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On the surjective (injective) envelope of strictly (co-) singular operators

by

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Abstract. The surjective envelope of strictly singular operators equals the injective envelope of strictly cosingular operators and an operator $T: X \to Y$ belongs to these envelopes if and only if one of the following equivalent conditions is fulfilled:

(a) For every bounded sequence $(x_n) \subset X$ there is a weak Cauchy subsequence of (Tx_n).

(b) $T(X)$ contains no subspace isomorphic to l_1.

(c) $T(X)$ has no quotient space isomorphic to l_2 with 2-absolutely summing quotient map.

Introduction. A. Pietsch has remarked that the strictly singular operators form an injective but not surjective operator ideal ([7], 5.1.4, 5.1.5) and that the ideal of strictly cosingular operators is surjective but not injective ([7], 5.2.4, 5.2.5). In this paper we shall prove the characterization of the surjective envelope of strictly singular operators and the injective envelope of strictly cosingular operators cited in the abstract (Theorem 3).

First, we explain some notions and definitions.

Let X, Y, Z always be Banach spaces and let U_X be the closed unit ball of X. By $B(X, Y)$ [$W(X, Y)$] we mean the *space of all continuous linear* [*weakly compact*] *mappings from X to Y*. An operator $T \in B(X, Y)$ is an *isomorphism*, if it is one-to-one with closed range. For every X, X^∞ is the space $l^\infty(U_{X'})$ and $J_X^\infty: X \to X^\infty$ is the usual embedding. Furthermore, X^1 is $l_1(U_X)$ and by Q_X^1 we denote the usual quotient map $Q_X^1: X^1 \to X$. Let M be a normed space and let $1 \leqslant p \leqslant \infty$. An operator $T: M \to Z$ is *p-integral* [6], if T admits the following factorization:

$$\begin{tikzcd} M \arrow[r, "T"] \arrow[d, "J_M^\infty"'] & Z \\ C(U_{M'}) \arrow[r, "I"] & L_p(U_{M'}, \mu) \arrow[u] \end{tikzcd}$$

where μ is a probability measure and I the canonical injection.

An *operator ideal* $\boldsymbol{A}$ (see: [7], 1.1.1 or [12]) is called *proper*, if for all infinite-dimensional Banach spaces X we have:

$$\boldsymbol{A}(X, X) \neq \boldsymbol{B}(X, X).$$

An operator ideal $\boldsymbol{A}$ is called *surjective* [*injective*], if for all surjective operators $Q \in \boldsymbol{B}(Z, X)$ [isomorphisms $J \in \boldsymbol{B}(Y, Z)$] and $S \in \boldsymbol{B}(X, Y)$ we have: $SQ \in \boldsymbol{A}(Z, Y)$ [$JS \in \boldsymbol{A}(X, Z)$] implies $S \in \boldsymbol{A}(X, Y)$.

By [7], 3.2.2, 3.6.2, the *surjective envelope* $\boldsymbol{A}^s$ [*injective envelope* $\boldsymbol{A}^j$] of an operator ideal $\boldsymbol{A}$, that is the smallest surjective [injective] operator ideal containing $\boldsymbol{A}$, is characterized as follows

(*) $$\boldsymbol{A}^s(X, Y) = \{T \in \boldsymbol{B}(X, Y):\ TQ_X^1 \in \boldsymbol{A}(X^1, Y)\},$$
$$\boldsymbol{A}^j(X, Y) = \{T \in \boldsymbol{B}(X, Y):\ J_Y^\infty T \in \boldsymbol{A}(X, Y^\infty)\}.$$

In order to describe the envelopes of strictly singular and strictly cosingular operators, we shall need the following notion from [2]: Let $\mathscr{F}$ be a class of infinite-dimensional Banach spaces.

(a) An operator $T \in \boldsymbol{B}(X, Y)$ is said to be *$\mathscr{F}$-singular* provided that for no $F \in \mathscr{F}$ does there exist isomorphisms $J_1\colon F \to X$, $J_2\colon F \to Y$ such that the diagram

$$\begin{tikzcd} X \arrow[rr, "T"] & & Y \\ & F \arrow[ul, "J_1"] \arrow[ur, "J_2"'] & \end{tikzcd}$$

commutes.

(b) An operator $T \in \boldsymbol{B}(X, Y)$ is said to be *$\mathscr{F}$-cosingular*, provided that for no $F \in \mathscr{F}$ does there exist surjective operators $Q_1\colon X \to F$, $Q_2\colon Y \to F$, such that the diagram

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "Q_1"'] & & Y \arrow[dl, "Q_2"] \\ & F & \end{tikzcd}$$

commutes.

If $\mathscr{F}$ contains all infinite-dimensional Banach spaces, then the $\mathscr{F}$-singular operators are called *strictly singular* (or *semi-compact*; [8], p. 252, or *Kato-operators* [7], 5.1.1) and the $\mathscr{F}$-cosingular operators are called *strictly cosingular* (or *cosemicompact*, [8], p. 257, or *Pełczyński-operators*, [7], 5.2.1).

By $\boldsymbol{S}$, resp. $\boldsymbol{C}$ we denote the ideal of strictly singular resp. strictly cosingular operators.

Characterization of $\boldsymbol{S}^s$ and $\boldsymbol{C}^j$.

1. Proposition. *Let $T \in \boldsymbol{B}(X, Y)$. Then the following are equivalent:*

(a) $T \in \boldsymbol{S}^s(X, Y)$;

(b) *$T(X)$ contains no subspace isomorphic to l_1;*

(c) *T is l_1-singular.*

Proof. (a) $\Rightarrow$ (b). Let M be a subspace of $T(X)$, isomorphic to l_1 and let $Q\colon Z \to X$ be a surjective operator. Define $N = Q^{-1}T^{-1}(M)$. Since $TQ|_N\colon N \to M$ is open, there exists a sequence $(z_n) \subset N$, $\|z_n\| \leqslant C < \infty$ such that $(y_n) = (TQz_n)$ is equivalent to the unit vector basis of l_1. Therefore, we can choose a constant C' such that

$$\Big\|\sum_{n=1}^\infty \lambda_n y_n\Big\| \leqslant \|TQ\| \cdot \Big\|\sum_{n=1}^\infty \lambda_n z_n\Big\| \leqslant C \cdot \|TQ\| \cdot \sum_{n=1}^\infty |\lambda_n| \leqslant C' \cdot C \cdot \|TQ\| \cdot \Big\|\sum_{n=1}^\infty \lambda_n y_n\Big\|$$

for all $(\lambda_n) \in l_1$. Hence, TQ is not strictly singular.

(b) $\Rightarrow$ (c). Trivial.

(c) $\Rightarrow$ (a). If (a) is false we have (by (*)): $TQ_X^1 \notin \boldsymbol{S}(X^1, Y)$. Then TQ_X^1 is not l_1-singular because (e.g. by [9], Corollary 1, p. 29) every closed infinite-dimensional subspace of $l_1(\Gamma)$ contains a subspace isomorphic to l_1. Thus T is not l_1-singular.

2. Proposition. *We have $T \in \boldsymbol{C}^j(X, Y)$ if and only if there is no infinite-dimensional complete quotient space of $T(X)$ with p-integral quotient map for some $2 \leqslant p < \infty$.*

Proof. a) Let $T \notin \boldsymbol{C}^j(X, Y)$. By (*) there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[drr, "h_2"'] & Y \arrow[r, "J_Y^\infty"] & Y^\infty \arrow[d, "h_1"] \\ & & Z \end{tikzcd}$$

where h_1, h_2 are surjective operators and $\dim Z = \infty$.

By [9], Theorem 3.7, h_1 is either weakly compact or there is a subspace M of Y^∞, isomorphic to l^∞, such that $h_1|_M$ is an isomorphism. Then $h_1(M)$ is complemented in Z and has a reflexive quotient space since l^∞ has such a quotient space. Thus, in the above diagram, we may assume without loss of generality that Z is reflexive.

Now from [10], Corollary 11, it follows that h_1 is p-absolutely summing for some $p \geqslant 2$. Since Y^∞ is isomorphic to some space $C(K)$, K compact, h_1 must even be p-integral ([6], Satz 45). So $h_1 \cdot J_Y^\infty|_{T(X)}$ is p-integral and surjective.

b) Let $h\colon T(X) \to Z$ be a p-integral surjective operator. In order to prove $T \notin \boldsymbol{C}^j(X, Y)$ we may assume $\overline{T(X)} = Y$ (because $\boldsymbol{C}^j$ is injective).

Now choose an operator $h_1\colon C(U_{Y'})\to Z$ such that $h = h_1 J_Y^\infty$. As a p-integral operator, h must be weakly compact. So we get the following commutative diagram

$$\begin{tikzcd}
X \arrow[r, "T"] \arrow[drr, "h_2"'] & Y \arrow[r, "J_Y^\infty"] & C(U_{Y'}) \arrow[r, hook] & C(U_{Y'})'' \arrow[dl, "h_1''"] \\
 & & Z &
\end{tikzcd}$$

where $h_2 = hT$ and $C(U_{Y'})''$ is a $\mathscr{P}$-space. Hence, by $(*)$, $T \notin C^j(X, Y)$.

3. Theorem. *Let $T \in B(X, Y)$. Then the following are equivalent*:

(a) *For every sequence $(x_n) \subset U_X$, (Tx_n) has a weak Cauchy subsequence.*

(b) $T \in C^j(X, Y)$.

(c) *There is no quotient space of $T(X)$ isomorphic to l_2 with 2-absolutely summing quotient map.*

(d) $T \in S^s(X, Y)$.

(e) *No subspace of $T(X)$ is isomorphic to l_1.*

Proof. (a) $\Rightarrow$ (b). Let $h\colon T(X)\to Z$ be a surjective p-integral operator. As a p-integral operator h transforms weak Cauchy sequences into Cauchy sequences with respect to the norm ([6], Satz 20). Then, by (a), $hT\colon X\to Z$ is surjective and compact. Consequently, Z is finite dimensional.

(b) $\Rightarrow$ (c): see Proposition 2. (d) $\Leftrightarrow$ (e): see Proposition 1.

(c) $\Rightarrow$ (e). Let M be a subspace of $T(X)$, isomorphic to l_1. Consider a surjective operator $h\colon M\to l_2$. By [3], Theorem 4.1, h must be 1-absolutely summing, hence 2-integral ([6], p. 43) and may be factored as shown by the following diagram:

$$\begin{tikzcd}
T(X) \arrow[drr, dashed, "g"] & & \\
M \arrow[u, phantom, "\cup" description] \arrow[rr, hook] \arrow[d, "h"'] & & C(U_{M'})'' \arrow[dll, "h_1"] \\
l_2 & &
\end{tikzcd}$$

where h_1 is 2-integral. Since $C(U_{M'})''$ is a $\mathscr{P}$-space, there is an operator g making the diagram commutative. Thus, (c) is false.

(e) $\Rightarrow$ (a). This follows from a deep result of Rosenthal ([11], [1]): If no subsequence of $(y_n) \subset Y$ is a weak Cauchy sequence, then there is a subsequence (y_{n_k}) of (y_n), equivalent to the unit vector basis of l_1.

Part of these equivalences also follows from [4], Proposition 3, which was discovered independently.

4. Corollary. *There is no class $\mathscr{F}$ of Banach spaces such that the class of $\mathscr{F}$-cosingular operators equals C^j. But for every $T \in B(X, Y)$ we have*:

$$T\ l_2\text{-}cosingular \Rightarrow T \in C^j \Rightarrow T\ l^\infty\text{-}cosingular.$$

Proof. If $T \notin C^j$, we see as in the proof of Proposition 2 (by 3(c)) that T cannot be l_2-cosingular. If T is not l^∞-cosingular, there is an operator h, making the diagram

$$\begin{tikzcd}
X \arrow[r, "T"] \arrow[drr] & Y \arrow[r, "J_Y^\infty"] & Y^\infty \arrow[d, dashed, "h"] \\
 & & l^\infty
\end{tikzcd}$$

commutative. By $(*)$ it follows, that $T \notin C^j$.

Finally, we assume that C^j is equal to a class of $\mathscr{F}$-cosingular operators. Since the embedding $l_1 \hookrightarrow l^\infty$ does not belong to C^j (3.(e)), $\mathscr{F}$ contains a separable quotient space Z of l^∞. By [9], Theorem 3.7, Z must be reflexive and we conclude: $\mathrm{Id}_Z \in C^j$((3(a)). On the other hand, Id_Z is not $\mathscr{F}$-cosingular contradicting our assumption.

5. Corollary. *Let X be a Banach space. Then the following are equivalent*:

(a) *X contains no subspace isomorphic to l_1,*

(b) *For every Y, we have $B(X, Y) = S^s(X, Y)$,*

(c) *For every Y, we have $B(X, Y) = C^j(X, Y)$.*

Proof. If there is some $T \notin S^s(X, Y)$, then, by 1(c), X contains a subspace isomorphic to l_1. Conversely, if M is a subspace of X isomorphic to l_1, we extend the embedding $M \hookrightarrow l^\infty$ to X.

Since every $\mathscr{L}_1$-space (see [3]) is weakly complete and has the Dunford–Pettis property (see [8], C II, § 7), Corollary 6 is a partial generalization of [5], Theorem 1.

6. Corollary. *Let Y be weakly complete. Then for every X we have*

$$S^s(X, Y) = C^j(X, Y) = W(X, Y).$$

Moreover, if in addition X, resp. Y, has the Dunford–Pettis property we have

$$S(X, Y) = W(X, Y), \quad resp.\ C(X, Y) = W(X, Y).$$

Proof. Theorem 3 and [8], C II Theorem 7.7.

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A criterion for compositions of (p, q)-absolutely summing operators to be compact

by

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Abstract. If $(S_j)_{j=1,2,\ldots,M}$ are $(p_j, 2)$-absolutely summing operators, and $\sum_{j=1}^{M} p_j^{-1} > 1/2$, then the composition $S_M S_{M-1} \ldots S_1$ is compact.

Results. Let $\infty > p \geqslant q \geqslant 1$ and let X, Y be normed linear spaces. Recall that a bounded linear operator $S\colon X \to Y$ is said to be *(p, q)-absolutely summing* if there exists a positive constant C such that for all finite sequences $x_1, x_2, \ldots, x_n$ in X $(n = 1, 2, \ldots)$

$$\Big(\sum_{j=1}^{n} \|Sx_j\|^p\Big)^{1/p} \leqslant C \sup_{x^* \in X^*, \|x^*\| \leqslant 1} \Big(\sum_{j=1}^{n} |x^*(x_j)|^q\Big)^{1/q}.$$

The greatest lower bound of the constants C satisfying the above inequality is denoted by $\pi_{p,q}(S)$.

The main result of the present paper is:

THEOREM 1. *Let M be a positive integer, let X_k be Banach spaces $(k = 0, 1, \ldots, M)$, and let $S_k\colon X_{k-1} \to X_k$ be $(p_k, 2)$-absolutely summing operators $(2 \leqslant p_k < \infty$ for $k = 1, 2, \ldots, M)$. Then the condition*

$$\sum_{k=1}^{M} p_k^{-1} > 2^{-1}$$

implies the compactness of the composition $S_M S_{M-1} \cdot \ldots \cdot S_1$.

Combining Theorem 1 with the well-known fact (cf. Kwapień [4]) that if $T\colon X \to Y$ is a (p, q)-absolutely summing operator, then T is also $(\tilde{p}, \tilde{q})$-absolutely summing for every pair $(\tilde{p}, \tilde{q})$ such that $p^{-1} - q^{-1} = \tilde{p}^{-1} - \tilde{q}^{-1}$ and $\tilde{p} > p$, we get

COROLLARY 1. *Let M be a positive integer, let X_k be Banach spaces $(k = 0, 1, \ldots, M)$, let $S_k\colon X_{k-1} \to X_k$ be (p_k, q_k)-absolutely summing operators, and let $0 \leqslant q_k^{-1} - p_k^{-1} < 2^{-1}$, $1 \leqslant q_k \leqslant 2$ for $k = 1, \ldots, M$. Then the condition*

$$\sum_{k=1}^{M} 2^{-1} + p_k^{-1} - q_k^{-1} > 2^{-1}$$