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Admissible translates of stable measures

by

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Abstract. We investigate the structure of the set of admissible translates of a stable measure, and we obtain bounds on the size of this set. We then apply this to show that certain stable stochastic processes have no non-trivial admissible translates.

1. Introduction. It is the purpose of this paper to examine the set A_μ of the admissible translates of a stable measure μ on a real separable Hilbert space. For the special case of a Gaussian measure the set A_μ can be described, completely, through the characteristic functional of μ (see [5], Theorem 4.1). For a general infinitely divisible meaure ν Gikhman and Skorokhod ([5], Theorem 6.1) have obtained sufficient conditions for an element of the Hilbert space to be an admissible translate of ν. However, the conditions of Theorem 6.1 [5] are difficult to verify. Theorem 6.2 [5] simplifies the conditions in the case of a stable measure, but unfortunately Theorem 6.2 is false. In contrast to [5] our main goal is to obtain information on the structure of the set A_μ (see Pitcher [13]) and to obtain measure theoretic and algebraic bounds on the size of A_μ. For example, we show that (i) A_μ is a cone in H, and (ii) A_μ is a Borel set of μ-measure zero.

The organization of the paper is as follows. Section 2 contains the preliminaries and Section 3 contains some general theorems on the structure of A_μ. In Section 4 we specialize to stable measures and in Section 5 we restrict our attention to stable measures on a real separable Hilbert space. Section 6 contains some results, which are useful for the applications given in Section 7. We conclude in Section 8 with some questions, and some remarks on these questions.

2. Preliminaries. X (and Y) always denote a real, Hausdorff, topological vector space (RHTVS). $\mathscr{B}(X)$ will denote the σ-algebra generated by the open sets of X, and the sets in $\mathscr{B}(X)$ will be referred to as *Borel sets*. μ will always represent a probability measure on $\mathscr{B}(X)$ and $\mathscr{B}_\mu(X)$ will denote the μ-completion of $\mathscr{B}(X)$. $\mathscr{M}(X)$ will denote the set of probability measures on $\mathscr{B}(X)$, and $\mathscr{M}(X)$ will be given the weak star topology.

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If Φ is a measurable mapping of $(X, \mathscr{B}(X))$ into $(Y, \mathscr{B}(Y))$ and $\mu \in \mathscr{M}(X)$, then $\mu^{\Phi} \in \mathscr{M}(Y)$ will be defined by

$$\mu^{\Phi}(E) = \mu(\Phi^{-1}(E))$$

for all $E \in \mathscr{B}(Y)$. If $\Phi(x) = x+z$ (for some $z \in X$), then we write μ_z for μ^{Φ}. If $\Phi(x) = \tau x$ (for some $\tau \in \boldsymbol{R} \setminus \{0\}$), then we write μ^{τ} for μ^{Φ}.

Definition 1. An element $a \in X$ is said to be an *admissible translate* (resp., *singular translate*) of μ, if μ_a is absolutely continuous (resp., singular) with respect to μ (denoted by $\mu_a \ll \mu$ and $\mu_a \perp \mu$, respectively).

Throughout A_μ (resp., S_μ) will denote the set of admissible (resp., singular) translates of μ. In the case where X has a non-trivial topological dual X^*, the characteristic functional of μ is the function on X^* given by

$$\hat{\mu}(x^*) = \int e^{i\langle x^*, y\rangle} \mu(dy),$$

for all $x^* \in X^*$, where $\langle x^*, y\rangle = x^*(y)$ for $y \in X$ and $x^* \in X^*$. If X is a Hilbert space, then we identify X^* with X and $\langle x, y\rangle$ means the *inner product* of x and y. For probability measures μ and ν on X, $\mu * \nu$ is defined by

$$\mu * \nu(E) = \int \mu(E-x)\, \nu(dx)$$

for all $E \in \mathscr{B}(X)$.

When we refer to a set G as being a subgroup of X, we, of course, mean that G is a subgroup under addition. Finally, m will denote Lebesgue measure on the real line.

3. In this section we present some general results on A_μ.

Let G_μ (resp., G^μ) $= \bigcap G$ where the intersection is taken over subgroups G, of X such that $G \in \mathscr{B}_\mu(X)$ and $\mu(G) > 0$ (resp., $\mu(G) = 1$).

Definition 2. A set C in X is said to be a *cone* if $c \in C$ and $\lambda > 0$ implies $\lambda c \in C$, and $c_1, c_2 \in C$ implies $c_1 + c_2 \in C$.

Proposition 1. *Let μ be a regular, tight probability measure on X.*

(i) *If $x \notin G^\mu$, then $\mu_x \perp \mu$.*

(ii) *If A_μ is a cone in X, then $A_\mu - A_\mu \subseteq G_\mu$.*

Proof. (i) Let G be a subgroup of X such that $G \in \mathscr{B}_\mu(X)$ and $\mu(G) = 1$. Then $x \notin G$ implies $G \cap (G-x) = \varphi$. Hence $\mu_x(G) = \mu(G-x) = 0$.

(ii) Let G be a subgroup of X such that $G \in \mathscr{B}_\mu(X)$ and $\mu(G) > 0$, and let $a \in A_\mu$. We are to show $a \in G$. Since μ is regular and tight, we may choose $G_0 \leqslant G$ such that $G_0 \in \mathscr{B}(X)$ and $\mu(G_0) > 0$ (see [19], Corollary 1.1). Let $H = \{\lambda \in \boldsymbol{R}^1 \colon \lambda a \in G_0\}$. H is a Borel set in $\boldsymbol{R}^1$, since X is an RHTVS.

If $m(H) = 0$, then there exists an uncountable collection of positive real numbers $\{\lambda_a\}$ such that $\{H - \lambda_a a\}$ are pairwise disjoint. Hence $\{G_0 - \lambda_a a\}$ are pairwise disjoint. However, $\mu(G_0) > 0$ implies $\mu(G_0 - \lambda_a a) > 0$, since A_μ is a cone. This is impossible. Hence $m(H) > 0$ and therefore $H = \boldsymbol{R}$. This in turn implies $a \in A_\mu$.

For the remainder of this section we assume that X is also a complete, separable metric space.

Proposition 2. $A_\mu \in \mathscr{B}(X)$.

Proof. Since X is a complete separable metric space, there exists a compact metric space K, and a continuous injection $F\colon X \to K$ (see [6], Theorem 2-46, pp. 68–69). Note that $F(X)$ is a Borel set of K (see, e.g., Theorem 3.9 [12]). The map $T\colon X \to \mathscr{M}(K)$ defined by $T(x) = (\mu_x)^F$ is continuous. By Theorems 2.10 and 3.1, [1], we know that the map $\Lambda\colon \mathscr{M}(K) \to \mathscr{M}(K)$ defined by $\Lambda(\nu) =$ absolutely continuous part of ν with respect to μ^F, is $\mathscr{B}(X)$-measurable. Hence $\{x \in X\colon \Lambda \circ T(x) = T(x)\} \in \mathscr{B}(X)$. We are done since

$$A_\mu = \{x \in X\colon \Lambda \circ T(x) = T(x)\}.$$

Proposition 3. *Suppose that A_μ is a cone in X. Then either $\mu(A_\mu) = 0$ or A_μ is finite dimensional.*

Proof. By Proposition 2, A_μ is a Borel set in X.

Let $\nu = \mu * \mu^{-1}$. Then A_ν, which is also a Borel set, contains $A_\mu - A_\mu$. If $\mu(A_\mu) > 0$, then $\nu(A) \geqslant \mu^2(A_\mu - A_\mu) > 0$. If $\gamma = \nu$ restricted to A_ν, then by Feldman [4] (see also Sudakov [18]), A_ν is finite dimensional.

Proposition 4. *Let X and Y be given and let μ be a probability measure on $\mathscr{B}(X)$. Assume that $\Lambda\colon X \to Y$ is measurable and linear. Then* (a) $\Lambda^{-1}(S_{\mu^\Lambda}) \subseteq S_\mu$ *and therefore* (b) $A_\mu \subseteq \Lambda^{-1}(S^c_{\mu^\Lambda})$. *Note that if Λ is an injection then* (c) $\Lambda^{-1}(A_{\mu^\Lambda}) = A_\mu$.

4. In this section we restrict ourselves to stable measures (defined below).

Definition 3. A probability measure μ on $\mathscr{B}(X)$ is said to be *stable of index a* if for any $\lambda, \tau > 0$, there exists $y \in X$ (y depends on λ and τ) such that $\mu^\lambda * \mu^\tau = (\mu^\gamma)_y$, where $\gamma^a = \lambda^a + \tau^a$.

Definition 4. A probability measure μ on $\mathscr{B}(x)$ is *symmetric* if $\mu(A) = \mu(-A)$ for all $A \in \mathscr{B}(x)$.

Proposition 5. *A_μ is a cone in X.*

Proof. Fix $0 < \lambda < 1$ and choose $\tau > 0$ such that $\lambda^a + \tau^a = 1$. Suppose that $a \in A_\mu$ and $\mu(E) = 0$. Now there exists $z = z(\lambda, \tau)$ such that $\mu_z = \mu^\lambda * \mu^\tau$. Hence

$$0 = \mu(E) = \mu_z(E+z) = \int \mu^\lambda(E+z-x)\, \mu^\tau(dx)$$
$$= \int \mu\big(\lambda^{-1}(E+z) - \lambda^{-1}\tau x\big)\, \mu(dx).$$

Therefore $\mu\big(\lambda^{-1}(E+z)-\lambda^{-1}\tau x\big)=0$ for μ-almost all x. Since $a\in A_\mu$, we have $0=\mu\big(\lambda^{-1}(E+z)-\lambda^{-1}\tau x-a\big)=\mu\big(\lambda^{-1}(E+z-\lambda a)-\lambda^{-1}\tau x\big)$ for μ-almost all x. This yields

$$\mu_{\lambda a}(E)=\mu_z(E+z-\lambda a)=\int\mu\big(\lambda^{-1}(E+z-\lambda a)-\lambda^{-1}\tau a\big)\mu(dx)=0.$$

We have just shown $a\in A_\mu$ implies $\lambda a\in A_\mu$ for $0<\lambda<1$. Since $0\in A_\mu$ and $G=\{\lambda:\ \mu_{\lambda a}\ll\mu\}$ is a semigroup in $\boldsymbol{R}^1$, G contains $[0,\infty)$.

COROLLARY 5.1. *If μ is a symmetric stable measure on $\mathscr{B}(X)$, then A_μ is a linear subspace of X.*

Proof. Since μ is symmetric, $A_\mu=-A_\mu$.

COROLLARY 5.2. *If μ is a stable measure on $\mathscr{B}(X)$, which is regular and tight, then $A_\mu-A_\mu\leqslant G_\mu$.*

Proof. Apply Proposition 1 (ii).

Remark. Let M_μ(resp., M^μ) be the intersection of all linear subspaces M of X such that $M\in\mathscr{B}(X)$ and $\mu(M)>0$ (resp., $\mu(M)=1$). It has been shown by Dudley and Kanter [3] that $\mu(M)>0$ implies $\mu(M)=1$. Hence $M_\mu=M^\mu$. Hence in order to prove that for μ symmetric we have, for every $x\in X$, either $\mu_x\perp\mu$ or $\mu_x\sim\mu$, it is sufficient to show that $A_\mu=M_\mu$. (To see the sufficiency apply Proposition 1.) For a Gaussian measure on a real separable Hilbert space $A_\mu=M_\mu$ (see [19], Theorem 5) and hence for such measures we have the above-mentioned dichotomy.

COROLLARY 5.3. *If μ is a stable measure, then either $\mu(A_\mu-A_\mu)=0$ or A_μ is finite dimensional.*

Proof. Apply Proposition 3.

Remark. If $A_\mu-A_\mu$ is finite dimensional and $\mu(A_\mu-A_\mu)>0$, then $A_\mu-A_\mu=$ support of μ (by [3]). Hence either $\mu(A_\mu-A_\mu)=0$ or $A_\mu-A_\mu$ $=$ support of μ.

5. From this point on X will be separable Hilbert space.

Let μ be a symmetric stable measure of index a on $\mathscr{B}(X)$. In [9] Kuelbs has shown that there exists a symmetric, finite, positive Borel measure Γ on the unit sphere

$$S=\{x\in X:\ \|x\|=1\}$$

such that

$$\hat{\mu}(y)=\exp\Big\{-\int_S|\langle y,\theta\rangle|^a\Gamma(d\theta)\Big\}\quad\text{(see also [8]).}$$

We will use the notation $\mu=[a,\Gamma]$.

At this point we give a couterexample to Theorem 6.2 [5]. Choose a finite positive Borel measure Γ on the unit sphere of an infinite-dimensional (separable) Hilbert space, such that the support of Γ is all of S.

Then, for $a\in H$,

$$a=\lim\frac{\|a\|}{\Gamma(E_n)}\int_{E_n}\theta\Gamma(d\theta),\quad\text{where}\quad E_n=\Big\{\theta\varepsilon S:\Big\|\frac{a}{\|a\|}-\theta\Big\|<\frac{1}{n}\Big\}.$$

Now, Theorem 6.2 [5] implies that $a\in A_\mu$, i.e., $A_\mu=H$, contradicting Proposition 3. For another counterexample see [20].

Now let $H_0=\{y\in X:\ \int_S|\langle y,\theta\rangle|^a\Gamma(d\theta)=0\}$. H_0 is a closed subspace of X. Let $H=H_0^\perp$. Then it is easy to see that $\mu(H)=1$. Now complete H with respect to the metric $\|\cdot\|_{a,\Gamma}$ given by

$$\|y\|_{a,\Gamma}=\begin{cases}\big[\int_S|\langle y,\theta\rangle|^a\Gamma(d\theta)\big]^{1/a} & \text{if}\quad a\geqslant 1,\\ \int_S|\langle y,\theta\rangle|^a\Gamma(d\theta) & \text{if}\ 0<a<1.\end{cases}$$

Note that

$$\|y\|_{a,\Gamma}\leqslant\begin{cases}\|y\|\,\Gamma^{1/a}(S) & \text{if}\quad a\geqslant 1,\\ \|y\|^a\Gamma(S) & \text{if}\quad 0<a<1,\end{cases}$$

and hence $\|\cdot\|_{a,\Gamma}$ is continuous. Let $B(a,\Gamma)$ denote this completed space. We, therefore, have the continuous injection i: $H\to B(a,\Gamma)$. Since i is one to one and has dense range, and since H is a Hilbert space, the adjoint map i^*: $B^*(a,\Gamma)\to H^*=H$ is one to one and if $a\geqslant 1$ has dense range. Note, also, that i is a compact operator, and hence so is i^*.

THEOREM 6. *If $a\in X\setminus i^*\big(B^*(a,\Gamma)\big)$, then $\mu_a\perp\mu$.*

Proof. It is enough to show that $\mu_a\perp\mu$ for $a\in H\setminus i^*\big(B^*(a,\Gamma)\big)$, since $\mu(H)=1$. We claim that there exists a sequence $\{b_n\}_{n=1}^\infty\subseteq H$ such that $\|i(b_n)\|_{a,\Gamma}\to 0$ and $\langle a,b_n\rangle=1$ for all n. Suppose not. Therefore for every sequence $\{b_n\}\subseteq H$ such that $\|i(b_n)\|_{a,\Gamma}\to 0$, we have $\langle a,b_n\rangle\to 0$. Hence if we define $\bar{a}$ on $i(H)$ by $\bar{a}\big(i(x)\big)=\langle a,x\rangle$, then the above assumption can be rephrased as saying that $\bar{a}$ is continuous on $i(H)$ in the metric $\|\cdot\|_{a,\Gamma}$. Therefore we can extend $\bar{a}$ to a continuous linear functional on $B(a,\Gamma)$. But note that $\langle x,i^*\bar{a}\rangle=\langle ix,\bar{a}\rangle=\langle a,x\rangle$ for all $x\in H$. Hence $i^*(\bar{a})=a$ or $a\in i^*\big(B^*(a,\Gamma)\big)$, a contradiction.

Now choose $\{b_n\}\subseteq H$ such that $\|i(b_n)\|_{a,\Gamma}\to 0$ and $\langle a,b_n\rangle=1$ for all n. We have

$$\int e_1^{it\langle x,b_n\rangle}\mu(dx)=\hat{\mu}(tb_n)=\exp\{-|t|^a\|i(b_n)\|_{a,\Gamma}^{\varepsilon(a)}\}\to 1$$

as $n\to\infty$, where

$$\varepsilon(a)=\begin{cases}1 & \text{if}\quad 0<a<1,\\ a & \text{if}\quad 1\leqslant a\leqslant 2.\end{cases}$$

Therefore $\langle\cdot, b_n\rangle\to 0$ in μ-measure and hence some subsequence $\{b_{n_k}\}$ of $\{b_n\}$ converges to zero for μ-almost all x. On the other hand,

$$\int e^{it\langle x,b_{n_k}\rangle}\mu_a(dx) = e^{it\langle a,b_{n_k}\rangle}\hat{\mu}(tb_{n_k}) = e^{it}\hat{\mu}(tb_{n_k})\to e^{it}.$$

Therefore $\langle\cdot, b_{n_k}\rangle\to 1$ in μ_a-measure, and thus a subsequence of $\{b_{n_k}\}$ converges to one for μ_a-almost all x. Hence $\mu\perp\mu_a$.

COROLLARY 6.1. $M^\mu\subseteq i^*(B^*(a,\Gamma))$.

Proof. Fix $a\in H\setminus i^*(B^*(a,\Gamma))$. For any sequence $\{y_n\}\subseteq H$, define

$$M\{y_n\} = \{z\in H:\ \lim\langle z, y_n\rangle = 0\}.$$

$M\{y_n\}$ is clearly a measurable set which is also a linear subspace. By the proof of Theorem 6 we see that there exists a sequence $\{b_n\}\subseteq H$ such that $\langle a, b\rangle_n = 1$, $\|i(b_n)\|_{a,\Gamma}\to 0$ and $\mu(M\{b_n\}) = 1$. However, $a\notin M\{b_n\}$ and hence $a\notin M^\mu$.

In the next proposition we find a sufficient condition for the singularity of the symmetric stable measures. Let $\mu_i = [a_i, \Gamma_i]$ $(i = 1, 2)$ be given.

PROPOSITION 7. *If* $\{\|\cdot\|_{a_i,\Gamma_i}\}$, $i = 1, 2$ *are not equivalent metrics on* X; *then* $\mu_1\perp\mu_2$.

Proof. If $\{\|\cdot\|_{a_i,\Gamma_i}\}$, $i = 1, 2$ are not equivalent, there exists (for example) $\{x_n\}\subseteq X$ such that $\|x_n\|_{a_1,\Gamma_1}\to 0$ and $\|x_n\|_{a_2,\Gamma_2} = 1$ for all n. Hence $\hat{\mu}_1(x_n)\to 1$.

Thus there exists a subsequence $\{x_{n_k}\}$ such that $\mu(M\{x_{n_k}\}) = 1$ where $M\{x_{n_k}\}$ is defined as in Corollary 6.1). By Dudley and Kanter's zero-one law [3], $M\{x_{n_k}\}$ has μ_2-measure zero or one. If $\mu_2(M\{x_{n_k}\}) = 1$, then $\hat{\mu}(x_{n_k})\to 1$, by the Bounded Convergence Theorem. However, $\hat{\mu}_2(x_{n_k}) = \exp\{-\|x_{n_k}\|^{s(a_2)}_{a_2,\Gamma_2}\} = \exp\{-1\}\neq 1$, a contradiction. Hence $\mu_2(M\{x_{n_k}\}) = 0$ and thus $\mu_1\perp\mu_2$.

6. In this section we present results which will be useful in applications to stable processes.

PROPOSITION 8. *Let* $\{\xi_k\}_{k=1}^\infty$ *be a sequence of independent random variables such that* $E[e^{it\xi_k}] = e^{-|t|^a}$, *for some fixed* a, $0 < a\leqslant 2$. *Let* μ *be the measure on* $\boldsymbol{R}^\infty$ *induced by the sequence* $\{\xi_k\}$. *Then* $\boldsymbol{A}_\mu = \{a\in\boldsymbol{R}^\infty:\ \sum_{k=1}^\infty a_k^2 < \infty\}$.

Proof. This follows from Shepp [15] (or LeCam [10]) and the fact that the stable density has finite Fisher information.

In the applications to stable processes we will only need that $\sum_{k=1}^\infty a_k^2 = \infty$ implies $\mu_a\perp\mu$. This follows more easily from Dudley [2] (Theorem 2).

COROLLARY 8.1. *Let* $\mu = [a, \Gamma]$ *be given where the support of* Γ *is the orthonormal set* $\{e_k\}$ *in* X *and* $\Gamma\{e_k\} = \lambda_k$. *Then*

$$A_\mu = \left\{x\in X:\ \sum_{k=1}^\infty\frac{\langle x, e_k\rangle^2}{\lambda_k^{2/a}} < \infty\right\}.$$

Proof. Consider the map Λ: $X\to\boldsymbol{R}^\infty$ defined by

$$\Lambda(x) = \left\{\frac{\langle x, e_k\rangle}{\lambda_k^{1/a}}\right\}_{k=1}^\infty,$$

and note that the random variables ξ_k on $\boldsymbol{R}^\infty$ given by $\xi_k(x) = x_k$ satisfy the hypotheses of Proposition 8.

Remark. Under the hypotheses of Corollary 8.1 it is easy to see that if $a\geqslant 1$

$$i^*(B^*(a,\Gamma)) = \left\{x\in X:\ \sum_{k=1}^\infty\frac{|\langle x, e_k\rangle|^\beta}{\lambda_k^{\beta/a}} < \infty\right\}$$

where $1/a + 1/\beta = 1$. Hence (in this case) $i^*(B^*(a,\Gamma))\neq A_\mu$ unless $a = 2$.

DEFINITION 5. A stochastic process $\{Y_t:\ 0\leqslant t\leqslant 1\}$ is said to be a *stable process of index* a if the finite-dimensional distributions of $\{Y_t\}$ are all stable (of index a).

Let $\{X_t:\ 0\leqslant t\leqslant 1\}$ be the stable process of type a such that

(i) $\{X_t\}$ *has stationary and independent increments* and

(ii) $E[e^{uiX(t)}] = e^{-t|u|^a}$.

For the remainder of this paper $\{X_t\}$ will always denote such a process.

Let $D[I^2]$ be the Skorokhod space of real-valued function on the square $I^2 = [0,1]\times[0,1]$, which has been studied by Straf [17] and Neuhaus [11]. Similary let $D[I]$ be the usual Skorokhod space (again, see, e.g. [17]). For a function $g\in L^a = L^a([0,1], m)$ the stochastic integral $\int_0^1 g(t)\,dX(t)$ has been defined by Schilder [14]. Hence for $f\in D[I^2]$ we may define the process $\{Y_t:\ 0\leqslant t\leqslant 1\}$ by the formula

$$Y(t) = \int_0^1 f(t,s)\,dX(S).$$

It is not hard to see that Y is a symmetric stable process with sample paths in $L^2[I]$. We now prove a Fubini-type result.

PROPOSITION 9. *Let* $\{X_t:\ t\in I\}$ *be as above. Then if* $f\in D[I^2]$, *we have*

$$(*)\qquad \int_0^1\left[\int_0^1 f(t,s)\,dX(s)\right]dt = \int_0^1\left[\int_0^1 f(t,s)\,dt\right]dX(s)\ \text{ a.s.}$$

Proof. Choose $\{f_n\} \subseteq D[I^2]$ such that

$$f_n(t,s) = \sum_{j=1}^{N_n} c_{j,n} 1_{I_{j,n}}(t) 1_{I_{j,n}}(s)$$

and (see Straf [17]) f_n converges to f uniformly. Since (*) holds trivially for f_n, we need only show:

(i) $$\int_0^1 \Big[\int_0^1 f_n(t,s)\,dX(s)\Big]\,dt \to \int_0^1 \Big[\int_0^1 f(t,s)\,dX(s)\Big]\,dt$$

in probability and

(ii) $$\int_0^1 \Big[\int_0^1 f_n(t,s)\,dt\Big]\,dX(s) \to \int_0^1 \Big[\int_0^1 f(t,s)\,dt\Big]\,dX(s)$$

in probability.

To show (i) we shall compute the *characteristic function* of

$$\int_0^1 \Big[\int_0^1 \big(f_n(t,s)-f(t,s)\big)\,dX(s)\Big]\,dt$$

and show that *it* converges to 1. But

$$\sum_{j=1}^{N} \Big[\int_0^1 \big(f_n(t_{j-1},s)-f(t_{j-1},s)\big)\,dX(s)\Big](t_j-t_{j-1})$$
$$= \int_0^1 \sum_{j=1}^{N} (t_j-t_{j-1})\big(f_n(t_{j-1},s)-f(t_{j-1},s)\big)\,dX(s)$$

has the characteristic function

$$\Phi(u) = \exp\Big\{-|u|^\alpha \int_0^1 \Big|\sum_{j=1}^{N}(t_j-t_{j-1})\big(f_n(t_{j-1},s)-f(t_{j-1},s)\big)\Big|^\alpha ds\Big\}.$$

Now since f_n and f are bounded, we have (by approximating the integrals and taking limits):

$$\int_0^1 \Big[\int_0^1 \big(f_n(t,s)-f(t,s)\big)\,dX(s)\Big]\,dt$$

has the characteristic function

$$\Psi(u) = \exp\Big\{-|u|^\alpha \int_0^1 \Big|\int_0^1 \big(f_n(t,s)-f(t,s)\big)\,dt\Big|^\alpha ds\Big\}.$$

Therefore, since $f_n \to f$ uniformly,

$$\int_0^1 \Big|\int_0^1 \big(f_n(t,s)-f(t,s)\big)\,dt\Big|^\alpha ds \to 0.$$

This yields

$$\int_0^1 \Big[\int_0^1 f_n(t,s)\,dX(s)\Big]\,dt \to \int_0^1 \Big[\int_0^1 f(t,s)\,dX(s)\Big]\,dt,$$

in probability. (ii) follows even more simply.

Since the paths of Y (for $f \in D[I^2]$) are in $L^2[I]$, Y induces a measure μ on $L^2[I]$ which is symmetric stable of index α. Hence $\mu = [\alpha, \Gamma]$. We shall now describe Γ in terms of the given f, if $\alpha < 2$.

For $z \in C(I)$ the characteristic function of

$$\int_0^1 z(t)Y(t)\,dt = \int_0^1 \Big[\int_0^1 z(t)f(t,s)\,dX(s)\Big]\,dt$$

is

$$\Phi(u) = \exp\Big\{-|u|^\alpha \int_0^1 \Big|\int_0^1 z(t)f(t,s)\,dt\Big|^\alpha ds\Big\} \quad \text{(apply Proposition 9).}$$

Now define $\Phi\colon I \to S$ by

$$\Phi(s) = \frac{f^s}{\|f^s\|_2}, \quad \text{where} \quad f^s(t) = f(t,s)$$

(we will also define $f_t(s) = f(t,s)$) and $\|\cdot\|_2$ is the L^2-norm). For $A \in \mathscr{B}(S)$, let

$$\Gamma_0(A) = \int_{\Phi^{-1}(A)} \|f^s\|_2^\alpha\,ds.$$

Then

$$\int_S |\langle z,\theta\rangle|^\alpha \Gamma_0(d\theta) = \int_0^1 \Big|\int_0^1 z(t)\frac{f(t,s)}{\|f^s\|_2}\,dt\Big|^\alpha \|f^s\|_2^\alpha\,ds$$
$$= \int_0^1 \Big|\int_0^1 z(t)f(t,s)\,dt\Big|^\alpha ds.$$

Hence, since the symmetric measure on the sphere is uniquely determined by μ, we have $\Gamma = \frac{1}{2}[\Gamma_0 + \Gamma_0^{-1}]$.

For the rest of the paper we make the following assumptions:

(i) $f \in D[I^2]$,

(ii) span$\{f_t\colon t \in I\}$ *is dense in* $L^2[I]$,

(iii) span$\{f^s\colon s \in I\}$ *is dense in* $L^2[I]$.

Consider the map $\Lambda\colon L^2 \to L^2$ defined by

$$(\Lambda x)(s) = \int_0^1 x(t) f(t, s)\, dt.$$

(i) and (iii) imply that Λ is an injection. (Note that Λ is clearly continuous.) By (i) and (ii) we have that the range of Λ is dense in L^2 and hence in L^α. Since $\|x\|_{\alpha,\Gamma} = \|\Lambda x\|_{L^\alpha}$, we obtain $B(\alpha, \Gamma) = L^\alpha[0, 1]$.

Now if $a = i^*(b^*)$, then

$$\int_0^1 b^*(s)\Big[\int_0^1 z(t) f(t, s)\, dt\Big] ds = \int_0^1 z(t)\Big[\int_0^1 b^*(s) f(t, s)\, ds\Big] dt.$$

Hence

$$(i^* b^*)(t) = \int_0^1 b^*(s) f(t, s)\, ds.$$

We record the above remarks as

PROPOSITION 10. *If f satisfies* (i), (ii) *and* (iii) (*above*), *then* $B(\alpha, \Gamma) = L^\alpha[I]$ *and*

$$i^*\big(B^*(\alpha, \Gamma)\big) = \Big\{x \in L^2[I]\colon x(t) = \int_0^1 b^*(s) f(t, s)\, ds \text{ for some } b^* \in \big[L^\alpha[I]\big]^*\Big\}.$$

COROLLARY 10.1. *If f satisfies* (i), (ii) *and* (iii) *and* $0 < \alpha < 1$, *then* $A_\mu = (0)$ *and moreover,* $a \neq 0$ *implies* $\mu_a \perp \mu$.

Proof. $(L^\alpha[I])^* = (0)$ for $0 < \alpha < 1$. Now apply Proposition 6.

COROLLARY 10.2. *Let* $\{X_t^{(\alpha)}\}$ *and* $\{X_t^{(\beta)}\}$ *be stable processes with indices* α *and* β ($\neq \alpha$), *respectively, such that* $\{X_t^{(\alpha)}\}$ *and* $\{X_t^{(\beta)}\}$ *have stationary, independent increments. Let*

$$Y(t) = \int_0^1 f(t, s)\, dX^{(\alpha)}(s), \quad \text{and} \quad Z(t) = \int_0^1 g(t, s)\, dX^{(\beta)}(s),$$

where f and g satisfy (i), (ii) *and* (iii). *Then the measures μ and ν induced by Y and Z, respectively, are singular.*

Proof. Apply Proposition 7.

7. In this section we will show that the set of admissible translates of the measure associated with the process X_t (defined previously) is trivial. Note that $X(t) = \int_0^1 1_{[0,t)}(s)\, dX(s)$, and $f(t, s) = 1_{[0,t)}(s)$. Therefore $x \in i^*(B^*(\alpha, \Gamma))$ if and only if $x(t) = \int_0^t g(s)\, ds$ where $g \in (L^\alpha[I])^*$. By Corollary 10.1, $i^*(B^*(\alpha, \Gamma)) = (0)$ if $0 < \alpha < 1$.

For $1 \leqslant \alpha < 2$ we must do a little more work. For $t_0 \in [0, 1)$ and $t_i \downarrow t_0$ define the map $\Lambda\colon D(I) \to \boldsymbol{R}^\infty$ by

$$\Lambda(x) = \left\langle \frac{x(t_i) - x(t_{i+1})}{(t_i - t_{i+1})^{1/\alpha}} \right\rangle_{i=1}^\infty .$$

By Proposition 8, $A_{\mu^\wedge} = \{a \in \boldsymbol{R}^\infty\colon \sum_k a_k^2 < \infty\}$. By Kakutani [7], $A_{\mu^\wedge} = (S_{\mu^\wedge})^c$ and therefore, by Proposition 4, $A_\mu \subseteq \Lambda^{-1}(A_{\mu^\wedge})$.

We now conclude that $x \in A_\mu$ implies that

$$\sum_{i=1}^\infty \frac{|x(t_i) - x(t_{i+1})|^2}{(t_i - t_{i+1})^{2/\alpha}} < \infty$$

for all sequences $\{t_i\}_{i=1}^\infty \subseteq I$, which are strictly decreasing.

If

$$(*) \qquad \frac{x(t_i) - x(t_{i+1})}{t_i - t_{i+1}}$$

converges to a non-zero constant, and

$$(**) \qquad \sum_{i=1}^\infty (t_i - t_{i+1})^{2/\beta} = \infty \quad \left(\frac{1}{\alpha} + \frac{1}{\beta} = 1\right),$$

then

$$\sum_{i=1}^\infty \frac{|x(t_i) - x(t_{i+1})|^2}{(t_i - t_{i+1})^{2/\alpha}} = \sum_{i=1}^\infty \frac{|x(t_i) - x(t_{i+1})|^2}{(t_i - t_{i+1})} (t_i - t_{i+1})^{2/\beta}$$
$$\geqslant \sum_{i=N_0}^\infty \left(\frac{c}{a}\right)^2 (t_i - t_{i+1})^{2/\beta} = \infty.$$

This would contradict $x \in A_\mu$. However,

$$\frac{x(t_i) - x(t_{i+1})}{t_i - t_{i+1}} = \frac{1}{t_i - t_{i+1}} \int_{t_{i+1}}^{t_i} g(s)\, ds \qquad (g \in L^\beta(I)).$$

If $g \neq 0$, it is not hard to construct a sequence $t_i \downarrow t_0$ such that $(*)$ and $(**)$ hold. Hence $g = 0$ a.e. and $x = 0$.

Remark. Let $\{Y_t\colon 0 \leqslant t \leqslant 1\}$ be a symmetric stable process with independent increments. Suppose also that $\{Y_t\}$ is stochastically continuous. Then $E[e^{iuY(t)}] = \exp\{-\gamma(t)|u|^\alpha\}$, where

(1) $\gamma(0) = 0$,

(2) γ is non-decreasing,

(3) γ is continuous.

If we let $\delta(t) = \inf\{s: \gamma(s) \geqslant t\}$, then $Z(t) = Y(\delta(t))$ is equivalent to $X(t)$. Define

$$A: D(I) \to D([0, \delta(1)]) \quad \text{by} \quad A(y)(t) = y(\delta(t)) \quad \text{for} \quad y \in D(I).$$

By Proposition 4(c), $A^{-1}(A_{\mu^\wedge}) = A_\mu$, where μ is the measure on $D(I)$ induced by Y. By the above results $A_{\mu^\wedge} = (0)$ and hence $A_\mu = (0)$.

Note that the non-existence of non-trivial admissible translates of X_t (or Y_t) also follows from Theorem 7.3 [5].

8. We end this paper with some questions and remarks.

We can always write $\Gamma = \Gamma_f + \Gamma_\infty$, where Γ_f sits on finite-dimensional sets, and $\Gamma_\infty(F) = 0$ for any finite-dimensional set. Let $\mu_f = [a, \Gamma_f]$ and $\mu_\infty = [a, \Gamma_\infty]$. Then $\mu = \mu_f * \mu_\infty$. If $\Gamma_f(F) > 0$ for some finite-dimensional set F, then $\Gamma_f = \Gamma_f^{(1)} + \Gamma_f^{(2)}$, where $\Gamma_f^{(1)}$ is Γ_f restricted to F and $\Gamma_f^{(2)} = \Gamma_f - \Gamma_f^{(1)}$. Hence $\mu_f = \mu_f^{(1)} * \mu_f^{(2)}$ and certainly $A_{\mu_f^{(1)}} \neq (0)$. Therefore $A_{\mu_f} \neq (0)$. Also, $A_\mu \geqslant A_{\mu_f} + A_{\mu_\infty}$.

QUESTION 1. *Is* $A_\mu = A_{\mu_f} + A_{\mu_\infty}$?

Note that in the case of $\{X(t)\}$, $\Gamma_f = 0$ and $A_\mu = (0)$.

QUESTION 2. *Is* A_{μ_∞} *always trivial?*

Recall that via Theorem 5 [16], if $a \in A_\mu$, then $\mu \sim \mu^P \times \mu^Q \equiv \nu$, where P is the projection of X onto the one-dimensional subspace generated by a, and $Q = I - P$ (I is the identity). Hence the measure on the sphere Γ_ν associated with ν has an atom, and the rest of its support is contained in the orthogonal complement of the span of $\{a\}$. Assume that one could show that $\mu_i = [a, \Gamma_i]$ and $\mu_1 \sim \mu_2$ implies $(\Gamma_{\mu_1})_\infty \sim (\Gamma_{\mu_2})_\infty$. Then since $a \in A_{\mu_\infty}$ implies $\mu \sim \mu^P \times \mu^Q \equiv \nu$, we would have $(\Gamma_\mu)_\infty \sim (\Gamma_\nu)_\infty$, which is impossible. Hence we would have $A_{\mu_\infty} = (0)$.

It is easy to see that Theorem 6 is directly related to Theorem 1 [2]. In [2] Dudley applies Theorem 1 to obtain a better bound on A_μ in the case where Γ sits on an orthonormal set. However, in the proof (Theorem 2) Dudley uses some non-linear functionals. It would be interesting to know if one could prove Theorem 2 [2] using only linear functionals.

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