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Weighted bounded mean oscillation and the Hilbert transform*

by

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Abstract. A weighted analogue of the class of functions of bounded mean oscillation of John and Nirenberg is defined. The definition is motivated by the behavior of the Hilbert transform of any function which is bounded by a multiple of the weight function. Estimates for the distribution of values of functions in the class are given.

§ 1. Introduction. In this paper, we consider a few results related o a weighted version of the class of functions of bounded mean oscillation of F. John and L. Nirenberg [6]. More specifically, we will study the distribution of values of functions in this class and give an analogue of Stein's result [10] that the Hilbert transform of a bounded function is of bounded mean oscillation. Our version of this last result is a natural extension of the weighted L^p norm inequalities for the Hilbert transform given in [5]. Let $f(x)$ and $w(x)$ be locally integrable in $\boldsymbol{R}^n$ and let $w \geqslant 0$. We then say that f is of *bounded mean oscillation with weight* w if there is a constant c such that

$$\int_I |f(x)-f_I|\,dx \leqslant c\int_I w(x)\,dx \tag{1.1}$$

for all n-dimensional "cubes" I whose edges are parallel to the coordinate axes. Here, f_I is a constant depending on I which we will show can always be chosen as the mean value of f on I:

$$f_I = \frac{1}{|I|}\int_I f(x)\,dx.$$

In fact, for any constant α,

$$\int_I |f(x)-f_I|\,dx \leqslant \int_I |f(x)-\alpha|\,dx + |I|\,|\alpha - f_I|,$$

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and since

$$|I|\,|a-f_I| = \Big|\int_I [f(x)-a]\,dx\Big| \leqslant \int_I |f(x)-a|\,dx,$$

we have

$$\int_I |f(x)-f_I|\,dx \leqslant 2\int_I |f(x)-a|\,dx.$$

In case (1.1) only holds for those I lying in some fixed cube I_0, we say that f is of *bounded mean oscillation on* I_0 *with weight* w. We will often use the same c to denote positive constants which may be different at each occurrence.

The case $w\equiv 1$ of (1.1) corresponds to that of John and Nirenberg. Other weighted definitions of this class might be given: for example,

$$\int_I |f(x)-c_I|\,w(x)\,dx \leqslant c\int_I w(x)\,dx \tag{1.2}$$

$$\text{where}\quad c_I = \int_I f(x)w(x)\,dx\Big/\int_I w(x)\,dx,$$

for all I. We will show in § 3 (Theorem 5) that if w satisfies a mild condition, the class of f for which (1.2) is true is identical to the class of John and Nirenberg. Our definition (1.1) is motivated by results about singular and fractional integrals. In order to state some of these results, we will need several definitions. First, a weight w is said *to belong to* A_p, $1<p<\infty$, if there is a constant c such that

$$\left(\frac{1}{|I|}\int_I w(x)\,dx\right)\left(\frac{1}{|I|}\int_I w(x)^{-1/(p-1)}\,dx\right)^{p-1} \leqslant c$$

for all I; and is said *to belong to* A_1 if there is a constant c such that $w^*(x)\leqslant cw(x)$ for almost all x, where w^* is the Hardy–Littlewood maximal function of w. (See [7].) The condition that $w\in A_1$ is the same as requiring that

$$\frac{1}{|I|}\int_I w(x)\,dx \leqslant c\,\operatorname{ess\,inf}_I w$$

for all I. On the other hand, w is said *to belong to* A_∞ if there exist α and β such that $0<\alpha,\ \beta<1$ and for every cube I and every measurable subset E of I, $\int_E w\,dx < \beta\int_I w\,dx$ whenever $|E|<\alpha|I|$. The statement that $w\in A_\infty$ is equivalent to either of the following two: (1) $w\in A_p$ for some p; (2) there are positive constants c_1, c_2, δ and η such that

$$c_1\left(\frac{|E|}{|I|}\right)^{\eta} \leqslant \int_E w\,dx\Big/\int_I w\,dx \leqslant c_2\left(\frac{|E|}{|I|}\right)^{\delta} \tag{1.3}$$

for every $E\subset I$. (See [2], [8] and [4].) If I is a cube and c is a positive constant, let cI denote the cube concentric with I which is c times as long. It follows from (1.3) that if $w\in A_\infty$ and $c>0$, there are constants $c_1, c_2>0$ such that

$$c_1 \leqslant \int_{cI} w\,dx\Big/\int_I w\,dx \leqslant c_2 \quad \text{for all } I.$$

Finally, we say that w *belongs to* B_2 if there is a constant c such that

$$|I|\int_{t\notin I}\frac{w(t)}{|x_I-t|^{2n}}\,dt \leqslant c\frac{1}{|I|}\int_I w(t)\,dt$$

for all I, where x_I is the center of I. It is known that if $w\in A_p$, $1\leqslant p\leqslant 2$, then $w\in B_2$. (For the case $n=1$, see [5] Lemma 1; the argument for $n>1$ is similar.)

For our results about singular integrals, we consider only the case $n=1$, and let $\tilde f$ denote the Hilbert transform of f. It will be convenient to use a modified version of $\tilde f$: let

$$Hf(x) = \lim_{\varepsilon\to 0+}\int_{|x-y|>\varepsilon}\left[\frac{1}{x-y}+\frac{\chi(y)}{y}\right]f(y)\,dy,$$

where $\chi(y)$ is the characteristic function of $|y|>1$. The reason for using Hf is that it may exist while $\tilde f$ may not. If $\tilde f$ and Hf both exist, as would be the case if, for example, $f\in L^p$, $1\leqslant p<\infty$, then they differ by a constant.

In the following theorem, $\|f\|_\infty$ denotes the usual L^∞ norm of f.

THEOREM 1. *Let w be non-negative and locally integrable. Then a necessary and sufficient condition that there exist a constant c such that*

$$\int_I |Hf-(Hf)_I|\,dx \leqslant c\,\|f/w\|_\infty\int_I w\,dx \tag{1.4}$$

for all intervals I and all f for which f/w is bounded is that $w\in A_\infty\cap B_2$.

In particular, if $w\in A_\infty\cap B_2$ and $|f|\leqslant cw$ a.e., then Hf exists and is of bounded mean oscillation with weight w. In case $w\in A_p$, $1<p<\infty$, the factor $\int_I w\,dx$ on the right-hand side may be replaced by

$$|I|\left(\frac{1}{|I|}\int_I w^{-1/(p-1)}\,dx\right)^{-(p-1)},$$

while if $w\in A_1$, it is equivalent to

$$|I|\operatorname{ess\,inf}_I w = |I|\left(\operatorname{ess\,sup}_I \frac{1}{w}\right)^{-1}.$$

THEOREM 2. *Let w be non-negative and locally integrable.*

(i) *A necessary and sufficient condition that there exists a constant c such that*

$$(1.5)\qquad \Big(\operatorname*{ess\,sup}_I \frac{1}{w}\Big)\frac{1}{|I|}\int_I |Hf-(Hf)_I|\,dx \leqslant c\|f/w\|_\infty$$

for all I and all f for which f/w is bounded is that $w \in A_1$.

(ii) *Let $1<p<\infty$. A necessary and sufficient condition that there exist a constant c such that*

$$(1.6)\qquad \Big(\frac{1}{|I|}\int_I w^{-1/(p-1)}\,dx\Big)^{p-1}\frac{1}{|I|}\int_I |Hf-(Hf)_I|\,dx \leqslant c\|f/w\|_\infty$$

for all I and all f for which f/w is bounded is that $w\in A_p \cap B_2$.

The first part of Theorem 2 is a natural extension of the weighted L^p norm inequalities for $\tilde f$ and is an analogue of a result (Theorem 7) from [9] for fractional integrals. For $1<p<\infty$, the weighted L^p norm inequality for $\tilde f$ is

$$\Big(\int_{-\infty}^{\infty} |\tilde f|^p w\,dx\Big)^{1/p} \leqslant c\Big(\int_{-\infty}^{\infty}|f|^p w\,dx\Big)^{1/p}$$

if $w\in A_p$. This may be rewritten as

$$\Big(\int_{-\infty}^{\infty} |\tilde f w^{-1}|^p\,dx\Big)^{1/p} \leqslant c\Big(\int_{-\infty}^{\infty}|f w^{-1}|^p\,dx\Big)^{1/p} \quad \text{if} \quad w^{-p}\in A_p;$$

that is, if

$$\Big(\frac{1}{|I|}\int_I w^{-p}\,dx\Big)^{1/p}\Big(\frac{1}{|I|}\int_I w^{p'}\,dx\Big)^{1/p'} \leqslant c,$$

where $1/p+1/p' = 1$. As $p\to\infty$,

$$\Big(\frac{1}{|I|}\int_I w^{-p}\,dx\Big)^{1/p} \to \operatorname*{ess\,sup}_I \frac{1}{w},$$

so the natural condition is

$$\Big(\operatorname*{ess\,sup}_I \frac{1}{w}\Big)\Big(\frac{1}{|I|}\int_I w\,dx\Big) \leqslant c.$$

This is exactly the requirement that $w\in A_1$.

An important feature of functions in the class of John and Nirenberg is the exponential nature of their distribution functions. There is an analogue of this property for functions of weighted bounded mean oscillation which depends on the A_p class to which the weight belongs. In stating this result, we will use the notation $m_w(E) = \int_E w\,dx$ for the w-measure of a set E, and $m_w(I\colon g>a)$ for $m_w(x\in I\colon g(x)>a)$.

THEOREM 3. *Let f be of bounded mean oscillation with weight w on I_0:*

$$\int_I |f-f_I|\,dx \leqslant c\int_I w\,dx;\qquad f_I = \frac{1}{|I|}\int_I f\,dx,\qquad I\subset I_0.$$

If $w\in A_1$, there are positive constants c_1 and c_2 such that

$$(1.7)\qquad m_w(I\colon |f-f_I|w^{-1}>a) \leqslant c_1 e^{-c_2 a} m_w(I)$$

for $a>0$ and $I\subset I_0$. If $w\in A_p$, $1<p<\infty$, and $1/p+1/p'=1$, there is a constant c_3 such that

$$(1.8)\qquad m_w(I\colon |f-f_I|w^{-1}>a) \leqslant c_3(1+a)^{-p'} m_w(I)$$

for $a>0$ and $I\subset I_0$.

As a simple corollary of Theorem 3, we will derive the next result.

THEOREM 4. *Let $1\leqslant p<\infty$ and $w\in A_p$. Then f is of bounded mean oscillation on I_0 with weight w if and only if there is a constant c such that*

$$(1.9)\qquad \int_I |f-f_I|^r w^{1-r}\,dx \leqslant c\int_I w\,dx$$

for all $I\subset I_0$ and every r which satisfies $1\leqslant r\leqslant p'$ and $r<\infty$.

Actually, if (1.9) holds for any w and any r, $1<r<\infty$, it follows easily from Hölder's inequality that f is of weighted bounded mean oscillation. The opposite implication is the important part of the theorem. For this, we will see that given $w\in A_p$ and f satisfying (1.1), there are values of r slightly larger than p' for which (1.9) is true. However, we will show that given p, no r larger than p' can be used in (1.9) for all f and w which satisfy (1.1) and $w\in A_p$. Moreover, given $p>1$, there exist f and w for which (1.1) holds and $w\in A_q$ for all $q>p$, but for which (1.9) fails when $r=p'$.

The results dealing with the Hilbert transform are proved in § 2 and those dealing with the distribution of $|f-f_I|$ in § 3. At the end of § 3, we show that the class of functions satisfying (1.2) is identical to that of John and Nirenberg. We would like to point out that some of these results and others have been obtained independently by J. G. Cuerva [3] using duality arguments.

§ 2. Proofs of Theorems 1 and 2. We will prove the sufficiency of Theorem 1 first. Thus, let $w\in A_\infty\cap B_2$ and suppose that there is a constant M such that $|f|\leqslant Mw$ a.e. Since $w\in B_2$, $w(y)|y|^{-2}$ is integrable over $|y|>1$. Therefore, $f(y)|y|^{-2}$ is integrable over $|y|>1$, and $\int_{|x-y|>\varepsilon} [1/(x-y)+$

$+\chi(y)/y]f(y)\,dy$ converges absolutely at infinity for any $\varepsilon > 0$. Since $w \in A_\infty$, there exist $p_0 > 1$ and a constant c such that

$$\left(\frac{1}{|I|}\int_I w^{p_0}\,dx\right)^{1/p_0} \leqslant c\frac{1}{|I|}\int_I w\,dx \tag{2.1}$$

for all I (see [7]). In particular, f is locally in L^p for $1 \leqslant p \leqslant p_0$. It follows that Hf exists a.e. To show that Hf satisfies (1.4), fix I and write $g = f\chi_{2I}$ and $h = f\chi_{(2I)^c}$, so that $f = g+h$ and $Hf = Hg+Hh$. Since $g \in L^{p_0}$ for some $p_0 > 1$, standard results about the Hilbert transform show that both Hg and $\tilde{g}$ are locally integrable. Since Hg and $\tilde{g}$ differ by a constant,

$$\int_I |Hg-(Hg)_I|\,dx = \int_I |\tilde{g}-(\tilde{g})_I|\,dx \leqslant 2\int_I |\tilde{g}|\,dx. \tag{2.2}$$

By Hölder's inequality and the norm inequality for the Hilbert transform,

$$\int_I |\tilde{g}|\,dx \leqslant |I|\left(\frac{1}{|I|}\int_I |\tilde{g}|^{p_0}\,dx\right)^{1/p_0} \leqslant c|I|\left(\frac{1}{|I|}\int_{-\infty}^{\infty} |g|^{p_0}\,dx\right)^{1/p_0}$$
$$= c|I|\left(\frac{1}{|I|}\int_{2I} |f|^{p_0}\,dx\right)^{1/p_0}.$$

Since $|f| \leqslant Mw$ a.e., it follows from (2.1) that

$$\int_I |\tilde{g}|\,dx \leqslant cM\int_{2I} w\,dx.$$

Since w satisfies the doubling condition $\int_{2I} w\,dx \leqslant c\int_I w\,dx$ (see (1.3)), we obtain by combining this estimate with (2.2) that

$$\int_I |Hg-(Hg)_I|\,dx \leqslant cM\int_I w\,dx. \tag{2.3}$$

To prove an analogous result for Hh, let x_0 be the center of I. Then

$$\int_I |Hh(x)-Hh(x_0)|\,dx = \int_I \left|\int_{(2I)^c}\left[\frac{1}{x-y}-\frac{1}{x_0-y}\right]f(y)\,dy\right|dx$$
$$\leqslant \int_{(2I)^c} |f(y)|\left(\int_I \left|\frac{1}{x-y}-\frac{1}{x_0-y}\right|dx\right)dy.$$

Since $|f| \leqslant Mw$ a.e. and, for $y \in (2I)^c$,

$$\int_I \left|\frac{1}{x-y}-\frac{1}{x_0-y}\right|dx \leqslant c|I|^2/(x_0-y)^2,$$

the last integral is majorized by a constant times

$$M|I|^2\int_{I^c}\frac{w(y)}{(x_0-y)^2}\,dy.$$

Combining inequalities and using condition B_2 gives

$$\int_I |Hh(x)-Hh(x_0)|\,dx \leqslant cM\int_I w\,dx.$$

Therefore, by the argument following (1.1),

$$\int_I |Hh(x)-(Hh)_I|\,dx \leqslant cM\int_I w\,dx. \tag{2.4}$$

It follows immediately from (2.3) and (2.4) that Hf satisfies (1.4).

We shall now prove the necessity of the condition in Theorem 1. Thus, let w be non-negative and locally integrable and assume that whenever $f/w \in L^\infty$, Hf exists and satisfies

$$\int_I |Hf-(Hf)_I|\,dx \leqslant c\|f/w\|_\infty\int_I w\,dx. \tag{2.5}$$

To show that $w \in A_\infty$, let $w_1 = w$ on the left half of I and $w_1 = 0$ elsewhere, and let $w_2 = w$ on the right half of I and $w_2 = 0$ elsewhere. Letting $f = w\chi_I$, we have $f = w_1+w_2$ and $Hf = Hw_1+Hw_2$. Consider Hw_1. For x in the right quarter RI of I, $|\tilde{w}_1(x)| \leqslant \frac{c}{|I|}\int_I w\,dx$ by a simple estimate. Therefore, since by hypothesis

$$\int_I |\tilde{w}_1-(\tilde{w}_1)_I|\,dx = \int_I |Hw_1-(Hw_1)_I|\,dx \leqslant c\int_I w\,dx, \tag{2.6}$$

we obtain by extending the integration in the integral on the left side of (2.6) only over RI that

$$\int_{RI} |(\tilde{w}_1)_I|\,dx \leqslant c\int_I w\,dx.$$

Thus, $|(\tilde{w}_1)_I| \leqslant c\frac{1}{|I|}\int_I w\,dx$, and by a similar argument, $|(\tilde{w}_2)_I| \leqslant c\frac{1}{|I|}\int_I w\,dx$. It follows that

$$|(\tilde{f})_I| = |(\tilde{w}_1)_I+(\tilde{w}_2)_I| \leqslant c\frac{1}{|I|}\int_I w\,dx.$$

Since by hypothesis

$$\int_I |\tilde{f}-(\tilde{f})_I|\,dx = \int_I |Hf-(Hf)_I|\,dx \leqslant c\int_I w\,dx,$$

we obtain

$$\int_I |\tilde{f}|\,dx \leqslant c\int_I w\,dx = c\int_I f\,dx.$$

Since $f\geqslant 0$, it follows from [11], Vol. 1, (2.10) on p. 254 and (2.25) on p. 256 that

$$\frac{1}{c}\int_I f\left(1+\log^+\frac{f}{f_I}\right)dx \leqslant \int_I (|\tilde{f}|+f)\,dx \leqslant c\int_I f\,dx. \tag{2.7}$$

Now let $0<\beta<1$, $E\subset I$, $\int_E w\,dx > \beta\int_I w\,dx$ and $E_1=\left\{x\in E\colon w(x) > \frac{\beta}{2|E|}\int_I w\,dt\right\}$. From (2.7),

$$\int_{E_1} w\left(1+\log^+\frac{w}{w_I}\right)dx \leqslant c\int_I w\,dx,$$

so that

$$\left(1+\log^+\frac{\beta|I|}{2|E|}\right)\int_{E_1} w\,dx \leqslant c\int_I w\,dx. \tag{2.8}$$

Since $\int_E w\,dx > \beta\int_I w\,dx$ and $\int_{E-E_1} w\,dx \leqslant \frac{1}{2}\beta\int_I w\,dx$, it follows that $\int_{E_1} w\,dx \geqslant \frac{1}{2}\beta\int_I w\,dx$. Substituting into (2.8), we obtain

$$\left(1+\log^+\frac{\beta|I|}{2|E|}\right)\frac{\beta}{2}\int_I w\,dx \leqslant c\int_I w\,dx,$$

or $\left(1+\log^+(\beta|I|/2|E|)\right) < 2c/\beta$. Hence, there exists α such that $0<\alpha<1$ and $|I|/|E|\leqslant 1/\alpha$. This shows that $w\in A_\infty$, since if $|E|<\alpha|I|$ then $\int_E w\,dx\leqslant\beta\int_I w\,dx$.

To show that $w\in B_2$, fix I and let $f=f_k=\min[k, w\chi_J]$, where k is a positive integer and J is the part of $(2I)^c$ which lies to the left of I. By (2.5),

$$\int_I |Hf-(Hf)_I|\,dx \leqslant c\int_I w\,dx,$$

with c independent of k and I. Therefore,

$$\int_I \left|\frac{1}{|I|}\int_I\left[\int_J\left(\frac{1}{x-y}-\frac{1}{t-y}\right)f(y)\,dy\right]dt\right|dx\leqslant c\int_I w\,dx.$$

Since f is bounded, we may interchange the two inner integrals in this expression to obtain

$$\frac{1}{|I|}\int_I\left|\int_J f(y)\left[\int_I\left(\frac{1}{x-y}-\frac{1}{t-y}\right)dt\right]dy\right|dx \leqslant c\int_I w\,dx.$$

This inequality is true a fortiori if the outermost integration on the left is extended only over the left quarter LI of I. But for $x\in LI$ and $y\in J$, it is not hard to see that

$$\int_I\left(\frac{1}{x-y}-\frac{1}{t-y}\right)dt \geqslant \frac{c|I|^2}{(x_0-y)^2},$$

where x_0 is the center of I. Since $f\geqslant 0$, we obtain

$$|I|\int_{LI}\left(\int_J \frac{f(y)}{(x_0-y)^2}\,dy\right)dx \leqslant c\int_I w\,dx,$$

or

$$|I|^2\int_J \frac{f(y)}{(x_0-y)^2}\,dy\leqslant c\int_I w\,dx.$$

Since an analogous result holds for the part of $(2I)^c$ which lies to the right of I, it follows that

$$|I|^2\int_{(2I)^c} \frac{f(y)}{(x_0-y)^2}\,dy\leqslant c\int_I w\,dx.$$

Letting $k\to\infty$, we obtain from the monotone convergence theorem and the doubling condition that $w\in B_2$. The proof of Theorem 1 is now complete.

Proof of Theorem 2. To prove Theorem 2, we first note that the sufficiency follows from Theorem 1. In fact, if $w\in A_1$ or if $w\in A_p\cap B_2$, then $w\in A_\infty\cap B_2$ and so, by Theorem 1, (1.4) holds. However, if $w\in A_1$, then

$$\int_I w\,dx\leqslant c|I|\operatorname*{ess\,inf}_I w = c|I|/\operatorname*{ess\,sup}_I \frac{1}{w},$$

and if $w\in A_p$, $1<p<\infty$, then

$$\int_I w\,dx \leqslant c|I|\left(\frac{1}{|I|}\int_I w^{-1/(p-1)}\,dx\right)^{1-p}.$$

Correspondingly, (1.5) or (1.6) holds.

To prove the necessity of the condition in Theorem 2, let $f=w\chi_I$ for a given I. Then since

$$|\tilde{f}(x)| = \left|\int_I \frac{w(y)}{x-y}\,dy\right|,$$

it follows that for $x\in 10I-9I$

$$|\tilde{f}(x)|\leqslant\frac{1}{4}\cdot\frac{1}{|I|}\int_I w\,dy, \tag{2.9}$$

and for $x \in 5I-4I$,

$$\frac{1}{3}\cdot\frac{1}{|I|}\int_I w\,dy \leqslant |\tilde{f}(x)|. \tag{2.10}$$

We have

$$\int_{10I}|\tilde{f}-(\tilde{f})_{10I}|\,dx \geqslant \int_{10I-9I}|\tilde{f}-(\tilde{f})_{10I}|\,dx + \int_{5I-4I}|\tilde{f}-(\tilde{f})_{10I}|\,dx.$$

In view of (2.9) and (2.10), the integrand of at least one of the integrals on the right is bounded below by $\frac{1}{24}\cdot\frac{1}{|I|}\int_I w\,dy$. Therefore,

$$\int_{10I}|\tilde{f}-(\tilde{f})_{10I}|\,dx \geqslant \tfrac{1}{24}\int_I w\,dy. \tag{2.11}$$

If (1.5) holds, then

$$\frac{1}{|10I|}\int_{10I}|\tilde{f}-(\tilde{f})_{10I}|\,dx = \frac{1}{|10I|}\int_{10I}|Hf-(Hf)_{10I}|\,dx$$
$$\leqslant c\,\operatorname*{ess\,inf}_{10I} w \leqslant c\,\operatorname*{ess\,inf}_{I} w,$$

and if (1.6) holds,

$$\frac{1}{|10I|}\int_{10I}|\tilde{f}-(\tilde{f})_{10I}|\,dx \leqslant c\left(\frac{1}{|10I|}\int_{10I} w^{-1/(p-1)}\,dx\right)^{-(p-1)}$$
$$\leqslant c'\left(\frac{1}{|I|}\int_I w^{-1/(p-1)}\,dx\right)^{-(p-1)}.$$

Using (2.11), we see that $w\in A_1$ in the first case and $w \in A_p$ in the second. To show that $w\in B_2$, we need only consider the second case. Since the inequality

$$1 \leqslant \left(\frac{1}{|I|}\int_I w\,dx\right)\left(\frac{1}{|I|}\int_I w^{-1/(p-1)}\,dx\right)^{p-1}$$

is always true, (1.6) implies (1.4). Therefore, by Theorem 1, (1.6) implies that $w \in B_2$. This completes the proof of Theorem 2.

§ 3. Proofs of Theorems 3–5. To prove the remaining results, we will need the following two lemmas. We recall the notation $m_w(E) = \int_E w\,dx$.

LEMMA (3.1). *Let $1<p<\infty$ and $1/p+1/p' = 1$. If $w\in A_p$, there is a constant c such that*

$$m_w(x\in I:\ w(x)<\beta) \leqslant c\left(\beta\frac{|I|}{m_w(I)}\right)^{p'} m_w(I)$$

for all I and every $\beta>0$.

Proof. We have

$$m_w(I:\ w<\beta) = m_w\left(I:\ \frac{1}{w}>\frac{1}{\beta}\right) \leqslant \beta^{p'}\int_I w^{1-p'}\,dx = \beta^{p'}\int_I w^{-1/(p-1)}\,dx.$$

Since $w\in A_p$,

$$\left(\frac{1}{|I|}\int_I w^{-1/(p-1)}\,dx\right)^{p-1} \leqslant c\left(\frac{1}{|I|}\int_I w\,dx\right)^{-1} = c\,|I|/m_w(I).$$

Hence,

$$m_w(I:\ w<\beta) \leqslant \beta^{p'}\left[\frac{c\,|I|}{m_w(I)}\right]^{1/(p-1)}|I| = \left[\frac{c\beta\,|I|}{m_w(I)}\right]^{p'} m_w(I).$$

The second lemma is a weighted version of the Calderón–Zygmund decomposition lemma [1]. By (1.3), any $w \in A_\infty$ satisfies the doubling condition $m_w(2I)\leqslant c m_w(I)$.

LEMMA (3.2). *Let w satisfy the doubling condition, and let f be a non-negative function which satisfies $\frac{1}{m_w(I)}\int_I fw\,dx \leqslant s$ for a given I and s. Then there exist non-overlapping subcubes I_k, $k = 1, 2, \ldots$, of I, and a constant c' depending only on w such that $f\leqslant s$ a.e. in $I-\bigcup I_k$ and $s \leqslant \frac{1}{m_w(I_k)}\int_{I_k} fw\,dx \leqslant c's$.*

Proof. The proof is nearly identical to that of the usual Calderón–Zygmund lemma. Subdivide I into 2^n non-overlapping subcubes of equal size and collect those subcubes I' with $\frac{1}{m_w(I')}\int_{I'} fw\,dx > s$. Continue subdividing each of those for which this average is at most s, collecting at each stage those cubes for which the average exceeds s. The totality of collected cubes, ordered in any way, is $\{I_k\}$. By construction,

$$\frac{1}{m_w(I_k)}\int_{I_k} fw\,dx > s,$$

while for each k there is a cube J_k containing I_k with

$$|J_k| = 2^n|I_k| \quad\text{and}\quad \frac{1}{m_w(J_k)}\int_{J_k} fw\,dx \leqslant s.$$

Therefore,

$$\frac{1}{m_w(J_k)}\int_{I_k} fw\,dx \leqslant s,$$

and so, by the doubling condition,

$$\frac{1}{m_w(I_k)}\int_{I_k} fw\,dx \leqslant c's.$$

Finally, every $x \in I-\bigcup I_k$ belongs to a sequence of cubes J_m with $|J_m|\to 0$ and $\frac{1}{m_w(J_m)}\int_{J_m} fw\,dx \leqslant s$. For almost every such x, Lebesgue's differentiation theorem gives

$$f(x) = \frac{1}{w(x)}[f(x)w(x)] = \lim_{m\to\infty}\frac{|J_m|}{m_w(J_m)}\,\frac{1}{|J_m|}\int_{J_m} fw\,dx \leqslant s.$$

This completes the proof.

Proof of Theorem 3. The proof of Theorem 3 is based on the method of John and Nirenberg [6]. The arguments for $p=1$ and $1<p<\infty$ are slightly different. Let f satisfy

$$\frac{1}{m_w(I)}\int_I |f-f_I|\,dx \leqslant c, \qquad f_I = \frac{1}{|I|}\int_I f\,dx,$$

for all subcubes I of a given I_0. Replacing f by f/c, we may assume from the beginning that $c=1$. Thus, let $\mathscr{F}$ denote the class of all pairs f and I_0 such that

$$\frac{1}{m_w(I)}\int_I |f-f_I|\,dx \leqslant 1 \quad \text{for all } I\subset I_0.$$

The cube I_0 is not fixed; it may vary with f. For any pair f and I_0 in $\mathscr{F}$, let

$$\lambda(\alpha; I) = \lambda(f; I_0; \alpha; I) = m_w\big(x\in I:\ |f(x)-f_I|/w(x) > \alpha\big)$$

for $\alpha>0$ and $I\subset I_0$. Then, by Tschebyshev's inequality,

$$\lambda(\alpha; I) \leqslant \frac{1}{\alpha}\int_I |f-f_I|\,dx \leqslant \frac{1}{\alpha}m_w(I), \qquad I\subset I_0.$$

Let $\mathscr{F}(\alpha) = \sup \lambda(\alpha; I)/m_w(I)$, where the supremum is taken over all $I\subset I_0$ for all pairs f and I_0 in $\mathscr{F}$. Thus, $\mathscr{F}(\alpha)\leqslant \operatorname{Min}[1/\alpha, 1]$. We will prove Theorem 3 by showing that, given w, there are constants c_0, c_1, c_2, and c_3 larger than 1 such that if $\alpha>c_0$ and $w\in A_1$ then $\mathscr{F}(\alpha)\leqslant c_1 e^{-c_2\alpha}$, and if $\alpha>c_0$ and $w\in A_p$, $1<p<\infty$, then $\mathscr{F}(\alpha)\leqslant c_3\alpha^{-p'}$.

For a given pair f and I_0 in $\mathscr{F}$, we have

$$\frac{1}{m_w(I)}\int_I |f-f_I|\frac{1}{w}w\,dx \leqslant 1, \qquad I\subset I_0.$$

Apply Lemma (3.2) to I, $|f-f_I|\frac{1}{w}$ and s, $s\geqslant 1$, to obtain non-overlapping $\{I_k\}$ in I such that

$$|f-f_I|\frac{1}{w} \leqslant s \text{ a.e. in } I-\bigcup I_k, \tag{3.3}$$

$$s \leqslant \frac{1}{m_w(I_k)}\int_{I_k}|f-f_I|\,dx \leqslant c's. \tag{3.4}$$

If $s\leqslant\alpha$, (3.3) implies that $|f-f_I|\frac{1}{w}\leqslant\alpha$ a.e. in $I-\bigcup I_k$, so that

$$\lambda(\alpha; I) = m_w\Big(I:\ |f-f_I|\frac{1}{w}>\alpha\Big) \leqslant \sum_k m_w\Big(I_k:\ |f-f_I|\frac{1}{w}>\alpha\Big).$$

Hence, for $1\leqslant s\leqslant\alpha$ and $0\leqslant\gamma\leqslant\alpha$,

$$\begin{aligned}\lambda(\alpha; I) &\\ &\leqslant \sum_k m_w\Big(I_k:\ |f-f_{I_k}|\frac{1}{w}>\alpha-\gamma\Big)+\sum_k m_w\Big(I_k:\ |f_{I_k}-f_I|\frac{1}{w}>\gamma\Big).\end{aligned} \tag{3.5}$$

We will estimate each of these two sums. Since the pair f, I_k belongs to $\mathscr{F}$, the terms of the first sum on the right of (3.5) are $\lambda(\alpha-\gamma; I_k)$. Since $\lambda(\alpha-\gamma; I_k)\leqslant\mathscr{F}(\alpha-\gamma)m_w(I_k)$, we obtain for the first sum at most $\mathscr{F}(\alpha-\gamma)\sum m_w(I_k)$. By (3.4),

$$\sum m_w(I_k) \leqslant \frac{1}{s}\sum\int_{I_k}|f-f_I|\,dx \leqslant \frac{1}{s}\int_I |f-f_I|\,dx \leqslant \frac{1}{s}m_w(I). \tag{3.6}$$

Therefore, combining inequalities, our estimate for the first sum (3.5) is

$$\sum m_w\Big(I_k:\ |f-f_{I_k}|\frac{1}{w}>\alpha-\gamma\Big) \leqslant \frac{\mathscr{F}(\alpha-\gamma)}{s}m_w(I). \tag{3.7}$$

For the second sum on the right in (3.5), we have by (3.4)

$$|f_{I_k}-f_I| = \left|\frac{1}{|I_k|}\int_{I_k}(f-f_I)\,dx\right| \leqslant \frac{1}{|I_k|}\int_{I_k}|f-f_I|\,dx \leqslant c's\frac{m_w(I_k)}{|I_k|}. \tag{3.8}$$

We will estimate the terms of the second sum differently for $p=1$ and $1<p<\infty$. If $w\in A_1$, then $m_w(I_k)/|I_k|\leqslant c\operatorname*{ess\,inf}_{I_k} w$. Therefore, combining inequalities, we obtain

$$m_w\Big(I_k:\ |f_{I_k}-f_I|\frac{1}{w}>\gamma\Big) \leqslant m_w\Big(I_k:\ w<\frac{c'cs}{\gamma}\operatorname*{ess\,inf}_{I_k} w\Big). \tag{3.9}$$

Now choose $s = 2$, $\gamma = 2c'c$. Then if $a > \gamma$, we have $1 < s < a$ and $0 < \gamma < a$ as required. (Note that $c, c' > 1$.) From (3.9)

$$m_w\left(I_k\colon |f_{I_k}-f_I|\frac{1}{w} > \gamma\right) \leqslant m_w(I_k\colon w < \operatorname*{ess\,inf}_{I_k} w) = 0$$

so that the second sum in (3.5) is zero. Therefore, by (3.5) and (3.7), $\lambda(a; I) \leqslant \frac{1}{2}\mathscr{F}(a-\gamma)m_w(I)$ if $a > \gamma$ and $I \subset I_0$. Hence, $\mathscr{F}(a) \leqslant \frac{1}{2}\mathscr{F}(a-\gamma)$ if $a > \gamma$. If m is a positive integer and a satisfies $m\gamma < a < (m+1)\gamma$, it follows that $\mathscr{F}(a) \leqslant 2^{-1}\mathscr{F}(a-\gamma) \leqslant \ldots \leqslant 2^{-m}\mathscr{F}(a-m\gamma)$. Since $\mathscr{F}(a-m\gamma) \leqslant 1$ and $m \geqslant a/\gamma - 1$ for such a, we obtain

$$\mathscr{F}(a) \leqslant 2^{-m} \leqslant 2^{1-\frac{a}{\gamma}} = 2e^{-\left(\frac{1}{\gamma}\log 2\right)a}.$$

Therefore, with $c_1 = 2$ and $c_2 = \dfrac{1}{\gamma}\log 2$, $\mathscr{F}(a) \leqslant c_1 e^{-c_2 a}$ if $a > \gamma$ and $w \in A_1$. This completes the proof of Theorem 3 in case $w \in A_1$.

Next, suppose $w \in A_p$, $1 < p < \infty$. By (3.8),

$$m_w\left(I_k\colon |f_{I_k}-f_I|\frac{1}{w} > \gamma\right) \leqslant m_w\left(I_k\colon w < \frac{c's}{\gamma}\,\frac{m_w(I_k)}{|I_k|}\right). \tag{3.10}$$

By Lemma (3.1), the right side of (3.10) is at most $c[c's/\gamma]^{p'}m_w(I_k)$. If we now add over k and use (3.6), our estimate for the second sum (3.5) is

$$\sum m_w\left(I_k\colon |f_{I_k}-f_I|\frac{1}{w} > \gamma\right) \leqslant c\left[\frac{c's}{\gamma}\right]^{p'}\sum m_w(I_k)$$
$$\leqslant c\left[\frac{c's}{\gamma}\right]^{p'}\frac{1}{s}m_w(I) = c''\frac{s^{p'-1}}{\gamma^{p'}}m_w(I).$$

Combining this and (3.7), we obtain from (3.5) that

$$\lambda(a; I) \leqslant c''\left[\frac{\mathscr{F}(a-\gamma)}{s} + \frac{s^{p'-1}}{\gamma^{p'}}\right]m_w(I) \tag{3.11}$$

if $1 \leqslant s \leqslant a$, $0 < \gamma < a$, $I \subset I_0$ and $w \in A_p$, $1 < p < \infty$. Now choose $s = 4^{p'}c''$, $\gamma = a/2$ and $c_0 = \max[s, c''s^{p'-1}2^{p'}]$. Then (3.11) implies

$$\mathscr{F}(a) \leqslant 4^{-p'}\mathscr{F}\left(\frac{a}{2}\right) + c_0 a^{-p'} \quad \text{if} \quad a > c_0. \tag{3.12}$$

We will show by induction that if $c_0 < a \leqslant 2c_0$ and m is a non-negative integer then

$$\mathscr{F}(2^m a) \leqslant (2c_0)^{p'}(2^m a)^{-p'}. \tag{3.13}$$

When $m = 0$, $\mathscr{F}(2^m a) \leqslant 1 \leqslant (2c_0)^{p'}a^{-p'}$. Assuming (3.13) for $m-1$, we have from (3.12) that

$$\mathscr{F}(2^m a) \leqslant 4^{-p'}\mathscr{F}(2^{m-1}a) + c_0(2^m a)^{-p'}$$
$$\leqslant 4^{-p'}(2c_0)^{p'}(2^{m-1}a)^{-p'} + c_0(2^m a)^{-p'}$$
$$= (2c_0)^{p'}(2^m a)^{-p'}[2^{-p'} + 2^{-p'}c_0^{1-p'}].$$

Since $2^{-p'} + 2^{-p'}c_0^{1-p'} < 2^{-p'} + 2^{-p'} < 1$, (3.13) follows. Therefore, if $a > c_0$, $\mathscr{F}(a) \leqslant (2c_0)^{p'}a^{-p'}$, which completes the proof of Theorem 3.

Proof of Theorem 4. By Hölder's inequality,

$$\int_I |f-f_I|\,dx \leqslant \left(\int_I |f-f_I|^r w^{1-r}\,dx\right)^{1/r}\left(\int_I w\,dx\right)^{1/r'}$$

for $1 < r < \infty$, $1/r + 1/r' = 1$. Therefore, if f satisfies (1.9), that is, if $\int_I |f-f_I|^r w^{1-r}\,dx \leqslant c\int_I w\,dx$ for some r, then

$$\int_I |f-f_I|\,dx \leqslant c^{1/r}\left(\int_I w\,dx\right)^{1/r+1/r'} = c^{1/r}\int_I w\,dx.$$

Therefore, f is of bounded mean oscillation with weight w.

Conversely, suppose that f is of bounded mean oscillation with weight w. Then, for $0 < r < \infty$,

$$\int_I |f-f_I|^r w^{1-r}\,dx = \int_I (|f-f_I|w^{-1})^r w\,dx = r\int_0^\infty a^{r-1}\lambda(a; I)\,da,$$

where $\lambda(a; I) = m_w(I\colon |f-f_I|w^{-1} > a)$. If $w \in A_1$, then by Theorem 3, $\lambda(a; I) \leqslant c_1 e^{-c_2 a}m_w(I)$. Therefore,

$$\int_I |f-f_I|^r w^{1-r}\,dx \leqslant c_1 r\int_0^\infty a^{r-1}e^{-c_2 a}\,da\, m_w(I) = c m_w(I).$$

On the other hand, if $w \in A_p$, $1 < p < \infty$, then by [7], Lemma 5 (extended to n dimensions on p. 222), there exists $\varepsilon > 0$ such that $w \in A_{p-\varepsilon}$. Therefore, by Theorem 3,

$$\int_I |f-f_I|^r w^{1-r}\,dx \leqslant c_3 r\int_0^\infty a^{r-1}(1+a)^{-(p-\varepsilon)'}\,da\, m_w(I).$$

If $0 < r < (p-\varepsilon)'$, this is a constant times $m_w(I)$. Since $p' < (p-\varepsilon)'$, it follows that f satisfies (1.9) for $0 < r \leqslant p'$, which completes the proof of Theorem 4.

Note that the proof shows that if f is of bounded mean oscillation with weight w and $w \in A_p$, $1 < p < \infty$, then (1.9) even holds for r slightly larger than p': $r < (p-\varepsilon)'$ for some $\varepsilon > 0$. However, we will now give an example which shows that, given p, no r larger than p' can be used in (1.9) for all f and w which satisfy (1.1) and $w \in A_p$. Given p and r satisfying $1 < p < \infty$ and $r > p'$, let

$$w = x^{r'-1}(1-x)^{r'-1}.$$

Then $w \in A_p$ on $[0, 1]$; in fact, $x^{q-1}(1-x)^{q-1} \in A_p$ on $[0, 1]$ if $1 < q < p < \infty$. Let $f = 1$ on $[0, \frac{1}{2}]$ and $f = 2$ on $[\frac{1}{2}, 1]$. Then f is of bounded mean oscillation on $[0, 1]$ with weight w. To see this, note that if $I \subset [0, 1]$ and $\frac{1}{2} \notin I$, then $\int_I |f-f_I|\,dx = 0$, while if $\frac{1}{2} \in I$, then

$$\int_I |f-f_I|\,dx \leqslant \int_I (f+f_I)\,dx \leqslant 4|I| \quad \text{and} \quad m_w(I) \geqslant c_{r'}|I|.$$

On the other hand, taking $I = [0, 1]$, we have $f_I = \frac{3}{2}$ and

$$\int_I |f-f_I|^r w^{1-r}\,dx \geqslant \int_0^{1/2} (\tfrac{1}{2})^r [x^{r'-1}(\tfrac{1}{2})^{r'-1}]^{1-r}\,dx = +\infty,$$

since $(r'-1)(1-r) = -1$. Therefore, f and w do not satisfy (1.9) for r. Changing the roles of the indices, we see from the same argument that f satisfies (1.1) with $w = x^{p-1}(1-x)^{p-1}$, but that f and w do not satisfy (1.9) with $r = p'$ even though w belongs to A_q for *all* $q > p$.

We conclude by sketching the proof of the result related to (1.2) which was mentioned in the introduction.

THEOREM 5. *Let $w \in A_\infty$. Then f satisfies* (1.2) *for all $I \subset I_0$ if and only if f is of bounded mean oscillation on I_0; that is, if and only if there is a constant c' such that for all $I \subset I_0$,*

$$\int_I |f-f_I|\,dx \leqslant c'|I|.$$

Proof. If f is of bounded mean oscillation on I_0, then by the result of John and Nirenberg (Theorem 3 above with $w = 1$), $|(I\colon |f-f_I| > a)| \leqslant c_1 e^{-c_2 a}|I|$ for $I \subset I_0$. Since $w \in A_\infty$, (1.3) implies $m_w(I\colon |f-f_I| > a) \leqslant cc_1 e^{-(c_2\delta)a} m_w(I)$ for $I \subset I_0$. Therefore,

$$\int_I |f-f_I|\,w\,dx = \int_0^\infty m_w(I\colon |f-f_I| > a)\,da \leqslant c' m_w(I).$$

A simple argument like that following (1.1) now shows that

$$\int_I |f-c_I|\,w\,dx \leqslant c'' m_w(I) \quad \text{where} \quad c_I = \frac{1}{m_w(I)}\int_I fw\,dx.$$

Conversely, suppose that $w \in A_\infty$ and f satisfies (1.2) for $I \subset I_0$. An argument exactly like that of John and Nirenberg (the proof of Theorem 3 corresponding to A_1 and $w \equiv 1$) using w-measure *everywhere* instead of Lebesgue measure, and using the Calderón–Zygmund lemma in the form of Lemma (3.2), shows that $m_w(I\colon |f-c_I| > a) \leqslant c_1 e^{-c_2 a} m_w(I)$ for $I \subset I_0$. Since $w \in A_\infty$, (1.3) implies

$$|(I\colon |f-c_I| > a)| \leqslant ce^{-(c_2/\eta)a}|I| \quad \text{for} \quad I \subset I_0,$$

which of course implies that f is of bounded mean oscillation. This completes the proof of Theorem 5.

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