Since for $\lambda < 0$ we have

$$\|\lambda k_\lambda\|_1 = 1$$

(cf. e.g. [2]) an easy application of Hille–Yoshida theorem (cf. e.g. [2]) shows that

$$\lim_{n\to-\infty}\left\|\left(\frac{n}{t}k_{n/t}\right)^{*-n}*f-p_t*f\right\|_1 = 0 \tag{11}$$

for all f in $L_1(G, m)$. Therefore, since $\{p_t\}_{t\to 0}$ is a bounded approximate identity in $L_1(G, m)$,

$$\left\{\left(\frac{n}{t}k_{n/t}\right)^{*-n}\right\}_{t\to 0,\ n\to-\infty}$$

is an approximate identity in $L_1(G, m)$. Putting $f = k_\lambda$, $\lambda < 0$, in (11), we see that the real algebra generated by the k_λ's, $\lambda < 0$, is dense in the real algebra generated by p_t, $t > 0$. Thus we see that $\mathrm{Sp}_1\, p_t$ is real and so, since $p_t = p_{t/2}*p_{t/2}$, it is non-negative. From this we easily infer that for each f in A $\mathrm{Sp}_1 f^**f$ is real non-negative, which completes the proof of Theorem 2.

Since G is amenable, Proposition 5.3 of [2] thus yields our main result.

COROLLARY. *If L is the Laplacian on G defined by* (1), *then*

$$\mathrm{Sp}_p L = \mathrm{Sp}_2 L \quad \text{for all } 1 \leqslant p < \infty.$$

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On Kadec–Klee norms on Banach spaces

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Abstract. If E is a non-reflexive Banach space with a Kadec–Klee norm, then "many" (in particular, if E^* is separable, then all) proper total subspaces V of E^* have characteristic $r(V) < 1$. Application: Every non-reflexive Banach space E admits an equivalent norm for which there exists no projection of norm 1 of E^{**} onto $\varkappa(E)$.

0. Definition of Kadec–Klee norms. Terminology and notations.

In the present paper we shall study some properties and give some applications of Kadec–Klee norms, which are defined as follows:

DEFINITION 0.1. Let E be a Banach space and W a separable subspace (by *subspace* we shall always mean: norm-closed linear subspace) of the conjugate space E^*. We shall say that the norm of E is a *Kadec–Klee norm* (or, briefly, a (KK)-*norm*) *with respect to* W if for every net $\{g_d\}_{d\in D} \subset E^*$ and every $g \in W$ such that $g_d \xrightarrow{w^*} g$ and $\|g_d\| \to \|g\|$ we have $\|g_d - g\| \to 0$.

In the particular case when E^* is separable, a (KK)-norm with respect to $W = E^*$ will be simply called a (KK)-*norm* (in this case, clearly, the above nets can be replaced by sequences).

M. I. Kadec [5] and V. Klee [7] have proved that every Banach space E with separable conjugate space admits an equivalent (KK)-norm (for other proofs see also [6], [9]). More generally, W. J. Davis and W. B. Johnson have proved ([2], lemma 1) that if E is a Banach space and W a separable subspace of E^*, then E admits a (KK)-norm with respect to W, equivalent to the initial norm (actually, their result is slightly stronger, but we shall use only this version of it).

We recall (see [3]) that the *characteristic* of a subspace V of a conjugate Banach space E^* is the greatest number $r = r(V)$ such that the

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unit cell $S_V = \{f \in V \mid \|f\| \leqslant 1\}$ of V is $\sigma(E^*, E)$-dense in the r-cell $rS_{E^*} = \{f \in E^* \mid \|f\| \leqslant r\}$ of E^* (clearly, $0 \leqslant r(V) \leqslant 1$) and that we have

$$(0.1)\qquad r(V) = \frac{1}{\sup\limits_{x \in \Sigma_E} \|x\|} = \inf_{\substack{x \in E \\ x \neq 0}} \sup_{\substack{f \in V \\ \|f\| \leqslant 1}} \left| f\left(\frac{x}{\|x\|}\right)\right| = \inf_{\substack{x \in E, \|x\|=1 \\ \Phi \in V^\perp}} \|\varkappa(x) - \Phi\|,$$

where Σ_E is the closure of the unit cell $S_E = \{x \in E \mid \|x\| \leqslant 1\}$ for the weak topology $\sigma(E, V)$, $\varkappa$ is the canonical embedding of E into E^{**} and $V^\perp = \{\Phi \in E^{**} \mid \Phi(f) = O(f \in V)\}$. It is immediate (see e.g. [11], the footnote on p. 239) that the last number in formula (0.1) is equal to $1/\|p\|$, where p is the projection of $\varkappa(E) \oplus V^\perp$ onto $\varkappa(E)$ along $V^\perp$.

Throughout this paper, the terminology will be the usual one (see e.g. [11]). If $\{x_n\}$ is a basis of a Banach space E and $\{f_n\} \subset E^*$, $f_i(x_j) = \delta_{ij}$ $(i, j = 1, 2, \ldots)$, then the norm closed linear span $[f_n]$ of $\{f_n\}$ in E^* will be called the *coefficient subspace* for the basis $\{x_n\}$ (since $x = \sum_{i=1}^{\infty} f_i(x) x_i$ for all $x \in E$). Also, we recall that a basis $\{x_n\}$ is called *asymptotically monotone* if for the associated partial sum operators we have $\|s_n\| \leqslant 1 + \varepsilon_n$ $(n = 1, 2, \ldots)$, where $\varepsilon_n > 0$ $(n = 1, 2, \ldots)$, $\lim\limits_{n \to \infty} \varepsilon_n = 0$.

Finally, if $|||\cdot|||$ is another norm on E and u an operator on E, we shall denote $|||u||| = \sup\limits_{\substack{x \in E \\ |||x||| \leqslant 1}} |||u(x)|||$ and we shall use the notation $r_{|||\cdot|||}(V)$ (where $V \subset E^*$) in its obvious sense.

1. On characteristics of subspaces of conjugate Banach spaces. Our first theorem shows the influence of (KK)-norms on the characteristics of subspaces of conjugate spaces.

THEOREM 1.1. *Let E be a Banach space and W a separable subspace of E^* such that the norm of E is a* (KK)*-norm with respect to W. Then for every subspace V of E^* which does not contain W, we have $r(V) < 1$.*

Proof. Assume, a contrario, that V is a (norm closed linear) subspace of E^* with $V \not\supset W$, $r(V) = 1$. Let $g \in W \setminus V$. Since $r(V) = 1$, there exists a net $\{g_d\}_{d \in \mathscr{D}} \subset V$ with $\|g_d\| \leqslant \|g\|$ for all $d \in \mathscr{D}$, such that $g_d \xrightarrow{w^*} g$. Then

$$\|g\| \leqslant \underline{\lim}\, \|g_d\| \leqslant \overline{\lim}\, \|g_d\| \leqslant \|g\|,$$

whence $\|g_d\| \to \|g\|$. Consequently, since the norm is a (KK)-norm with respect to W, $\|g_d - g\| \to 0$, whence $g \in V$, in contradiction with our choice of g. This completes the proof.

Remark 1.1. (a) The same argument also shows that under the above assumptions, if $g \in W \setminus V$, $\|g\| = 1$, then $g \notin \tilde{S}_V$, the $\sigma(E^*, E)$-closure of $S_V = \{f \in V \mid \|f\| \leqslant 1\}$.

(b) From Theorem 1.1 it follows that, at least when E is separable, the obvious extension of the notion of (KK)-norm for non-separable subspaces W of E^* is void. Indeed, it is well known (see e.g. [6], Lemma 4) that for every separable Banach space E there exists a separable subspace V of E^* with $r(V) = 1$. Clearly, if $W \subset E^*$ is non-separable, then $V \not\supset W$, whence, if the norm of E is a (KK)-norm with respect to W, then, by Theorem 1.1, $r(V) < 1$, a contradiction.

COROLLARY 1.1. *Let E be a non-reflexive Banach space with separable conjugate space. Then there exists an equivalent norm on E, which is not a* (KK)*-norm.*

Proof. Let $\Phi \in E^{**} \setminus \varkappa(E)$ and let $V = \operatorname{Ker} \Phi (\subsetneqq E^*)$. Then $V^\perp = \langle \Phi \rangle$, the one-dimensional subspace of E^{**} spanned by Φ, whence $\varkappa(E) \oplus V^\perp$ is closed in E^{**} and hence [3] $r(V) > 0$. Put

$$(1.1)\qquad \|x\|_1 = \sup_{\substack{f \in V \\ \|f\| \leqslant 1}} |f(x)| \qquad (x \in E).$$

Then $\|\cdot\|_1$ is an equivalent norm on E such that $r_{\|\cdot\|_1}(V) = 1$ [3] and hence, by Theorem 1.1 (with $W = E^*$), $\|\cdot\|_1$ is not a (KK)-norm on E, which completes the proof.

Combining Theorem 1.1 with the result of Davis and Johnson mentioned in the Introduction, we obtain

THEOREM 1.2. *Let E be a Banach space and W a separable subspace of E^*. Then there exists a norm $|||\cdot|||$ on E, equivalent to the initial norm of E, such that in this new norm for every subspace V of E^* which does not contain W, we have $r_{|||\cdot|||}(V) < 1$. In fact, any equivalent* (KK)*-norm with respect to W has this property.*

Remark 1.2. One can also give the following simpler direct proof of the first statement of Theorem 1.2: Since W is separable, let $\{f_n\} \subset W$, $[f_n] = W$. Put

$$(1.2)\qquad |||f||| = \|f\| + \sum_{n=1}^{\infty} \frac{1}{2^n} \operatorname{dist}(f, \langle f_n \rangle) \qquad (f \in E^*),$$

where, for each n, $\langle f_n \rangle$ denotes the 1-dimensional subspace of E^* spanned by f_n. It is readily seen (for a similar argument see [2], proof of Lemma 1) that $|||\cdot|||$ is the dual norm for some norm $|||\cdot|||$ on E equivalent to the initial norm of E. Assume now that V is a subspace of E^* with $V \not\supset W = [f_n]$ and with $r_{|||\cdot|||}(V) = 1$. Let $f_{n_0} \in W \setminus V$, hence

$$\inf_{\substack{f \in V, \|f\|=1 \\ \alpha = \text{scalar}}} \|f - \alpha f_{n_0}\| = \lambda > 0.$$

Since $r_{|||\cdot|||}(V) = 1$, there exists a net $\{g_d\}_{d \in \mathscr{D}} \subset V$ with $|||g_d||| \leqslant 1$, such that

$$g_d \xrightarrow{w^*} \frac{f_{n_0}}{|||f_{n_0}|||}.$$

Then $1 \leqslant \underline{\lim}\, |||g_d||| \leqslant \overline{\lim}\, |||g_d||| \leqslant 1$, whence $|||g_d||| \to 1$, and thus we may assume (considering $g_d/|||g_d|||$) that $|||g_d||| = 1$ for all $d \in \mathscr{D}$. Define now the auxiliary norm

$$(1.3)\qquad |||f|||_0 = \|f\| + \sum_{\substack{n=1\\ n\neq n_0}}^{\infty}{}' \frac{1}{2^n}\operatorname{dist}(f, \langle f_n\rangle) = |||f||| - \frac{1}{2^{n_0}}\operatorname{dist}(f, \langle f_{n_0}\rangle) \qquad (f \in E^*),$$

which is clearly also the dual of some norm on E. Then, since $g_d \in V$, we have

$$(1.4)\qquad |||g_d|||_0 = 1 - \frac{\|g_d\|}{2^{n_0}}\operatorname{dist}\left(\frac{g_d}{\|g_d\|}, \langle f_{n_0}\rangle\right) \leqslant 1 - \frac{\lambda\|g_d\|}{2^{n_0}} \qquad (d \in \mathscr{D}).$$

On the other hand,

$$\left|\left|\left|\frac{f_{n_0}}{|||f_{n_0}|||}\right|\right|\right|_0 = 1 - \frac{1}{2^{n_0}}\operatorname{dist}\left(\frac{f_{n_0}}{|||f_{n_0}|||}, \langle f_{n_0}\rangle\right) = 1,$$

whence, since $g_d \xrightarrow{w^*} f_{n_0}/|||f_{n_0}|||$ and since $|||\cdot|||_0$ is a dual norm, it follows that $1 \leqslant \underline{\lim}\, |||g_d|||_0$, in contradiction with (1.4), which completes the proof. This proof of Theorem 1.2 does not use the (KK)-property, but one can show that (1.2) is also a (KK)-norm with respect to W (actually, the proof of the latter is somewhat simpler than that of the fact that the norm constructed in [2], proof of Lemma 1, is a (KK)-norm with respect to W).

Let us mention separately, because of its importance, the particular case $W = E^*$ of Theorem 1.2:

COROLLARY 1.2. *Let E be a Banach space with separable conjugate space. Then there exists a norm on E equivalent to the initial norm, such that in this new norm for every proper subspace V of E^* we have $r(V) < 1$. In fact, any equivalent* (KK)*-norm on E has this property.*

Remark 1.3. For any $\varepsilon > 0$ one can choose the above equivalent (KK)-norms so that they satisfy, in addition,

$$(1.5)\qquad |||x||| \leqslant \|x\| \leqslant (1+\varepsilon)|||x||| \qquad (x \in E).$$

Indeed, it is enough to replace in (1.2) (or in [2], proof of Lemma 1) $1/2^n$ by $\varepsilon/2^n$.

In view of the above, let us raise

PROBLEM 1.1. (a) Let E be a non-reflexive Banach space with separable conjugate space. Does there exist an equivalent norm $|||\cdot|||$ on E such that for some constant $\alpha < 1$ (depending perhaps on the space E) we have $r_{|||\cdot|||}(V) \leqslant \alpha$ for all proper subspaces V of E^*?

(b) More generally (if E^* is not necessarily separable), one can ask the similar question for all subspaces V of E^* such that $V \not\supset W$, where W is a given separable subspace of E^*.

If for some E the answer to Problem 1.1 (a) is affirmative, then clearly for any equivalent norm $|||\cdot|||_1$ which is "sufficiently near" to $|||\cdot|||$ (in the sense of (1.5)), there exists a constant $\alpha_1 < 1$ such that $r_{|||\cdot|||_1}(V) \leqslant \alpha_1$ for all proper subspaces V of E^*. Let us also note that if for some $(E, |||\cdot|||)$ there exists an $\alpha < 1$ as in Problem 1.1 (a), then we must have $\alpha \geqslant 1/2$, as shown by

PROPOSITION 1.1. *Let E be a non-reflexive Banach space and let $0 < \varepsilon < \frac{1}{2}$. Then there exists a hyperplane V in E^* with $r(V) > \frac{1}{2} - \varepsilon$.*

Proof. Let $\Phi \in E^{**} \setminus \varkappa(E)$. Since $\varkappa(E)$ is a hyperplane in $\varkappa(E) \oplus \langle\Phi\rangle$, there exists (see e.g. [1]) a projection p of $\varkappa(E)\oplus\langle\Phi\rangle$ onto $\varkappa(E)$ with $\|p\| < 2/(1-2\varepsilon)$. Choose $\Psi \in \operatorname{Ker} p$ and let $V = \operatorname{Ker}\Psi \subset E^*$. Then $V^\perp = \langle\Psi\rangle$, whence, by the remark following (0.1),

$$r(V) = \frac{1}{\|p\|} > \frac{1-2\varepsilon}{2} = \frac{1}{2} - \varepsilon,$$

which completes the proof.

In connection with Problem 1.1 and Proposition 1.1, let us recall [3] that if $E = c_0$ (with its usual norm!), we have $r(V) \leqslant \frac{1}{2}$ for all proper subspaces V of $E^* \equiv l^1$.

In every Banach space E there exists a basic sequence $\{x_n\}$ such that $r([\varphi_n]) = 1$, where $[\varphi_n] \subset [x_n]^*$ is the coefficient subspace for $\{x_n\}$; indeed, it is well known that in E there exists an asymptotically monotone basic sequence $\{x_n\}$ and then, by the argument of the proof of [10], Theorem 1 or [11], p. 116, $r([\varphi_n]) = 1$. It is natural to ask whether for Banach spaces E with a basis one can replace in the above result "basic sequence" by "basis". The following theorem shows that the answer is negative:

THEOREM 1.3. *For every Banach space E with a basis and with separable conjugate space E^* having no basis, there exists an equivalent norm $|||\cdot|||$ on E such that for every basis $\{x_n\}$ of E we have $r_{|||\cdot|||}([f_n]) < 1$, where $[f_n]$ is the coefficient subspace for $\{x_n\}$.*

Proof. Note first that such spaces E exist, by the results of P. Enflo [4] and J. Lindenstrauss ([8], Corollary 3 and the remark preceding Corollary 4). Now, if $\{x_n\}$ is a basis of such a space E, with coefficient functionals $\{f_n\}$, then (e.g. by [11], p. 112, Theorem 12.1) $\{f_n\}$ is a basis of $[f_n]$ and hence $[f_n] \neq E^*$. Consequently, by Corollary 1.2, for any equivalent (KK)-norm $|||\cdot|||$ on E we have $r_{|||\cdot|||}([f_n]) < 1$, which completes the proof.

2. An application to projections of E^{**} onto $\varkappa(E)$. The following theorem solves in the affirmative a problem raised by W. J. Davis and W. B. Johnson [2].

THEOREM 2.1. *Let E be a non-reflexive Banach space. Then there exists a norm $|||\cdot|||$ on E, equivalent to the initial norm on E, such that there exists no projection p of norm $|||p|||=1$ of E^{**} onto $\varkappa(E)$.*

Proof. Since E is non-reflexive, by [2], Lemma 2 there exists a separable subspace W of E^* such that $E^{**}\neq\varkappa(E)+W^{\perp}$. We shall show that any norm $|||\cdot|||$ as in § 1, Theorem 1.2, with respect to this W, has the required property. Assume, a contrario, that p is a projection of E^{**} onto $\varkappa(E)$ with $|||p|||=1$. Since $E^{**}\neq\varkappa(E)+W^{\perp}$, there exists a $\Phi\in\operatorname{Ker}p\setminus W^{\perp}$. Then for $V=\operatorname{Ker}\Phi$ we have $V^{\perp}=\langle\Phi\rangle$, whence, by the remark following (0.1),

$$r_{|||\cdot|||}(V)=\frac{1}{|||p|_{\varkappa(E)\oplus\langle\Phi\rangle}|||}=1. \tag{2.1}$$

On the other hand, we have $V\not\supset W$ (since otherwise $W^{\perp}\supset V^{\perp}=\langle\Phi\rangle\ni\Phi$), whence, by § 1, Theorem 1.2, $r_{|||\cdot|||}(V)<1$, in contradiction with (2.1). This completes the proof.

Remark 2.1. If $(E,|||\cdot|||)$ is such that there exists no projection p of norm $|||p|||=1$ of E^{**} onto $\varkappa(E)$, then E is not isometric to any conjugate Banach space (see e.g. [3]). Thus from Theorem 2.1 we obtain again the result of W. J. Davis and W. B. Johnson ([2], Theorem) that every non-reflexive Banach space admits an equivalent norm under which it fails to be isometric to a conjugate Banach space.

PROBLEM 2.1. Let E be a non-reflexive Banach space. Does there exist an equivalent norm $|||\cdot|||$ on E such that for some constant $C>1$ (depending perhaps on the space E) there exists no projection p of norm $|||p|||\leqslant C$ of E^{**} onto $\varkappa(E)$?

The above proof of Theorem 2.1 shows that an affirmative answer to § 1, Problem 1.1 (b), would imply an affirmative answer to Problem 2.1.

In connection with Problem 2.1 one might ask whether for any $C>1$ every non-reflexive Banach space admits an equivalent norm $|||\cdot|||$ under which there exists no projection of norm $\leqslant C$ of E^{**} onto $\varkappa(E)$. However, the answer is negative, e.g. for any equivalent norm $|||\cdot|||$ on a quasi-reflexive space E of order 1 (i.e., with $\dim E^{**}/\varkappa(E)=1$) and for any $\varepsilon>0$ there exists a projection p of E^{**} onto $\varkappa(E)$ of norm $|||p|||<2+\varepsilon$. This also shows that if there exists a $C>1$ as in Problem 2.1, and independent of the space E, then we must have $C\leqslant 2$.

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