and (b) in Theorem 1.3 was proved in [1] for the case X is a completely regular Hausdorff space.

Let A be a subset in X as well as in Y, and let $\mathfrak{m}$ be the weight of Y. Since the weight of $A\leqslant\mathfrak{m}$, A is P-embedded in X iff A is $P^{\mathfrak{m}}$-embedded in X. Therefore Theorem 1.4 is an immediate consequence of Theorem 1.5.

Before proving Theorem 1.5 we need one more lemma. In the sequel γ denotes the initial ordinal number with $\mathrm{card}\,\mathfrak{m}$.

LEMMA 3.2. *Let Y be a completely regular Hausdorff space of weight $\mathfrak{m}$. Then for each $\alpha<\gamma$ there are subsets A_α, B_α, U_α and V_α of Y such that*

(a) A_α and B_α are zero-sets and $A_\alpha\subset U_\alpha$ and $B_\alpha\subset V_\alpha$,

(b) U_α and V_α are cozero-sets and disjoint,

(c) $\forall\beta<\alpha$, either $A_\alpha\not\subset U_\beta$ or $B_\alpha\not\subset V_\beta$.

In case it is only required that each of A_α and B_α be closed and each of U_α and V_α be open, Lemma 3.2 was proved by M. Starbird (cf. [12]), and his proof can be modified easily so as to yield our lemma.

Now we shall prove Theorem 1.5.

Proof of Theorem 1.5. Since the "only if" part follows readily from Theorem 1.3, we have only to prove the "if" part.

Suppose that Y is a compact Hausdorff space of weight $\mathfrak{m}$ and $A\times Y$ is C^*-embedded in $X\times Y$.

First let us note that A is C-embedded in X by Lemma 2.8. Let $\{G_\alpha|\ \alpha<\gamma\}$ be a discrete collection of open sets of A and $\{F_\alpha|\ \alpha<\gamma\}$ a collection of closed sets of A such that $\{G_\alpha, A-F_\alpha\}$ is a normal open cover of A for each $\alpha<\gamma$. Take a zero-set K_α and a cozero-set L_α of A so that

$$F_\alpha\subset K_\alpha\subset L_\alpha\subset G_\alpha\quad \text{for}\quad \alpha<\gamma\,.$$

Let A_α, B_α, U_α and V_α be the subsets of Y with the properties described in Lemma 3.2. Then, each $K_\alpha\times A_\alpha$ is a zero-set of $A\times Y$, $\{L_\alpha\times U_\alpha|\ \alpha<\gamma\}$ is a locally finite collection of cozero-sets of $A\times Y$ and $K_\alpha\times A_\alpha\subset L_\alpha\times U_\alpha$ for $\alpha<\gamma$. Hence the set

$$Z_1 = \bigcup\{K_\alpha\times A_\alpha|\ \alpha<\gamma\}$$

is a zero-set of $A\times Y$ by Lemma 2.3. Similarly the set

$$Z_2 = \bigcup\{K_\alpha\times(Y-U_\alpha)|\ \alpha<\gamma\}$$

is a zero-set of $A\times Y$, and it is easy to see that Z_1 and Z_2 are disjoint. By the same argument as above we also have that the sets

$$Z_3 = \bigcup\{K_\alpha\times B_\alpha|\ \alpha<\gamma\}$$

and

$$Z_4 = \bigcup\{K_\alpha\times(Y-V_\alpha)|\ \alpha<\gamma\}$$

are mutually disjoint zero-sets of $A\times Y$. Since $A\times Y$ is C^*-embedded in $X\times Y$, by Lemma 2.1 and Theorem 2.5 there exists a locally finite cozero-set cover $\mathscr{M} = \{M_\lambda|\ \lambda\in\Lambda\}$ of X with the property that for a suitable collection $\{\mathscr{N}_\lambda|\ \lambda\in\Lambda\}$ of open covers of Y $\{M_\lambda\times N|\ N\in\mathscr{N}_\lambda, \lambda\in\Lambda\}$ covers $X\times Y$ and that

$$\{(M_\lambda\cap A)\times N|\ N\in\mathscr{N}_\lambda, \lambda\in\Lambda\}$$

refines

$$\{A\times Y-Z_1, A\times Y-Z_2\}\quad\text{and}\quad\{A\times Y-Z_3, A\times Y-Z_4\}.$$

For a set M_λ of $\mathscr{M}$ we have

(1) $$M_\lambda\cap K_\alpha\neq\emptyset\quad\text{and}\quad M_\lambda\cap K_\beta\neq\emptyset\Rightarrow\alpha=\beta.$$

To prove this, suppose $\beta<\alpha$. Then by Lemma 3.2

$$\text{either}\quad A_\alpha-U_\beta\neq\emptyset\quad\text{or}\quad B_\alpha-V_\beta\neq\emptyset.$$

For example, let $A_\alpha-U_\beta\neq\emptyset$. Then there is a set N of the cover $\mathscr{N}_\lambda$ of Y with $(A_\alpha-U_\beta)\cap N\neq\emptyset$. We have then

$$((M_\lambda\cap A)\times N)\cap Z_1\supset((M_\lambda\cap A)\times N)\cap(K_\alpha\times A_\alpha)\neq\emptyset.$$

On the other hand, we have

$$((M_\lambda\cap A)\times N)\cap Z_2\supset((M_\lambda\cap A)\times N)\cap(K_\beta\times(Y-U_\beta))\neq\emptyset.$$

But this is a contradiction since $\{(M_\lambda\cap A)\times N|\ N\in\mathscr{N}_\lambda, \lambda\in\Lambda\}$ refines $\{A\times Y-Z_1, A\times Y-Z_2\}$. Thus (1) is proved.

Let us put

$$H_\alpha = \operatorname{St}(K_\alpha, \mathscr{M})\quad\text{for}\quad\alpha<\gamma;$$

then by (1) $\{H_\alpha|\ \alpha<\gamma\}$ is locally finite, and each H_α is a cozero-set of X containing K_α. Since A is clearly C^*-embedded in X, there is a cozero-set $\tilde{L}_\alpha$ of X such that $\tilde{L}_\alpha\cap A = L_\alpha$. Then $\{H_\alpha\cap\tilde{L}_\alpha|\ \alpha<\gamma\}$ is a locally finite collection of cozero-sets of X, and we have

$$F_\alpha\subset K_\alpha\subset H_\alpha\cap\tilde{L}_\alpha\cap A\subset L_\alpha\subset G_\alpha\quad\text{for}\quad\alpha<\gamma.$$

Therefore by Theorem 2.4 A is $P^{\mathfrak{m}}$-embedded in X. This completes the proof of Theorem 1.5.

As is known, a space X is said to be $\mathfrak{m}$-collectionwise normal if for every discrete collection $\{F_\alpha|\ \alpha\in\Omega\}$ of closed sets of X with $\operatorname{card}\Omega\leqslant\mathfrak{m}$ there exists a discrete collection $\{G_\alpha|\ \alpha\in\Omega\}$ of open sets of X such that $F_\alpha\subset G_\alpha$ for each $\alpha\in\Omega$. The following theorem can be obtained by putting together the several results in C. H. Dowker [2]. Here we shall give a direct proof.

THEOREM 3.3. *A space X is $\mathfrak{m}$-collectionwise normal iff every closed subset of X is $P^{\mathfrak{m}}$-embedded in X.*

Proof. The "only if" part follows from Theorem 2.4. To prove the "if" part, let $\{F_\alpha|\ \alpha\in\Omega\}$ be a discrete collection of closed sets of X with $\operatorname{card}\Omega\leqslant\mathfrak{m}$. Then $\{F_\alpha|\ \alpha\in\Omega\}$ is a locally finite cozero-set cover with $\operatorname{card}\leqslant\mathfrak{m}$ of the closed set $A = \bigcup\{F_\alpha|\ \alpha\in\Omega\}$. By assumption there is a locally finite cozero-set cover $\mathscr{U}$ of X such that $\mathscr{U}\cap A$ refines $\{F_\alpha|\ \alpha\in\Omega\}$. Since $\mathscr{U}$ is normal, there is a locally finite open cover $\mathscr{V}$ of X which is a star-refinement of $\mathscr{U}$. Let us put

$$G_\alpha = \operatorname{St}(F_\alpha, \mathscr{V})\quad\text{for}\quad\alpha\in\Omega.$$

Then it is easy to see that $\{G_\alpha|\ \alpha\in\Omega\}$ is a discrete collection of open sets such that $F_\alpha\subset G_\alpha$ for $\alpha\in\Omega$. Hence X is $\mathfrak{m}$-collectionwise normal, and this completes the proof.

Combining Theorem 1.5 with Theorem 3.3, we have readily the following theorem which was proved by M. Starbird [12] with a different method.

THEOREM 3.4. *Let Y be a compact Hausdorff space with weight $\mathfrak{m}$. Then a space X is $\mathfrak{m}$-collectionwise normal iff for every closed set A of X $A\times Y$ is C^*-embedded in $X\times Y$.*

Added in proof. In a letter (dated Mar. 29, 1975) to K. Morita, T. Przymusiński communicated, without proof, the equivalence of conditions (a) and (c) in our Theorem 1.3 and the validity of Theorem 1.4 in the case of Y being βX.

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