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# Topological conditional entropy

by

MICHAŁ MISIUREWICZ (Warszawa)

**Abstract.** A new invariant of discrete dynamical systems — topological conditional entropy — is introduced. Its vanishing (i.e. asymptotical $h$-expansiveness) implies the upper semi-continuity of measure-theoretic entropy, regarded as a function of an invariant regular normed Borel measure (and in particular — the existence of a measure with maximal entropy). All the endomorphisms of compact topological groups are shown to be asymptotically $h$-expansive. Expansiveness or $h$-expansiveness also imply asymptotical $h$-expansiveness.

A very short proof is given of the Goodwyn's theorem on the bounding of measure-theoretic entropy by topological entropy. Also a new proof is given of the formula for topological entropy of a product of transformations.

**§ 0. Introduction.** In the present paper we consider continuous transformations of compact Hausdorff spaces into themselves (cascades). We define a new invariant, which we call 'topological conditional entropy'. It is based on the notion of topological entropy ([1]), and it is also close to the notions of $h$-expansiveness ([5]) and asymptotical $h$-expansiveness ([15]). Some of its properties are the same as those of topological entropy. Topological conditional entropy is also connected with measure-theoretic entropy, regarded as a function of an invariant regular normed Borel measure. Unfortunately, it cannot be defined by means of the above function, and therefore topological tools cannot be avoided in some proofs (contrary to some results concerning usual topological entropy admitting a proof by a reduction to simple measure-theoretic facts).

**§ 1. Definitions.** We shall consider a continuous transformation $f\colon X \to X$ of a non-empty compact Hausdorff space $X$ into itself. Such a pair $(X, f)$ is called a *cascade*.

A set $Y \subset X$ is called *invariant* (*under* $f$) if $fY \subset Y$; it is called *strictly invariant* if $fY = Y$. If $(Z, g)$ is also a cascade, and if there exists a continuous surjection $p\colon X \to Z$ such that $g \circ p = p \circ f$, then $(Z, g)$ is called a *factor* of $(X, f)$.

Denote by $\mathscr{P}(X)$ the set of all covers of the space $X$ containing a finite subcover, and by $\mathfrak{A}(X)$ the set of all open finite covers of $X$ (we write simply $\mathscr{P}$ and $\mathfrak{A}$ if we consider only one space).

For $Y \subset X$, $A \in \mathscr{P}$, write $Y \prec A$ if there exists an $a \in A$ such that $Y \subset a$. For $A$, $B \in \mathscr{P}$ write $A \geqslant B$ ($A$ is a refinement of $B$) if $a \prec B$ for every $a \in A$. For a family $\{A_i\}_{i \in I}$, let

$$\bigvee_{i \in I} A_i = \{\bigcap_{i \in I} a_i\colon\ a_i \in A_i \text{ for } i \in I\}$$

(or $A_{i_1} \vee \ldots \vee A_{i_n}$ for a finite set $I = \{i_1, \ldots, i_n\}$).

$\operatorname{Card} I$ denotes the *cardinality* of the set $I$. If $\operatorname{Card} I < \infty$ and $A_i \in \mathscr{P}$ (resp. $\mathfrak{A}$) for $i \in I$ then, of course, also $\bigvee_{i \in I} A_i \in \mathscr{P}$ (resp. $\mathfrak{A}$). The operation '$\vee$' preserves the relation '$\leqslant$'. The sets $\mathscr{P}$ and $\mathfrak{A}$ are directed by the relation $\leqslant$. Clearly, for $A, B \in \mathscr{P}$, $A \vee B \geqslant A$. For $A \in \mathscr{P}$, $n \in \boldsymbol{Z}$, let $f^n A = \{f^n a\colon\ a \in A\}$; for $n \in \boldsymbol{N}$, $A_f^n = \bigvee_{i=0}^{n-1} f^{-i} A$ (if we consider only one transformation, we write simply $A^n$).

For a non-empty set $Y \subset X$ and a cover $A \in \mathscr{P}$ write

$$N(Y, A) = \min\{\operatorname{Card} C\colon\ C \subset A,\ Y \subset \bigcup C\}.$$

For the empty set put $N(\varnothing, A) = 1$. For $A$, $B \in \mathscr{P}$ write

$$N(A \mid B) = \max_{b \in B} N(b, A).$$

Further, $N(A) = N(X, A)$. We write also $\Theta(X) = \{X\} \in \mathfrak{A}$ (if we consider only one space, we write simply $\Theta$). Of course, we have

$$(1.1) \qquad N(A) = N(A \mid \Theta) \quad \text{for} \quad A \in \mathscr{P}.$$

Now we list the simplest properties of the function $N$, similar to the properties of the functions $\exp H$ for measure-theoretic entropy (see [2]).

Let $A, B, C, D \in \mathscr{P}$; $Y, Z \subset X$. The following inequalities hold:

$$(1.2) \qquad N(Y, A) \leqslant N(Z, B) \quad \text{for} \quad B \geqslant A,\ Y \subset Z,$$

$$(1.3) \qquad N(A \mid B) \leqslant N(C \mid D) \quad \text{for} \quad C \geqslant A,\ B \geqslant D,$$

$$\begin{array}{ll} (1.4) & N(f^{-1}A \mid f^{-1}B) \leqslant N(A \mid B), \\ (1.5) & N(f^{-1}Y, f^{-1}A) \leqslant N(Y, A), \end{array} \left| \begin{array}{l} \text{(if } f \text{ is surjective,} \\ \text{then the equalities hold)} \end{array} \right.$$

$$(1.6) \qquad N(A \vee B \mid C) \leqslant N(A \mid C) \cdot N(B \mid A \vee C),$$

$$(1.7) \qquad N(A \vee B \mid C \vee D) \leqslant N(A \mid C) \cdot N(B \mid D),$$

$$(1.8) \qquad N(Y, A \vee B) \leqslant N(Y, A) \cdot N(Y, B),$$

$$(1.9) \qquad N(A) \leqslant N(B) \cdot N(A \mid B),$$

$$(1.10) \qquad N(A \mid B) \leqslant N(A \mid C) \cdot N(C \mid B).$$

Inequalities (1.2)–(1.5) are obvious. For the proof of (1.6) let us fix $c \in C$. Then there exists $E \subset A$ such that $c \subset \bigcup E$ and $\operatorname{Card} E \leqslant N(A \mid C)$.



Now for any $a \in E$ there exists $B_a \subset B$ such that $c \cap a \subset \bigcup B_a$ and $\operatorname{Card} B_a \leqslant N(B \mid A \vee C)$. The family $\{a \cap b\colon\ a \in E,\ b \in B_a\}$ is a subfamily of $A \vee B$, its cardinality is not greater than $N(A \mid C) \cdot N(B \mid A \vee C)$ and its union contains $c$.

Inequalities (1.7)–(1.10) can easily be obtained from (1.6) by the use of (1.1) and (1.3).

In view of (1.4) and (1.7) the sequence $(\log N(A^n \mid B^n))_{n=1}^{\infty}$ is subadditive; therefore there exists a limit

$$\lim_{n\to\infty} \frac{1}{n} \log N(A^n \mid B^n) = h(f, A \mid B),$$

which will be called the *conditional entropy of $f$ on the cover $A$ with respect to the cover $B$*. Moreover, we have

$$(1.11) \qquad h(f, A \mid B) \leqslant \log N(A \mid B).$$

Notice that if $A$ is an open cover, then $h(f, A \mid \Theta)$ is equal to the familiar topological entropy of $f$ on the cover $A$ $h(f, A)$.

Further, let

$$h(f, Y, A) = \limsup_{n\to\infty} \frac{1}{n} \log N(Y, A^n)$$

for $Y \subset X$, $A \in \mathscr{P}$. This number is, in general, different from the analogous ones appearing in [4], [6].

Of course, $h(f, X, A) = h(f, A)$. From (1.2)–(1.10) we obtain the following inequalities (we list only those which will be necessary in the sequel):

$$(1.12) \qquad h(f, Y, A) \leqslant h(f, Z, B) \quad \text{for} \quad B \geqslant A,\ Y \subset Z,$$

$$(1.13) \qquad h(f, A \mid B) \leqslant h(f, C \mid D) \quad \text{for} \quad C \geqslant A,\ B \geqslant D,$$

$$\begin{array}{ll} (1.14) & h(f, f^{-1}A \mid f^{-1}B) \leqslant h(f, A \mid B), \\ (1.15) & h(f, f^{-1}Y, f^{-1}A) \leqslant h(f, Y, A), \end{array} \left| \begin{array}{l} \text{(if } f \text{ is surjective,} \\ \text{then the equalities hold)} \end{array} \right.$$

$$(1.16) \qquad h(f, A) \leqslant h(f, B) + h(f, A \mid B),$$

$$(1.17) \qquad h(f, A \mid B) \leqslant h(f, A \mid C) + h(f, C \mid B).$$

In view of (1.13) there exists a limit (finite or infinite)

$$\lim_{A \in \mathfrak{A}} h(f, A \mid B) = \sup_{A \in \mathfrak{A}} h(f, A \mid B) = h(f \mid B).$$

We call $h(f \mid B)$ the *conditional entropy of $f$ with respect to the cover $B$*. Notice that $h(f) = h(f \mid \Theta)$ is the topological entropy of $f$.

In view of (1.12) there also exists a limit

$$\lim_{A \in \mathfrak{A}} h(f, Y, A) = \sup_{A \in \mathfrak{A}} h(f, Y, A) = h(f, Y).$$

Of course, $h(f, X) = h(f)$.

From (1.12)–(1.17) we obtain:

$$h(f,Y)\leqslant h(f,Z)\quad\text{for}\quad Y\subset Z, \tag{1.18}$$

$$h(f|A)\leqslant h(f|B)\quad\text{for}\quad A\geqslant B, \tag{1.19}$$

$$\left.\begin{aligned} &h(f|f^{-1}B)=h(f|B), &(1.20)\\ &h(f,f^{-1}Y)=h(f,Y), &(1.21)\end{aligned}\right\}\ \text{if } f \text{ is a homeomorphism}$$

$$h(f)\leqslant h(f,B)+h(f|B). \tag{1.22}$$

To prove (1.20) and (1.21) it should be noted that $\{f^{-1}A:\ A\in\mathfrak{A}\}=\mathfrak{A}$ if $f$ is a homeomorphism.

We have, from the definition,

$$h(f,A|B)\leqslant h(f|B)\quad\text{for}\quad A\in\mathfrak{A}. \tag{1.23}$$

In view of (1.19) we can take the limit once more:

$$\lim_{B\in\mathfrak{A}} h(f|B)=\inf_{B\in\mathfrak{A}} h(f|B)=h^*(f).$$

We call $h^*(f)$ the (*topological*) *conditional entropy of the transformation* $f$ (or, more precisely, of the *cascade* $(X,f)$).

From the definition we have

$$h^*(f)\leqslant h(f|B)\quad\text{for}\quad B\in\mathfrak{A}. \tag{1.24}$$

Putting $B=\Theta$, we obtain

$$h^*(f)\leqslant h(f). \tag{1.25}$$

**§ 2. Connection with $h$-expansiveness.** We first recall some notions from Bowen's paper [5]. Let $(X,d)$ be a compact metric space and $f\colon X\to X$ a continuous transformation. The function $d_n$, defined by

$$d_n(x,y)=\max_{0\leqslant i\leqslant n-1} d(f^ix,f^iy),$$

is a metric on $X$, equivalent to $d$ $(n=1,2,\ldots)$. For a compact set $Y\subset X$, $a>0$, $r_n(Y,a)$ is the smallest cardinality of an $a$-network for $Y$ in the metric $d_n$ (an $(n,a)$-spanning set for $Y$). Further, put

$$\bar r(Y,a)=\limsup_{n\to\infty}\frac{1}{n}\log r_n(Y,a),\qquad \tilde h(f,Y)=\lim_{a\to 0}\bar r(Y,a).$$

$B_a^n(x)$ is a closed ball with the centre $x$ and radius $a$ in the metric $d_n$,

$$\Phi_a(x)=\bigcap_{n=1}^{\infty}B_a^n(x).$$

Then

$$h_f^*(a)=\sup_{x\in X}\tilde h(f,\Phi_a(x)).$$



The transformation $f$ is called *$h$-expansive* if there exists an $a>0$ such that $h_f^*(a)=0$, and is called *asymptotically $h$-expansive* ([15]) if $\lim_{a\to 0}h_f^*(a)=0$.

We shall point out the connection between these notions and the notions introduced in Section 1.

LEMMA 2.1. *Let $Y\subset X$; $A,E\in\mathfrak{A}$; $n\in N$; $\operatorname{diam}A<a<\frac{1}{2}L$, where $L$ is the Lebesgue number for $E$. Then*

$$N(Y,A^n)\leqslant r_n(Y,a)\leqslant N(Y,E^n).$$

Proof. Let $e$ be the minimal $(n,a)$-spanning set for $Y$. For $x\in e$, $k=0,\ldots,n-1$, we have $B_a(f^kx)\prec E$; hence $B_a^n(x)\prec E^n$. But $\bigcup_{x\in e}B_a^n(x)\supset Y$, and so $r_n(Y,a)\leqslant N(Y,E^n)$. Let $C\subset A^n$, $Y\subset\bigcup C$, $\operatorname{Card}C=N(Y,A^n)$. Take one point from every $c\in C$ and let $b$ denote the set thus obtained. If $x\in c=\bigcap_{k=0}^{n-1}f^{-k}a_k\in C$, $a_k\in A$ for $k=0,\ldots,n-1$, then $a_k\subset B_a(f^kx)$ for $k=0,\ldots,n-1$; therefore, $c\subset B_a^n(x)$. Hence $b$ is an $(n,a)$-spanning set and thus $N(Y,A^n)\leqslant r_n(Y,a)$. ■

THEOREM 2.1. *Let $X$ be a compact metric space, and let $(X,f)$ be a cascade, $a>0$, $A,E\in\mathfrak{A}$, $\operatorname{diam}A<a<\frac{1}{2}L$, where $L$ is the Lebesgue number for $E$. Then*

$$h(f|A)\leqslant h_f^*(a)<h(f|E).$$

Proof. Taking in Lemma 2.1 the limit with respect to $A,E\in\mathfrak{A}$, $a\to 0$, we obtain $\tilde h(f,Y)=h(f,Y)$ for a compact set $Y$. Thus, $h_f^*(a)=\sup_{x\in X}h(f,\Phi_a(x))$. Let $C\in\mathfrak{A}$. The number $2a$ is smaller than the Lebesgue number of $E$; therefore (we use (1.2)): $N(\Phi_a(x),C^n)\leqslant N(B_a^n(x),C^n)\leqslant N(C^n|E^n)$ for $n=1,2,\ldots$, and thus $h(f,\Phi_a(x),C)\leqslant h(f,C|E)$. Taking the limit with respect to $C\in\mathfrak{A}$, we obtain $h(f,\Phi_a(x))\leqslant h(f|E)$, and therefore $h_f^*(a)\leqslant h(f|E)$.

Let $\beta>0$, $\delta>0$. Proposition 2.2 from [5] states that there exists a $c>0$ such that $r_n(B_a^n(x),\delta)\leqslant c\exp[(\beta+h_f^*(a))n]$ for $n=1,2,\ldots$, $x\in X$. Let $D\in\mathfrak{A}$ be a cover with the Lebesgue number greater than $2\delta$. In view of Lemma 2.1, (1.2) and the inequality $\operatorname{diam}A<a$, we have

$$N(D^n|A^n)\leqslant\sup_{x\in X}N(B_a^n(x),D^n)\leqslant\sup_{x\in X}r_n(B_a^n(x),\delta)\leqslant c\exp[(\beta+h_f^*(a))n].$$

When $n\to\infty$, we obtain $h(f,D|A)\leqslant h_f^*(a)+\beta$. Let $\delta\to 0$, $\beta\to 0$; we have $h(f,D|A)\leqslant h_f^*(a)$ for every $D\in\mathfrak{A}$ and therefore $h(f|A)\leqslant h_f^*(a)$. ■

COROLLARY 2.1. *For a cascade $(X,f)$, where $X$ is a metric space,*

(a) *$f$ is $h$-expansive* $\Leftrightarrow$ $\exists_{A\in\mathfrak{A}}\ h(f|A)=0$,

(b) *$f$ is asymptotically $h$-expansive* $\Leftrightarrow$ $h^*(f)=0$. ■

In the general case, where the space $X$ is not necessarily metrizable, the right-hand sides of (a) and (b) can serve as definitions of $h$-expansiveness and asymptotical $h$-expansiveness, respectively.

**§ 3. Properties of conditional entropy.** In this section we examine the basic properties of conditional entropy. The methods employed will enable us to simplify the proofs of some theorems concerning usual topological entropy.

In some proofs we shall use arguments of the theory of uniform structures (see Kelley [14]). For a compact Hausdorff space $X$, denote the set of all open symmetric neighbourhoods of the diagonal in $X\times X$ by $\mathfrak{N}(X)$ (simply $\mathfrak{N}$ if we consider only one space). It is the base for the uniform structure on $X$. The topology in $X$ defined by this structure coincides with the original one. For any open cover $A$ of $X$ there exists $L\in\mathfrak{N}$ such that $\{y\colon (x,y)\in L\}\prec A$ for any $x\in X$. Every such $L$ will be called a *Lebesgue number for* $A$.

Two important properties of conditional entropy have already been proved: they are (1.22) and (1.25). The next ones are:

PROPOSITION 3.1. *$h^*(f^k) = |k|\cdot h^*(f)$ for $k\in \mathbf{N}$; or for $k\in \mathbf{Z}$ if $f$ is a homeomorphism.*

Proof. Observe that for a cover $A$ we have $(A_f^k)_{f^k}^n = A_f^{kn}$ and that for a fixed $k$ the family $\{A_f^k\colon A\in\mathfrak{A}\}$ is cofinal with $\mathfrak{A}$. This implies the proposition in the general case; in the case of a homeomorphism apply also (1.14). ■

PROPOSITION 3.2. *Let $Y$ be a non-empty closed invariant subset of $X$. Then $h^*(f|_Y)\leqslant h^*(f)$.*

Proof. Let $A, B\in\mathfrak{A}(X)$. Then we have $N\big((A|_Y)|(B|_Y)\big)\leqslant N(A|B)$, $(A|_Y)^n = A^n|_Y$, and therefore $h\big(f|_Y, (A|_Y)|(B|_Y)\big)\leqslant h(f, A|B)$. To end the proof notice that $\{A|_Y\colon A\in\mathfrak{A}(X)\} = \mathfrak{A}(Y)$. ■

From (1.22) immediately follows

PROPOSITION 3.3. *If $h(f) = \infty$, then also $h^*(f) = \infty$.* ■

PROPOSITION 3.4. *Let $X = Y_1\cup\ldots\cup Y_r$, where the sets $Y_i$ are non-empty, closed and invariant ($i = 1,\ldots,r$). Then $h^*(f) = \max\limits_{1\leqslant i\leqslant r} h^*(f|_{Y_i})$.*

Proof. The inequality $\max\limits_{1\leqslant i\leqslant r} h^*(f|_{Y_i})\leqslant h^*(f)$ follows from Proposition 3.2. Let $A, B\in\mathfrak{A}(X)$. For $n = 1, 2, \ldots$, $N(A^n|B^n)\leqslant \sum\limits_{i=1}^{r} N\big((A^n|_{Y_i})|(B^n|_{Y_i})\big)$; hence

$$h(f, A|B)\leqslant \max_{1\leqslant i\leqslant r} h\big((f|_{Y_i}), (A|_{Y_i})|(B|_{Y_i})\big)\leqslant \max_{1\leqslant i\leqslant r} h\big((f|_{Y_i})|(B|_{Y_i})\big).$$

To end the proof notice that for any $B_1,\ldots,B_r$ such that $B_i\in\mathfrak{A}(Y_i)$, $i = 1,\ldots,r$, there exists a $B\in\mathfrak{A}(X)$ such that $B|_{Y_i}\geqslant B_i$ for $i = 1,\ldots,r$. ■



A set $Y\subset X$ is called *wandering* if $f^{-n}Y\cap Y = \emptyset$ for $n = 1, 2, \ldots$ A point $x\in X$ is called *non-wandering* if it has no wandering open neighbourhood.

Let $\Omega$ denote the set of all non-wandering points for a transformation $f\colon X\to X$. Of course, $\Omega$ is non-empty, closed and invariant. We shall prove the following theorem, an analogue of Bowen's result concerning usual topological entropy (cf. Bowen [3]):

THEOREM 3.1. *Let $(X, f)$ be a cascade, and let $\Omega$ be the set of non-wandering points for $f$. Then the conditional entropy of $f$ is attained on $\Omega$, i.e., $h^*(f) = h^*(f|_\Omega)$.*

Proof. The inequality $h^*(f|_\Omega)\leqslant h^*(f)$ follows from Proposition 3.2 We shall prove the reverse inequality.

Let $B = \{b_1,\ldots,b_r\}\in\mathfrak{A}(\Omega)$. There exists a closed cover of the space $\Omega$, $\{q_1,\ldots,q_r\}$, such that $q_i\subset b_i$ for $i = 1,\ldots,r$. The sets $q_i$ and $\Omega\setminus b_i$ are compact disjoint, and therefore there exists an open subset $c_i$ of a space $X$ such that $q_i\subset c_i$ and $\bar{c}_i\cap(\Omega\setminus b_i) = \emptyset$ (i.e. $\bar{c}_i\cap\Omega\subset b_i$); $i = 1,\ldots,r$. The set $X\setminus\bigcup\limits_{i=1}^{r} c_i$ is compact and disjoint with $\Omega$. Hence there exists a finite family $F$ of wandering open subsets of $X$ such that $\bigcup F\supset X\setminus\bigcup\limits_{i=1}^{r} c_i$. Therefore, $C = F\cup\{c_1,\ldots,c_r\}$ belongs to $\mathfrak{A}(X)$. Now let us take arbitrary $E\in\mathfrak{A}(X)$ and $\varepsilon > 0$. Write $A = E|_\Omega\in\mathfrak{A}(\Omega)$. Take $p$ such that

$$\frac{1}{p}\log N(A^p|B^p)\leqslant h\big((f|_\Omega), A|B\big). \tag{3.1}$$

Let $A = \{a_1,\ldots,a_s\}$; write

$$a = N(A^p|B^p). \tag{3.2}$$

The definition of $N(A^p|B^p)$ tells us that there exists a mapping $T$ from $\{1,\ldots,r\}^p$ (the Cartesian product of $p$ copies of the set $\{1,\ldots,r\}$) into the family of all subsets of the set $\{1,\ldots,s\}^p$, such that for any $\sigma\in\{1,\ldots,r\}^p$:

$$\bigcup_{\tau\in T\sigma}\Big(\bigcap_{i=0}^{p-1} f^{-i}a_{\tau i}\Big)\supset\bigcap_{i=0}^{p-1} f^{-i}b_{\sigma i}\cap\Omega\quad\text{ and }\quad \operatorname{Card} T\sigma\leqslant a.$$

Let $d_i$ ($i = 1,\ldots,s$) be an element of $E$ for which $a_i = d_i\cap\Omega$. For $\sigma\in\{1,\ldots,r\}^p$ we have

$$\Big(\bigcap_{i=0}^{p-1} f^{-i}\bar{c}_{\sigma i}\Big)\cap\Omega\subset\Big[\bigcap_{i=0}^{p-1} f^{-i}(\bar{c}_{\sigma i}\cap\Omega)\Big]\cap\Omega\subset\Big(\bigcap_{i=0}^{p-1} f^{-i}b_{\sigma i}\Big)\cap\Omega$$
$$\subset\bigcup_{\tau\in T\sigma}\Big(\bigcap_{i=0}^{p-1} f^{-i}a_{\tau i}\Big)\subset\bigcup_{\tau\in T\sigma}\Big(\bigcap_{i=0}^{p-1} f^{-i}d_{\tau i}\Big)$$

and thus

$$\emptyset = \Big[\bigcap_{i=0}^{p-1} f^{-i}\overline{c_{\sigma i}} \setminus \bigcup_{\tau \in T\sigma} \Big(\bigcap_{i=0}^{p-1} f^{-i} d_{\tau i}\Big)\Big] \cap \Omega \supset \overline{\Big[\bigcap_{i=0}^{p-1} f^{-i} c_{\sigma i} \setminus \bigcup_{\tau \in T\sigma} \Big(\bigcap_{i=0}^{p-1} f^{-i} d_{\tau i}\Big)\Big]} \cap \Omega .$$

Hence we obtain

$$\bar{e} \cap \Omega = \emptyset, \tag{3.3}$$

where

$$e = \bigcup_{\sigma \in \{1,\ldots,r\}^p} \Big[\bigcap_{i=0}^{p-1} f^{-i} c_{\sigma i} \setminus \bigcup_{\tau \in T\sigma} \Big(\bigcap_{i=0}^{p-1} f^{-i} d_{\tau i}\Big)\Big]. \tag{3.4}$$

Therefore there exists a finite family $G$ of open wandering subsets of $X$ such that

$$\bigcup G \supset \bar{e} \cup \Big(X \setminus \bigcup_{i=1}^{s} d_i\Big) \tag{3.5}$$

and $G|_{\bigcup G} \geqslant E|_{\bigcup G}$. Write $D = G \cup \{d_1, \ldots, d_s\}$. We have $D \in \mathfrak{A}(X)$, $D \geqslant E$.

Now we fix $k \in N$ and estimate the number $N(D^{kp} | C^{kp})$ from above. Let $\emptyset \neq c \in C^{kp}$. Then $c = \bigcap_{j=0}^{kp-1} f^{-1} w_j$ for some sets $w_j \in C$. If a set $w$ belongs to $F$, then it occurs in the sequence $(w_0, \ldots, w_{kp-1})$ at most once. Therefore we can divide the set $\{0, \ldots, k-1\}$ into two sets $P$ and $Q$ in such a way that $\operatorname{Card} P \leqslant \operatorname{Card} F$ and there exist elements $\sigma_j \in \{1, \ldots, r\}^p$ for $j \in Q$ such that $c \subset \bigcap_{j=0}^{k-1} f^{-jp} v_j$, where

$$v_j = \begin{cases} X & \text{for} \quad j \in P, \\ \bigcap_{i=0}^{p-1} f^{-i} c_{\sigma_j i} & \text{for} \quad j \in Q. \end{cases}$$

In view of (3.4) and (3.5) we have

$$\bigcap_{i=0}^{p-1} f^{-i} c_{\sigma i} \setminus \bigcup_{\tau \in T\sigma} \Big(\bigcap_{i=0}^{p-1} f^{-i} d_{\tau i}\Big) \subset e \subset \bigcup G$$

for any $\sigma \in \{1, \ldots, r\}^p$, and thus for $j \in Q$

$$v_j \subset \bigcup G \cup \bigcup_{\tau \in T\sigma_j} \Big(\bigcap_{i=0}^{p-1} f^{-i} d_{\tau i}\Big) = \bigcup (G \vee f^{-1} D^{p-1}) \cup \bigcup_{\tau \in T\sigma_j} \Big(\bigcap_{i=0}^{p-1} f^{-i} d_{\tau i}\Big).$$

For $j \in P$ we have $v_j = X \subset \bigcup D^p$. Finally

$$c \subset \bigcap_{j \in Q} f^{-jp} \Big[\bigcup (G \vee f^{-1} D^{p-1}) \cup \bigcup_{\tau \in T\sigma_j} \Big(\bigcap_{i=0}^{p-1} f^{-i} d_{\tau i}\Big)\Big] \cap \bigcap_{j \in P} f^{-jp} \Big(\bigcup D^p\Big). \tag{3.6}$$



The right-hand side of (3.6) can be rewritten as the union of a certain family $H \subset D^{kp}$. If $\emptyset \neq d \in H$, $d = \bigcap_{i=0}^{kp-1} f^{-i} u_i$, $u_i \in D$, then $\operatorname{Card}\{i\colon u_i \in G\} \leqslant \operatorname{Card} G$. Therefore from (3.6) we obtain

$$\begin{aligned} &\operatorname{Card}(H \setminus \{\emptyset\}) \\ &\quad \leqslant \sum_{n=0}^{\operatorname{Card} G} \Big[\binom{\operatorname{Card} Q}{n} \cdot \big(\operatorname{Card} G \cdot \operatorname{Card}(D^{p-1})\big)^n \cdot \alpha^{\operatorname{Card} Q - n} \cdot \big(\operatorname{Card}(D^p)\big)^{\operatorname{Card} P}\Big] \\ &\quad \leqslant (\operatorname{Card} G + 1) \cdot k^{\operatorname{Card} G} \cdot \big(\operatorname{Card} G \cdot \operatorname{Card}(D^{p-1})\big)^{\operatorname{Card} G} \cdot \alpha^k \cdot \big(\operatorname{Card}(D^p)\big)^{\operatorname{Card} F} \end{aligned}$$

Hence we have $N(D^{kp} | C^{kp}) \leqslant \beta \cdot k^{\gamma} \cdot \alpha^k + 1$ for some constants $\beta$ and $\gamma$ independent of $k$ and thus

$$h(f, D | C) \leqslant \frac{1}{p} \log \alpha . \tag{3.7}$$

But $D \geqslant E$, and therefore (3.1), (3.2), (3.7), (1.13) and (1.23) imply

$$h(f, E | C) \leqslant h\big((f|_{\Omega}), A | B\big) + \varepsilon \leqslant h\big((f|_{\Omega}) | B\big) + \varepsilon .$$

We have thus obtained the following result:

$$\underset{B \in \mathfrak{A}(\Omega)}{\forall} \ \underset{C \in \mathfrak{A}(X)}{\exists} \ \underset{\substack{E \in \mathfrak{A}(X) \\ \varepsilon > 0}}{\forall} \ h(f, E | C) \leqslant h\big((f|_{\Omega}) | B\big) + \varepsilon .$$

Taking the limit with respect to $E \in \mathfrak{A}(X)$ and $\varepsilon \to 0$, we obtain $\underset{B \in \mathfrak{A}(\Omega)}{\forall} \ \underset{C \in \mathfrak{A}(X)}{\exists} \ h(f | C) \leqslant h\big((f|_{\Omega}) | B\big)$. But from this and (1.24) it follows that $h^*(f) \leqslant h^*(f|_{\Omega})$. ∎

In the proofs of the further properties we shall have to be able to refine a given open cover in an essential way. For this purpose define

(a) the *star of a set* $Y$ *with respect to a cover* $A$

$$\operatorname{st}(Y, A) = \bigcup \{a \in A\colon a \cap Y \neq \emptyset\},$$

(b) the *star of a cover* $A$

$$\operatorname{St} A = \{\operatorname{st}(a, A)\colon a \in A\}$$

(notice that the above definition is different from the one commonly used).

Of course, for $A \in \mathfrak{A}$ we have $\operatorname{St} A \in \mathfrak{A}$; also $A \geqslant \operatorname{St} A$.

Proposition 3.5. *The family* $\{\operatorname{St} A\colon A \in \mathfrak{A}\}$ *is cofinal with* $\mathfrak{A}$.

Proof. Let $A \in \mathfrak{A}$ and let $L \in \mathfrak{N}$ be a Lebesgue number for $A$. There exists a $U \in \mathfrak{N}$ such that $U \circ U \circ U \subset L$ (where $U \circ V = \{(x, y) \in X \times X\colon \underset{z \in X}{\exists} \ (x, z) \in U, \ (z, x) \in V\}$). Then any finite cover chosen from the open cover $\{\{y \in X\colon (x, y) \in U\}\}_{x \in X}$ is a refinement of $A$. ∎

A family $A$ of subsets of a space $X$ will be called a *discover* if elements of $A$ are pairwise disjoint.

The application of the notion of stars is based on the following three lemmas:

LEMMA 3.1. *Let $A$ be a finite cover of $X$, $Y \subset X$. Then there exists a discover $B \subset A$ such that*

(a) $$b \cap Y \neq \emptyset \quad \text{for} \quad b \in B,$$

(b) $$Y \subset \bigcup \{\mathrm{st}(b, A) \colon b \in B\}.$$

Proof. Assume that $B' \subset A$ is a discover fulfilling (a) but not (b). Then there exists a point $y \in Y$ which does not belong to the set $\bigcup\{\mathrm{st}(b, A)\colon b \in B'\}$. Take $a \in A$ such that $y \in a$. Then $B'' = B' \cup \{a\}$ is also a discover contained in $A$ and fulfilling (a). Now the lemma follows from the finiteness of $A$ (notice that $\emptyset$ is a discover fulfilling (a)). ■

LEMMA 3.2. *Let $A$ be a finite cover of $X$; $Y \subset X$; let $B$ be a discover contained in* $\mathrm{St}A$ *such that $b \cap Y \neq \emptyset$ for $b \in B$; $C \subset A$, $\mathrm{st}(Y, A) \subset \bigcup C$. Then* $\mathrm{Card}\, C \geqslant \mathrm{Card}\, B$.

Proof. Take $a \in A$ such that $\mathrm{st}(a, A) \in B$. Then $a \cap \mathrm{st}(Y, A) \neq \emptyset$, and thus there exists a $c_a \in C$ such that $a \cap c_a \neq \emptyset$. But then $c_a \subset \mathrm{st}(a, A) \in B$. $B$ is a discover and therefore $c_a \neq c_d$ provided $\mathrm{St}(a, A) \neq \mathrm{St}(d, A)$. Thus $\mathrm{Card}\, C \geqslant \mathrm{Card}\, B$. ■

LEMMA 3.3. *Let $A$ be a cover. Then $\mathrm{St}(A^k) \geqslant (\mathrm{St}A)^k$ for $k = 1, 2, \ldots$*

Proof. If $a_i \in A$ for $i = 0, \ldots, k-1$, then

$$\mathrm{st}\Big(\bigcap_{i=0}^{k-1} f^{-i} a_i, A^k\Big) \subset \mathrm{st}(f^{-j} a_j, f^{-j} A) = f^{-j}\big(\mathrm{st}(a_j, A)\big)$$

for $j = 0, \ldots, k-1$; therefore

$$\mathrm{st}\Big(\bigcap_{i=0}^{k-1} f^{-i} a_i, A^k\Big) \subset \bigcap_{j=0}^{k-1} f^{-j}\, \mathrm{st}(a_j, A).$$

Thus $\mathrm{St}(A^k) \geqslant (\mathrm{St}A)^k$. ■

For cascades $(X_i, f_i)$, $i \in I$, denote by $\prod_{i \in I} f_i\colon \prod_{i \in I} X_i \to \prod_{i \in I} X_i$ ($f_1 \times \ldots \times f_n$ in a finite case) the transformation given by the formula $\prod_{i \in I} f_i (x_i)_{i \in I} = (f_i x_i)_{i \in I}$. For families $A_i$ of subsets of spaces $X_i$ $(i = 1, 2)$, respectively, write $A_1 \times A_2 = \{a_1 \times a_2\colon a_i \in A_i,\ i = 1, 2\}$. The operations st and St, as can easily be seen, commute with the operation of taking a product.

LEMMA 3.4. *Let $(X_1, f_1)$, $(X_2, f_2)$ be cascades, $\emptyset \neq Y_i \subset X_i$ and let $A_i$ be a finite cover of $X_i$ $(i = 1, 2)$. Then*

(a) $$N(Y_1, \mathrm{St}\mathrm{St} A_1) \cdot N(Y_2, \mathrm{St}\mathrm{St} A_2) \leqslant N\big(\mathrm{st}(Y_1 \times Y_2, A_1 \times A_2), A_1 \times A_2\big),$$

(b) $$N(Y_1 \times Y_2, A_1 \times A_2) \leqslant N(Y_1, A_1) \cdot N(Y_2, A_2).$$



Proof. The inequality (b) is obvious. We shall prove (a).

In view of Lemma 3.1 there exist discovers $B_i \subset \mathrm{St}A_i$ such that $b \cap Y_i \neq \emptyset$ for $b \in B_i$ and $Y_i \subset \bigcup\{\mathrm{st}(b, \mathrm{St}A_i)\colon b \in B_i\}$ $(i = 1, 2)$. The last inclusion implies $\mathrm{Card}\, B_i \geqslant N(Y_i, \mathrm{St}\mathrm{St}A_i)$. We have $B_1 \times B_2 \subset \mathrm{St}A_1 \times \mathrm{St}A_2 = \mathrm{St}(A_1 \times A_2)$. $B_1 \times B_2$ is a discover; moreover $b \cap (Y_1 \times Y_2) \neq \emptyset$ for $b \in B_1 \times B_2$, and therefore, in view of Lemma 3.2, if $\bigcup C \supset \mathrm{st}(Y_1 \times Y_2, A_1 \times A_2)$ for some $C \subset A_1 \times A_2$, then $\mathrm{Card}\, C \geqslant \mathrm{Card}(B_1 \times B_2) = \mathrm{Card}\, B_1 \cdot \mathrm{Card}\, B_2$. ■

Now we are able to prove the next property of conditional entropy:

THEOREM 3.2. *Let $(X_1, f_1)$, $(X_2, f_2)$ be cascades. Then*

$$h^*(f_1 \times f_2) = h^*(f_1) + h^*(f_2).$$

Proof. From Lemma 3.4 (b) it follows that

$$h(f_1, A_1 | B_1) + h(f_2, A_2 | B_2) \geqslant h(f_1 \times f_2, A_1 \times A_2 | B_1 \times B_2)$$

for $A_i, B_i \in \mathfrak{A}(X_i)$, $i = 1, 2$. In view of the fact that $\{A_1 \times A_2\colon A_i \in \mathfrak{A}(X_i),\ i = 1, 2\}$ is cofinal with $\mathfrak{A}(X_1 \times X_2)$, this inequality yields $h^*(f_1 \times f_2) \leqslant h^*(f_1) + h^*(f_2)$. Let $A_i, B_i \in \mathfrak{A}(X_i)$, $A_i \geqslant B_i$ for $i = 1, 2$. We have

$$\mathrm{st}(b_1 \times b_2, A_1 \times A_2) \subset \mathrm{st}(b_1 \times b_2, B_1 \times B_2) \in \mathrm{St}(B_1 \times B_2)$$

for $b_i \in B_i$, $i = 1, 2$. Lemma 3.4 (a) implies for $n = 1, 2, \ldots$

$$N\big(\mathrm{St}\mathrm{St}(A_1^n) | B_1^n\big) \cdot N\big(\mathrm{St}\mathrm{St}(A_2^n) | B_2^n\big) \leqslant N\Big((A_1 \times A_2)^n | \mathrm{St}\big((B_1 \times B_2)^n\big)\Big).$$

Hence, in view of Lemma 3.3 and (1.3):

$$h(f_1, \mathrm{St}\mathrm{St}A_1 | B_1) + h(f_2, \mathrm{St}\mathrm{St}A_2 | B_2) \leqslant h\big(f_1 \times f_2, A_1 \times A_2 | \mathrm{St}(B_1 \times B_2)\big).$$

In view of Proposition 3.5 and the fact that the family $\{A_1 \times A_2\colon A_i \in \mathfrak{A}(X_i),\ i = 1, 2\}$ is cofinal with $\mathfrak{A}(X_1 \times X_2)$, the above inequality gives the inequality $h^*(f_1) + h^*(f_2) \leqslant h^*(f_1 \times f_2)$. ■

Remark 3.1. Theorem 3.2 can be generalized by induction to the product of an arbitrary finite number of transformations: $h^*(f_1 \times \ldots \times f_n) = \sum_{i=1}^{n} h^*(f_i)$.

Remark 3.2. Putting in the proof of Theorem 3.2 $B_i = \Theta(X_i)$, we obtain a simple proof of the theorem on the topological entropy of the product due to Goodwyn [11]: $h(f_1 \times f_2) = h(f_1) + h(f_2)$. This theorem can, in the same way, be generalized by induction to the product of an arbitrary finite number of transformations. The foregoing considerations result finally in a new proof of the following theorem (also due to Goodwyn [11]):

THEOREM 3.3. *Let* $(X_i, f_i)$ *be cascades* $(i = 1, 2, \ldots)$. *Then*

$$h\Big(\prod_{i=1}^{\infty} f_i\Big) = \sum_{i=1}^{\infty} h(f_i).$$

Proof. Write

$$\mathfrak{A}_k = \Big\{A \times \Big\{\prod_{i=k}^{\infty} X_i\Big\}:\ A \in \mathfrak{A}\Big(\prod_{i=1}^{k-1} X_i\Big)\Big\} \quad \text{for } k = 1, 2, \ldots; \quad \mathfrak{A}_0 = \mathfrak{A}\Big(\prod_{i=1}^{\infty} X_i\Big).$$

The family of open sets $\{a \times \prod_{i=k}^{\infty} X_i$: $a$ is an open subset of $\prod_{i=1}^{k-1} X_i$, $k = 1, 2, \ldots\}$ is a base of the space $\prod_{i=1}^{\infty} X_i$, and therefore the family $\bigcup_{k=1}^{\infty} \mathfrak{A}_k$ is cofinal with $\mathfrak{A}_0$. Denote by $\pi_k$: $\prod_{i=1}^{\infty} X_i \to \prod_{i=1}^{k-1} X_i$ the natural projection. Notice that the sequence $(\mathfrak{A}_n)_{n=1}^{\infty}$ is ascending and that $h\big(\prod_{i=1}^{\infty} f_i, A\big) = h\big(\prod_{i=1}^{k-1} f_i, \pi_k A\big)$ for $A \in \mathfrak{A}_k$. Hence

$$h\Big(\prod_{i=1}^{\infty} f_i\Big) = \lim_{k\to\infty} \lim_{A \in \mathfrak{A}_k} h\Big(\prod_{i=1}^{\infty} f_i, A\Big) = \lim_{k\to\infty} h\Big(\prod_{i=1}^{k-1} f_i\Big)$$
$$= \lim_{k\to\infty} \sum_{i=1}^{k-1} h(f_i) = \sum_{i=1}^{\infty} h(f_i). \ \blacksquare$$

Concerning conditional entropy, the following analogous theorem also holds:

THEOREM 3.4. *Let* $(X_i, f_i)$ *be cascades* $(i = 1, 2, \ldots)$, $h\big(\prod_{i=1}^{\infty} f_i\big) < \infty$. *Then*

$$h^*\Big(\prod_{i=1}^{\infty} f_i\Big) = \sum_{i=1}^{\infty} h^*(f_i).$$

Proof. We have

$$h^*\Big(\prod_{i=1}^{\infty} f_i\Big) = h^*\Big(\prod_{i=1}^{k-1} f_i\Big) + h^*\Big(\prod_{i=k}^{\infty} f_i\Big) = \sum_{i=1}^{k-1} h^*(f_i) + h^*\Big(\prod_{i=k}^{\infty} f_i\Big).$$

But $0 \leqslant h^*\big(\prod_{i=k}^{\infty} f_i\big) \leqslant h\big(\prod_{i=k}^{\infty} f_i\big) = \sum_{i=k}^{\infty} h(f_i)$ for $k = 1, 2, \ldots$, and therefore $\lim_{k\to\infty} h^*\big(\prod_{i=k}^{\infty} f_i\big) = 0$. Thus

$$h^*\Big(\prod_{i=1}^{\infty} f_i\Big) = \lim_{k\to\infty} \sum_{i=1}^{k-1} h^*(f_i) = \sum_{i=1}^{\infty} h^*(f_i). \ \blacksquare$$



**§ 4. Connection with measure-theoretic entropy.** Now we shall examine the connection between topological conditional entropy and measure-theoretic entropy.

Denote by $\mathfrak{M}(X)$ the space of all Borel regular normed measures on $X$ and by $\mathfrak{M}(X, f)$ the subspace of those measures from $\mathfrak{M}(X)$ which are invariant with respect to $f$. We shall consider these spaces with the weak-* topology. It is a well-known fact that they are both compact.

Now we prove an important fact, which allows us to simplify some proofs (e.g. Bowen [5], [6], Goodwyn [10]) and to give a very short proof of Goodwyn's theorem. This fact can be found in Denker's paper [7] though it is not explicitly formulated there.

PROPOSITION 4.1. *Let* $\mu \in \mathfrak{M}(X, f)$; *let* $A$ *be a finite Borel partition of* $X$. *Then there exist a finite Borel partition* $B$ *of* $X$ *and a cover* $C \in \mathfrak{A}(X)$ *such that*

(a) $$h_\mu(f, B) \geqslant h_\mu(f, A) - 1,$$

(b) $$N(B|C) \leqslant 2.$$

Proof. Let $A = \{a_1, \ldots, a_r\}$. There exists a number $\varepsilon > 0$ such that if $\mu(a_i \dot{-} b_i) \leqslant \varepsilon$ for $i = 1, \ldots, r$ for a partition $B = \{b_0, b_1, \ldots, b_r\}$, then the condition (a) holds (see e.g. Smorodinsky [16], Lemma 5.8). We choose compact sets $b_i$ contained in $a_i$ such that $\mu(a_i \setminus b_i) \leqslant \varepsilon$ for $i = 1, \ldots, r$ (this is possible because the measure $\mu$ is regular) and we take $b_0 = X \setminus \bigcup_{i=1}^{r} b_i$. Now we define $C = \{c_1, \ldots, c_r\}$, $c_i = b_0 \cup b_i$ for $i = 1, \ldots, r$. $C \in \mathfrak{A}$ and (b) also holds. $\blacksquare$

THEOREM 4.1 (Goodwyn [10]). *Let* $(X, f)$ *be a cascade,* $\mu \in \mathfrak{M}(X, f)$. *Then* $h_\mu(f) \leqslant h(f)$.

Proof. Let $A$ be a finite Borel partition of $X$, $B$ and $C$ as in Proposition 4.1. The well-known inequality $H_\mu(B^n) \leqslant \log N(B^n)$ for $n = 1, 2, \ldots$ gives $h_\mu(f, B) \leqslant h(f, B)$. In view of Proposition 4.1, (1.11) and (1.16) we get

$$h_\mu(f, A) \leqslant h_\mu(f, B) + 1 \leqslant h(f, B) + 1 \leqslant h(f, C) + h(f, B|C) + 1$$
$$\leqslant h(f) + (\log 2 + 1).$$

$A$ has been arbitrary, and therefore $h_\mu(f) \leqslant h(f) + (\log 2 + 1)$. But this is true for every continuous transformation of $X$ into itself for which $\mu$ is invariant, in particular for $f^n$ instead of $f$. Hence, for $n = 1, 2, \ldots$,

$$h_\mu(f) = \frac{1}{n} h_\mu(f^n) \leqslant \frac{1}{n} h(f^n) + \frac{1}{n}(\log 2 + 1) = h(f) + \frac{1}{n}(\log 2 + 1).$$

Therefore, $h_\mu(f) \leqslant h(f)$. $\blacksquare$

In the sequel we shall need the following simple inequality:

LEMMA 4.1. *For finite Borel partitions $A$ and $B$,*

$$H_\mu(A\,|\,B) \leqslant \log N(A\,|\,B).$$

Proof. For any $b \in B$

$$H_{\mu|_b/\mu(b)}(A\,|_b) \leqslant \log N(b, A) \leqslant \log N(A\,|\,B),$$

and therefore

$$H_\mu(A\,|\,B) = \sum_{b \in B} \mu(b) \cdot H_{\mu|_b/\mu(b)}(A\,|_b) \leqslant \log N(A\,|\,B). \ \blacksquare$$

Formula (1.22) is valid also for measure-theoretic entropy, namely:

PROPOSITION 4.2 (cf. Bowen [5]). *Let $D$ be a finite Borel partition of $X$, $\mu \in \mathfrak{M}(X, f)$. Then*

$$h_\mu(f) \leqslant h_\mu(f, D) + h(f\,|\,D).$$

Proof. Let $A$ be a finite Borel partition of $X$ and let $B$ and $C$ be as in Proposition 4.1. In view of Lemma 4.1,

$$H_\mu(B^n) \leqslant H_\mu(D^n) + H_\mu(B^n\,|\,D^n) \leqslant H_\mu(D^n) + \log N(B^n\,|\,D^n) \quad \text{for } n = 1, 2, \ldots,$$

and therefore $h_\mu(f, B) \leqslant h_\mu(f, D) + h(f, B\,|\,D)$. Applying Proposition 4.1, (1.11) and (1.17), we obtain

$$\begin{aligned} h_\mu(f, A) &\leqslant h_\mu(f, B) + 1 \leqslant h_\mu(f, D) + h(f, B\,|\,D) + 1 \\ &\leqslant h_\mu(f, D) + h(f, C\,|\,D) + h(f, B\,|\,C) + 1 \\ &\leqslant h_\mu(f, D) + h(f\,|\,D) + (\log 2 + 1). \end{aligned}$$

$A$ has been arbitrary, and therefore $h_\mu(f) \leqslant h_\mu(f, D) + h(f\,|\,D) + (\log 2 + 1)$. But this is true for every continuous transformation of $X$ into itself for which $\mu$ is invariant and for every finite Borel partition of $X$, and therefore for $n = 1, 2, \ldots$ (it is easy to see that $h_\mu(f^n, D_f^n) = n h_\mu(f, D)$ and $h(f^n\,|\,D_f^n) = n h(f\,|\,D)$):

$$\begin{aligned} h_\mu(f) &= \frac{1}{n} h_\mu(f^n) \leqslant \frac{1}{n} h_\mu(f^n, D_f^n) + \frac{1}{n} h(f^n\,|\,D_f^n) + \frac{1}{n}(\log 2 + 1) \\ &= h_\mu(f, D) + h(f\,|\,D) + \frac{1}{n}(\log 2 + 1). \end{aligned}$$

Thus $h_\mu(f) \leqslant h_\mu(f, D) + h(f\,|\,D)$. $\blacksquare$

Let us consider the measure-theoretic entropy regarded as a function of a measure: $h_{\cdot}(f)\colon \mathfrak{M}(X, f) \to \boldsymbol{R}$. Denote by $h_\mu^*(f) = \limsup_{\nu \to \mu} h_\nu(f) - h_\mu(f)$ (we assume $\infty - \infty = 0$) for $\mu \in \mathfrak{M}(X, f)$. Notice that $h_\mu^*(f) = 0$ iff $h_{\cdot}(f)$ is upper semicontinuous at the point $\mu$.



Now we generalize Theorem 3 of [15].

THEOREM 4.2. *Let $(X, f)$ be a cascade, $\mu \in \mathfrak{M}(X, f)$. Then $h_\mu^*(f) \leqslant h^*(f)$.*

Proof. If $h(f) = \infty$, then, in view of Proposition 3.3, also $h^*(f) = \infty$. Hence we can assume $h(f) < \infty$. Let $A = \{a_1, \ldots, a_r\} \in \mathfrak{A}$. Take a cover $B = \{b_1, \ldots, b_r\} \in \mathfrak{A}$ such that $\overline{b_i} \subset a_i$ for $i = 1, \ldots, r$. Then take continuous functions $\varphi_i\colon X \to [0, 1]$ such that $\varphi_i x = 1$ for $x \in \overline{b_i}$ and $\varphi_i x = 0$ for $x \notin a_i$, $i = 1, \ldots, r$. For some $\alpha \in [0, 1]$ we have $\mu\big(\bigcup_{i=1}^{r} \varphi_i^{-1}\{\alpha\}\big) = 0$; therefore the finite Borel partition $C = \{\varphi_1^{-1}[\alpha, 1], \varphi_2^{-1}[\alpha, 1] \setminus \varphi_1^{-1}[\alpha, 1], \ldots \ldots, \varphi_r^{-1}[\alpha, 1] \setminus \bigcup_{i=1}^{r-1} \varphi_i^{-1}[\alpha, 1]\}$ consists of sets with boundaries of measure 0. We also have $C \geqslant A$. Take $\varepsilon > 0$. For a fixed $n$, the partition $C^n$ consists also of sets with boundaries of measure 0, and therefore for some open neighbourhood $U_n \subset \mathfrak{M}(X, f)$ of $\mu$ we have $\frac{1}{n} H_\mu(C^n) \geqslant \frac{1}{n} H_\nu(C^n) - \varepsilon$ whenever $\nu \in U_n$. Thus, in view of Proposition 4.2,

$$\begin{aligned} h_\mu(f) \geqslant h_\mu(f, C) &= \lim_{n\to\infty} \frac{1}{n} H_\mu(C^n) \geqslant \limsup_{n\to\infty} \sup_{\nu \in U_n} \frac{1}{n} H_\nu(C^n) - \varepsilon \\ &\geqslant \limsup_{n\to\infty} \sup_{\nu \in U_n} h_\nu(f, C) - \varepsilon \geqslant \limsup_{n\to\infty} \sup_{\nu \in U_n} h_\nu(f) - h(f\,|\,C) - \varepsilon \\ &\geqslant \limsup_{\nu \to \mu} h_\nu(f) - h(f\,|\,A) - \varepsilon. \end{aligned}$$

$\varepsilon$ has been arbitrary, and therefore $h_\mu^*(f) \leqslant h(f\,|\,A)$. But $A$ has also been arbitrary, and thus $h_\mu^*(f) \leqslant h^*(f)$. $\blacksquare$

COROLLARY 4.1 (cf. [15]). *If $f$ is asymptotically $h$-expansive, then there exists a measure with maximal entropy for $f$.*

Proof. If $f$ is asymptotically $h$-expansive, then $h_{\cdot}(f)$ is an upper semicontinuous function on the compact space $\mathfrak{M}(X, f)$, and therefore it attains its supremum. $\blacksquare$

Remark 4.1. Dinaburg's theorem ([8], proof in the general case [9], [17]) asserts that the topological entropy of a transformation $f$ is the best estimation of the measure-theoretic entropies of $f$, i.e., $h(f) = \sup_{\mu \in \mathfrak{M}(X,f)} h_\mu(f)$. But an analogous theorem for $h^*$ and $h_\mu^*$ is not valid (see Example 6.4).

In some cases the following fact will be useful for the computation of topological conditional entropy:

PROPOSITION 4.3. *Let $(Y_n)_{n=1}^{\infty}$ be a descending sequence of invariant closed subsets of $X$. Let $(U_n)_{n=1}^{\infty}$ be a descending sequence of open subsets* *of $X$ such that $Y_n \subset U_n$ for $n = 1, 2, \ldots$ and $\bigcap_{n=1}^{\infty} U_n = \bigcap_{n=1}^{\infty} Y_n = Y \neq \emptyset$. Then*

$$h^*(f) \geqslant \lim_{n\to\infty} h(f|_{Y_n}) - h(f|_Y) \quad (\text{we assume } \infty - \infty = 0).$$

Proof. From Dinaburg's theorem it follows that for any $n$ there exists a measure $\mu_n \in \mathfrak{M}(X, f)$ such that $h_{\mu_n}(f) \geqslant h(f|_{Y_n}) - 1/n$ and $\operatorname{supp}\mu_n \subset Y_n$. Let $\mu \in \mathfrak{M}(X, f)$ be a cluster point of the set $\{\mu_n\}_{n=1}^{\infty}$. We have

$$\mu(X \setminus Y) = \mu\Big(\bigcup_{n=1}^{\infty} (X \setminus U_n)\Big) = \lim_{n\to\infty} \mu(X \setminus U_n).$$

But, for $n$ fixed, $\mu_k(X \setminus Y_n) = 0$ for almost every $k$, and therefore $\mu(X \setminus U_n) = 0$. Therefore, $\operatorname{supp}\mu \subset Y$. But in view of Goodwyn's theorem we have $h_\mu(f) \leqslant h(f|_Y)$. Applying Theorem 4.2, we obtain:

$$h^*_\mu(f) \geqslant \liminf_{n\to\infty} h_{\mu_n}(f) - h_\mu(f)$$
$$\geqslant \lim_{n\to\infty} [h(f|_{Y_n}) - 1/n] - h(f|_Y) = \lim_{n\to\infty} h(f|_{Y_n}) - h(f|_Y). \ \blacksquare$$

Remark 4.2. If $X$ is a metric space, then the assumption of the existence of the sets $U_n$ is fulfilled automatically; it suffices to put $U_n = \bigcup_{x \in Y_n} \operatorname{Int} B_{1/n}(x)$.

One can apply the above proposition in order to compute the topological conditional entropy in the examples from [12], [15]. The conditional entropy in these examples turns out to be equal to the usual topological entropy.

**§ 5. Flows.** Now let us consider a continuous flow $\{\varphi^t\}_{t\in\mathbf{R}}$ on a non-empty compact Hausdorff space $X$. We can write, as in the discrete case, $A_\varphi^T = \bigvee_{t\in[0,T)} \varphi^{-t}A$ for $A \in \mathfrak{A}$, $T > 0$. The application of this notion is based on the following proposition:

PROPOSITION 5.1. *For $A \in \mathfrak{A}$, $T > 0$, there exists a $B \in \mathfrak{A}$ such that $B \geqslant A^T$.*

Proof. Let $L \in \mathfrak{N}$ be a Lebesgue number for $A$. Let $\pi$: $X \times X \times [0, T] \to X \times X$ be the natural projection; let $\Phi$: $X \times X \times [0, T] \to X \times X$ be the mapping defined as follows: $\Phi(x, y, t) = (\varphi^t x, \varphi^t y)$. Clearly, $\Phi$ is continuous. Write $Z = \pi\big(\Phi^{-1}(X \times X \setminus L)\big)$. $Z$ is compact and symmetric. If there exists a point $x \in X$ with $(x, x) \in Z$, then there exists a number $t \in [0, T]$ such that $(\varphi^t x, \varphi^t x) \in X \times X \setminus L$, but this is impossible. Hence $U = X \times X \setminus Z$ is an element of $\mathfrak{N}$. For any $(x, y) \in U$ and $t \in [0, T]$ we have $(\varphi^t x, \varphi^t y) \in L$; therefore any finite cover $B$ chosen from the open cover $\big\{\{y\colon (x, y) \in U\}\big\}_{x\in X}$ is a refinement of $A_\varphi^T$. $\blacksquare$



In view of the above proposition, for $A \in \mathfrak{A}$ and $T > 0$, $A_\varphi^T$ belongs to $\mathscr{P}$. For $A, B \in \mathfrak{A}$ the function $\log N(A_\varphi^T | B_\varphi^T)$ is, as in the discrete case, subadditive; therefore there exists a limit

$$\lim_{T\to\infty} \frac{1}{T} \log N(A_\varphi^T | B_\varphi^T) = h_{\text{flow}}(\varphi, A | B).$$

Further, in the same way as in the discrete case, we obtain

$$h_{\text{flow}}(\varphi | B) = \lim_{A \in \mathfrak{A}} h_{\text{flow}}(\varphi, A | B) = \sup_{A\in\mathfrak{A}} h_{\text{flow}}(\varphi, A | B),$$
$$h^*_{\text{flow}}(\varphi) = \lim_{B\in\mathfrak{A}} h_{\text{flow}}(\varphi | B) = \inf_{B\in\mathfrak{A}} h_{\text{flow}}(\varphi | B),$$
$$h_{\text{flow}}(\varphi) = h_{\text{flow}}(\varphi | \Theta).$$

THEOREM 5.1. *Let $\{\varphi^t\}_{t\in\mathbf{R}}$ be a continuous flow on a compact Hausdorff space $X$. Then for $T \in \mathbf{R}$:*

(a) $$h(\varphi^T) = |T| \cdot h_{\text{flow}}(\varphi),$$

(b) $$h^*(\varphi^T) = |T| \cdot h^*_{\text{flow}}(\varphi).$$

Proof. We may assume that $T > 0$; for $T = 0$ the theorem is obvious; for $T < 0$ it is a consequence of the case $T > 0$ and the formulas $h(f) = h(f^{-1})$ and $h^*(f) = h^*(f^{-1})$.

Let $A, B, C, D \in \mathfrak{A}$, $B \geqslant A_\varphi^T$, $D \geqslant C_\varphi^T$. For $n \in \mathbf{N}$ we have

$$N\big((A_\varphi^T)^n_{\varphi^T} | D^n_{\varphi^T}\big) \leqslant N(A_\varphi^{nT} | C_\varphi^{nT}) \leqslant N\big(B^n_{\varphi^T} | (C_\varphi^T)^n_{\varphi^T}\big);$$

thus

$$h(\varphi^T, A_\varphi^T | D) \leqslant T \cdot h_{\text{flow}}(\varphi, A | C) \leqslant h(\varphi^T, B | C_\varphi^T) \leqslant h(\varphi^T | C_\varphi^T).$$

In view of Proposition 5.1 the family $\{A_\varphi^T\colon A \in \mathfrak{A}\}$ is cofinal with $\mathfrak{A}$; therefore we can take the limit with respect to $A$. We obtain

$$h(\varphi^T | D) \leqslant T \cdot h_{\text{flow}}(\varphi | C) \leqslant h(\varphi^T | C_\varphi^T) \quad \text{for} \quad D \geqslant C_\varphi^T. \tag{5.1}$$

Putting $C = D = \Theta$, we obtain $h(\varphi^T) \leqslant T \cdot h_{\text{flow}}(\varphi) \leqslant h(\varphi^T)$, whence (a) follows.

Taking in (5.1) the limit with respect to $D$, we obtain $h^*(\varphi^T) \leqslant T \cdot h_{\text{flow}}(\varphi | C) \leqslant h(\varphi^T | C_\varphi^T)$, and (b) follows, because $C$ is arbitrary. $\blacksquare$

COROLLARY 5.1. *Under the assumptions of the theorem*

(a) $$h(\varphi^T) = |T| \cdot h(\varphi^1),$$

(b) $$h^*(\varphi^T) = |T| \cdot h^*(\varphi^1). \ \blacksquare$$

Remark 5.1. The part (a) of Corollary 5.1 has already been known, but it has been proved only for metric spaces (Dinaburg [8], Bowen [4]).

Remark 5.2. In the whole of Section 5 (except the case $T<0$ in Theorem 5.1 and in Corollary 5.1) all the theorems are valid for a one-parameter semigroup $\{\varphi^t\}_{t\geqslant 0}$ of continuous transformations of a space $X$ into itself (a semiflow).

**§ 6. Counterexamples.** We have proved some properties of topological conditional entropy which are similar to the properties of usual topological conditional entropy. In this part we shall give examples which show that some other properties of this type are not valid.

In the whole of Section 6 the symbols for spaces and transformations will be fixed.

EXAMPLE 6.1. Let us consider the product of a countable number of copies of the compact discrete two-element topological group $\boldsymbol{Z}_2$: $Q=\prod_{-\infty}^{\infty}\boldsymbol{Z}_2$. $Q$ is a metric compact group. The shift, given by $T(x_i)_{i=-\infty}^{\infty}=(y_i)_{i=-\infty}^{\infty}$, $y_i=x_{i+1}$, is a continuous transformation $T\colon Q\to Q$. It is well known that $T$ is expansive and $h(T)=\log 2$.

Let $P=\{0\}\cup\{1/n\}_{n=1}^{\infty}\subset\boldsymbol{R}$. $P$ is a compact metric space. Take $X=P\times Q$; $f=\mathrm{id}_P\times T$. $f\colon X\to X$ is a homeomorphism. We have $h(f)=\log 2$ and $f$ is $h$-expansive (from the proof of Theorem 3.4 it follows that the product of two $h$-expansive transformations is also $h$-expansive).

Denote by $Y$ the space obtained by identifying all the points of $X$ with the first coordinate 0. $Y$ is a compact metrizable space. $\{0\}\times Q$ is a subset of $X$ strictly invariant with respect to $f$, and therefore there exists exactly one homeomorphism $g\colon Y\to Y$ such that $\pi\circ f=g\circ\pi$, where $\pi\colon X\to Y$ is the natural projection. We have $h(g)\leqslant h(f)=\log 2$. Besides, $g|_{\{1/n\}\times Q}$ is conjugate in a natural way with $T$. Hence $h(g)=\log 2$. In view of Proposition 4.3 and Remark 4.2 we have also $h^*(g)=\log 2$. Thus we see that a factor of an $h$-expansive transformation may have positive conditional entropy. ■

It is not a very surprising result, because the image of an open cover needs not be open (in fact, in the above example, if for some $A\in\mathfrak{A}(X)$ we have $h(f|A)=0$, then $\pi(A)$ is not open). But, as the next example shows, even the assumption that a projection from $X$ onto $Y$ is open is not sufficient.

EXAMPLE 6.2. Let $p_n\colon Q\to Q$ $(n=1,2,\ldots)$ be defined as follows: $p_n(x_i)_{i=-\infty}^{\infty}=(y_i)_{i=-\infty}^{\infty}$, $y_i=\sum_{j=i-n}^{i+n}x_j$ (addition is in $\boldsymbol{Z}_2$). The following properties of $p_n$ are easy to check:

(6.1) *$p_n$ is continuous,*

(6.2) *for any cylinder (i.e., a set $C_{\varepsilon_{-k},\ldots,\varepsilon_k}=\{(x_i)_{i=-\infty}^{\infty}\in Q\colon x_j=\varepsilon_j$ for $|j|\leqslant k\}$) we have $p_nC_{\varepsilon_{-k},\ldots,\varepsilon_k}=Q$ if $k<n$,*



(6.3) $p_n\circ T=T\circ p_n$.

Now let us define $p\colon X\to Y$ as follows: $p(0,x)=0$; $p(1/n,x)=(1/n,p_nx)$, $n=1,2,\ldots$ From properties (6.1)–(6.3) we obtain:

(6.4) *$p$ is continuous,*

(6.5) *$p$ is a surjection,*

(6.6) *$p\circ f=g\circ p$,*

(6.7) *$p$ is an open mapping.*

The properties (6.4)–(6.6) are obvious. To prove (6.7) notice that the family of all the sets of the form $a\times C_{\varepsilon_{-k},\ldots,\varepsilon_k}\subset X$, where $a$ is open in $P$, is an open base for $X$; the image of such a set under the mapping $p$ is, in view of (6.2), a set of the form

$$(b\times Q)\cup\bigcup_{j=1}^{r}\left\{\frac{1}{s_j}\right\}\times C_{\varepsilon_{-k_j,j},\ldots,\varepsilon_{k_j,j}},$$

where $b$ is open in $P$. ■

If $k\circ\psi=k$, then we have $h(\psi)=\sup h(\psi|_{k^{-1}(y)})$ (Bowen [4]). The next example shows that the above theorem is not valid for $h^*$ instead of $h$.

EXAMPLE 6.3. Let $\varrho\colon Y\to P$ be the natural projection. We have $\varrho\circ g=\varrho$, $h^*(g)=\log 2$, but for every $x\in P$ the transformation $g|_{\varrho^{-1}(x)}$ is expansive. ■

Now we shall show that the upper semi-continuity (even the constancy) of the measure-theoretic entropy does not imply asymptotical $h$-expansiveness of a transformation.

EXAMPLE 6.4. Let $R\subset Q$ be a closed, $T$-invariant subset such that $T|_R$ is uniquely ergodic and $h(T|_R)=\gamma>0$ (see Hahn, Katznelson [13]). Fix a point $\omega=(\omega_i)_{i=-\infty}^{\infty}\in R$ and let $\omega^n=(\omega_i^n)_{i=-\infty}^{\infty}$ $(i=1,2,\ldots)$ be defined by $\omega_{kn+j}^n=\omega_j$ for $j=0,1,\ldots,n-1$, $k\in\boldsymbol{Z}$. Now define a closed $(T\times g)$-invariant set $Z$ by

$$Z=\left\{(x,y)\in Q\times Y\colon\ (y=0,\ x\in R)\ \text{or}\ \left(y\in\left\{\frac{1}{n}\right\}\times R,\ x=T^j\omega^n\ \text{for some}\ j\right)\right\}.$$

$\varphi=T\times g|_Z$ is a homeomorphism of $Z$ onto itself. We shall compute the conditional entropy of $\varphi$. The family of all open finite covers of $Z$ of the form

$$\{C_{\varepsilon_{-k},\ldots,\varepsilon_k}\}_{(\varepsilon_{-k},\ldots,\varepsilon_k)\in\boldsymbol{Z}_2{}^{2k+1}}\times$$
$$\times\pi\left[\{a\times R\}\cup\left(\left\{\left\{\frac{1}{i}\right\}\right\}_{i=1}^{m-1}\times\{C_{\varepsilon_{-j},\ldots,\varepsilon_j}\}_{(\varepsilon_{-j},\ldots,\varepsilon_j)\in\boldsymbol{Z}_2{}^{2j+1}}\right)\right]\Big|_Z,$$

where $a = \{0\} \cup \{1/i\}_{i=m}^{\infty} \subset P$, is cofinal with $\mathfrak{A}(Z)$ (remember that $\{\{1/i\}\}_{i=1}^{m-1}$ is an open cover of $\{1/i\}_{i=1}^{m-1}$). But for a cover $B$ of the above form we have $\{\omega^m\} \times (\{1/m\} \times R) \prec B^n$ for $n = 1, 2, \ldots$, and therefore

$$h(\varphi|B) \geqslant h\left(\varphi, \{\omega^m\} \times \left(\left\{\frac{1}{m}\right\} \times R\right)\right) = \frac{1}{m} h\left(\varphi^m, \{\omega^m\} \times \left(\left\{\frac{1}{m}\right\} \times R\right)\right)$$

$$= \frac{1}{m} h(\varphi^m|_{\{\omega^m\} \times (\{1/m\} \times R)}) = \frac{1}{m} h(T^m|_R) = h(T|_R) = \gamma.$$

It is easy to check that $h(\varphi) = \gamma$, and thus also $h^*(\varphi) = \gamma$. Denote by $\mu_0$ the unique measure from $\mathfrak{M}(Z, \varphi)$ whose support is contained in $R \times \{0\}$, and by $\mu_n$ the unique measure from $\mathfrak{M}(Z, \varphi)$ whose support is contained in $[Q \times (\{1/n\} \times R)] \cap Z$ $(n = 1, 2, \ldots)$. It is easy to see that they are well defined, ergodic, and that there are no more ergodic measures in $\mathfrak{M}(Z, \varphi)$. Hence for any $\mu \in \mathfrak{M}(Z, \varphi)$ there exist numbers $q_n \geqslant 0$, $n = 0, 1, 2, \ldots$, such that $\sum_{i=0}^{\infty} q_i = 1$ and $\mu = \sum_{i=0}^{\infty} q_i \mu_i$. We have $h_\mu(\varphi) = \gamma$, because $h_{\mu_n}(\varphi) = \gamma$ for $n = 0, 1, 2, \ldots$ ∎

Remark 6.1. We can use the spaces and the transformations defined in Example 6.4 to obtain a slightly stronger result than that of Example 6.2. Namely, the transformation $g$ with positive conditional entropy may be a factor even of an expansive transformation under an open mapping. Take a factor of $\varphi$ under $\mathrm{id}_Q \times \varrho$. It is expansive, but $\mathrm{id}_P$ is its factor under an open mapping.

The next example shows that it is possible for an inverse limit of expansive transformations not to be even asymptotically $h$-expansive.

EXAMPLE 6.5. Write $Y_n = \{1/k\}_{k=1}^n \times Q \cup \{0\}$, $g_n\colon Y_n \to Y_n$, $g_n|_{\{1/k\}_{k=1}^n \times Q} = \mathrm{id} \times T$, $g_n 0 = 0$. We have projections $\tau_{nm}\colon Y_n \to Y_m$ for $n > m$ defined as follows:

$$\tau_{nm}\left(\frac{1}{k}, x\right) = \begin{cases} \left(\frac{1}{k}, x\right) & \text{for} \quad k \leqslant m, \\ 0 & \text{for} \quad k > m, \end{cases}$$

$\tau_{nm} 0 = 0$. It is easy to see that $(Y_n, g_n)_{n=1}^{\infty}$ with mappings $(\tau_{nm})_{n>m}$ form an inverse system with the limit $(Y, g)$. All $g_n$'s are expansive, but $h^*(g) > 0$. ∎

**§ 7. Cascades with a homogeneous measure.** Bowen in [5] proved that any continuous group endomorphism of a Lie group is $h$-expansive. We can see at once that this statement is not valid for an arbitrary compact group instead of a Lie group, because no countable infinite product of transformations with positive entropy is $h$-expansive. But we shall



show that every continuous group endomorphism of a compact group is asymptotically $h$-expansive if its entropy is finite. For this purpose we shall use the notion of a homogeneous measure.

Let $(X, f)$ be a cascade and let $\mu \in \mathfrak{M}(X)$. For $A \in \mathfrak{A}$ write

$$P(A) = \bigcup_{x \in X} \{a \in A\colon x \in a, \ [\forall_{b \in A} (x \in b \Rightarrow \mu(a) \geqslant \mu(b))]\}.$$

Of course, $P(A) \in \mathfrak{A}$. The measure $\mu$ is called *f-homogeneous* if there exist mappings $D(\cdot)\colon \mathfrak{A} \to \mathfrak{A}$ and $c(\cdot)\colon \mathfrak{A} \to (0, \infty)$ such that for any $E \in \mathfrak{A}$, $k \geqslant 1$, $a \in P(E^k)$, $d \in D(E)^k$ we have

$$\mu(d) \leqslant c(E) \cdot \mu(a). \tag{7.1}$$

It is easy to check that for $X$ being metric the above definition is equivalent to the definition given by Bowen [4].

For $A \in \mathfrak{A}$, let

$$M_k(A) = \max_{a \in A^k} \mu(a); \qquad m_k(A) = \min_{a \in P(A^k)} \mu(a).$$

For $B \geqslant A$ we have $M_k(B) \leqslant M_k(A)$, and therefore for an $f$-homogeneous measure

$$M_k(D) \leqslant c(E) \cdot m_k(E) \quad \text{for} \quad D, E \in \mathfrak{A},\ D \geqslant D(E),\ k = 1, 2, \ldots \tag{7.2}$$

LEMMA 7.1. *Let $A, B \in \mathfrak{A}$, $B \geqslant A$. Then*

$$N((\mathrm{St}\,B)^k | A^k) \cdot m_k(B) \leqslant M_k(\mathrm{St}\,A).$$

Proof. Let $a$ be the element of $A^k$ for which the number $N(a, \mathrm{St}(B^k))$ is the largest. In view of Lemma 3.3, $\mathrm{St}(B^k) \geqslant (\mathrm{St}\,B)^k$, and so this number is not smaller than $p = N((\mathrm{St}\,B)^k | A^k)$. In view of Lemma 3.1 there exists a discover $C \subset P(B^k)$ such that $c \cap a \neq \emptyset$ for $c \in C$ and that $a \subset \bigcup \{\mathrm{st}(c, B^k)\colon c \in C\}$ (because $\mathrm{st}(c, P(B^k)) \subset \mathrm{st}(c, B^k)$). We have $\mathrm{Card}\{\mathrm{st}(c, B^k)\colon c \in C\} \leqslant \mathrm{Card}\, C$, and therefore $\mathrm{Card}\, C \geqslant p$. Let $a = \bigcap_{i=0}^{k-1} f^{-i} a_i$, $a_i \in A$. We have

$$\bigcup C \subset \mathrm{st}(a, B^k)$$
$$\subset \bigcap_{i=0}^{k-1} \mathrm{st}(f^{-i} a_i, f^{-i} B) \subset \bigcap_{i=0}^{k-1} f^{-i} \mathrm{st}(a_i, B) \subset \bigcap_{i=0}^{k-1} f^{-i} \mathrm{st}(a_i, A) \in (\mathrm{St}\,A)^k.$$

Hence

$$p \cdot m_k(B) \leqslant \mu\left(\bigcup C\right) \leqslant M_k(\mathrm{St}\,A). \ ∎$$

THEOREM 7.1. *Let $(X, f)$ be a cascade, $h(f) < \infty$. If there exists an f-homogeneous measure $\mu \in \mathfrak{M}(X)$, then $f$ is asymptotically $h$-expansive.*

Proof. Putting in Lemma 7.1 $A = \Theta$, $B = E$, we obtain

$$N((\mathrm{St}\,E)^k) \cdot m_k(E) \leqslant 1 \quad \text{for} \quad E \in \mathfrak{A}. \tag{7.3}$$

The following inequality is obvious:

$$1 \leqslant N(D^k)\cdot M_k(D) \quad \text{for} \quad D \in \mathfrak{A}. \tag{7.4}$$

Hence $M_k(D) > 0$ for $D \in \mathfrak{A}$; from this and (7.2) it follows also that $m_k(E) > 0$ for $E \in \mathfrak{A}$. Thus from (7.2) and (7.4) we obtain

$$m_k(E) \geqslant \frac{1}{N(D^k)\cdot c(E)} \quad \text{for} \quad D, E \in \mathfrak{A},\ D \geqslant D(E), \tag{7.5}$$

and from (7.2) and (7.3) we obtain

$$M_k(D) \leqslant \frac{c(E)}{N\big((\mathrm{St}\,E)^k\big)} \quad \text{for} \quad D, E \in \mathfrak{A},\ D \geqslant D(E). \tag{7.6}$$

Putting $E = B$ in (7.5) and $D = \mathrm{St}\,A$ in (7.6) and applying Lemma 7.1 to them, we get

$$N\big((\mathrm{St}\,B)^k \mid A^k\big) \leqslant \frac{c(E)}{N\big((\mathrm{St}\,E)^k\big)}\cdot N(D^k)\cdot c(B) \tag{7.7}$$

$$\text{for} \quad A, B, D, E \in \mathfrak{A},\ \mathrm{St}\,A \geqslant D(E),\ D \geqslant D(B),\ B \geqslant A.$$

Hence

$$h(f, \mathrm{St}\,B \mid A) \leqslant h(f, D) - h(f, \mathrm{St}\,E) \tag{7.8}$$

$$\text{for} \quad A, B, D, E \in \mathfrak{A},\ \mathrm{St}\,A \geqslant D(E),\ D \geqslant D(B),\ B \geqslant A.$$

We may take the limit with respect to $D$:

$$h(f, \mathrm{St}\,B \mid A) \leqslant h(f) - h(f, \mathrm{St}\,E) \tag{7.9}$$

$$\text{for} \quad A, B, E \in \mathfrak{A},\ \mathrm{St}\,A \geqslant D(E),\ B \geqslant A.$$

Now we take the limit with respect to $B$, applying Proposition 3.5:

$$h(f \mid A) \leqslant h(f) - h(f, \mathrm{St}\,E) \quad \text{for} \quad A, E \in \mathfrak{A},\ \mathrm{St}\,A \geqslant D(E). \tag{7.10}$$

Further, take the limit with respect to $A$:

$$h^*(f) \leqslant h(f) - h(f, \mathrm{St}\,E) \quad \text{for} \quad E \in \mathfrak{A}. \tag{7.11}$$

Finally, taking the limit with respect to $E$ and applying Proposition 3.5 once more, we obtain (notice that here we use the assumption that $h(f) < \infty$):

$$h^*(f) \leqslant 0. \ \blacksquare \tag{7.12}$$

The assumptions of Theorem 7.1 are difficult to check directly. But they are a consequence of certain other conditions, pointed out below, which admit a much simpler verification.



Let $G$ be a group. Let $\Phi$ be a (left-hand) action of $G$ on a space $X$ (i.e., a homomorphism of $G$ into the group of all homeomorphisms of $X$). We shall write $g\cdot x = (\Phi g)x$ for $g \in G$, $x \in X$. An action is called *transitive* if $\underset{x,y\in X}{\forall}\ \underset{g\in G}{\exists}\ g\cdot x = y$. An action is called *equicontinuous* if the family $\Phi(G)$ of homeomorphisms of $X$ onto itself is equicontinuous, i.e., for any $V \in \mathfrak{N}(X)$ there exists a $W \in \mathfrak{N}(X)$ such that if $g \in G$ and $(x, y) \in W$, then $(g\cdot x, g\cdot y) \in V$. A measure $\mu \in \mathfrak{M}(X)$ is called *invariant* with respect to an action $\Phi$ if $\mu \in \bigcap_{g\in G} \mathfrak{M}(X, \Phi g)$.

THEOREM 7.2. *Let $(X, f)$ be a cascade. Let $G$ be a group and $T\colon G \to G$ a homomorphism. Further, let $\Phi$ be a transitive equicontinuous action of $G$ on $X$ such that $f(g\cdot x) = Tg\cdot fx$ for $x \in X$, $g \in G$. Let a measure $\mu \in \mathfrak{M}(X)$ be invariant with respect to $\Phi$. Then $\mu$ is $f$-homogeneous.*

Proof. Note that

$$f^n(g\cdot x) = T^n g\cdot f^n x \quad \text{for} \quad g \in G,\ x \in X,\ n = 0, 1, \ldots \tag{7.13}$$

Let $g \in G$, $Y \subset X$, $n \in \boldsymbol{N}$. We have

$$g\cdot(f^{-n}Y) = \{x\colon f^n(g^{-1}\cdot x) \in Y\} = \{x\colon T^n g^{-1}\cdot f^n x \in Y\} = f^{-n}(T^n g\cdot Y).$$

Hence

$$g\cdot(f^{-n}Y) = f^{-n}(T^n g\cdot Y) \quad \text{for} \quad g \in G,\ Y \subset X,\ n = 0, 1, \ldots \tag{7.14}$$

Now take $E \in \mathfrak{A}$. The action $\Phi$ is equicontinuous, and therefore there exists an open non-empty subset $U$ of $X$ such that the set $\{(x, y) \in X\times X\colon \underset{g\in G}{\exists}\ x, y \in g\cdot U\}$ is contained in some Lebesgue number of $E$. Write

$$U_x = \{y \in X\colon \underset{g\in G}{\exists}\ x, y \in g\cdot U\}, \qquad U_x^n = \bigcap_{i=0}^{n-1} f^{-i} U_{f^i x}$$

for $x \in X$, $n = 1, 2, \ldots$ From the definition of $U_x$ it follows that $U_x \prec E$ for $x \in X$. For $x \in X$ and $k \in G$ we have

$$k\cdot U_x = \{y \in X\colon \underset{g\in G}{\exists}\ x, k^{-1}\cdot y \in g\cdot U\} = \{z \in X\colon \underset{g\in G}{\exists}\ k\cdot x, z \in kg\cdot U\}.$$

But the left-hand side multiplication by $k$ is an isomorphism of $G$ onto itself, and therefore we obtain

$$k\cdot U_x = U_{k\cdot x} \quad \text{for} \quad x \in X,\ k \in G. \tag{7.15}$$

In view of (7.13)–(7.15), for $i = 0, \ldots, n-1$ we have

$$g\cdot f^{-i} U_{f^i x} = f^{-i}(T^i g\cdot U_{f^i x}) = f^{-i}(U_{T^i g\cdot f^i x}) = f^{-i} U_{f^i(g\cdot x)},$$

and thus

$$g\cdot U_x^n = U_{g\cdot x}^n \quad \text{for} \quad x \in X,\ g \in G,\ n = 1, 2, \ldots \tag{7.16}$$

Now we choose a finite cover $D(E)$ from the open cover $\{g\cdot U\}_{g\in G}$ of $X$. Let $x\in d\in (D(E))^n$. Then $x\in d=\bigcap_{i=0}^{n-1} f^{-i}(g_i\cdot U)$ for some $g_0,\ldots,g_{n-1}\in G$. Hence $f^ix\in g_i\cdot U$, and therefore $g_i\cdot U\subset U_{f^ix}$ for $i=0,\ldots,n-1$. Consequently, $d\subset U_x^n$. Thus we have

$$\mu(d)\leqslant \mu(U_x^n) \quad \text{if only} \quad x\in d\in (D(E))^n \ (n=1,2,\ldots). \tag{7.17}$$

Now fix $x\in X$. We have $U_{f^ix}\subset a_i$ for some $a_i\in E$, $i=0,\ldots,n-1$. Hence $x\in U_x^n\subset \bigcap_{i=0}^{n-1} f^{-i}a_i\in E^n$, and so

$$\mu(U_x^n)\leqslant \max\{\mu(a)\colon x\in a\in E^n\} \quad \text{for} \quad x\in X,\ n=1,2,\ldots \tag{7.18}$$

The measure $\mu$ is invariant with respect to $\Phi$ and $\Phi$ is transitive; thus from (7.16) it follows that

$$\mu(U_x^n)=\mu(U_y^n) \quad \text{for} \quad x,y\in X,\ n=1,2,\ldots \tag{7.19}$$

In view of (7.17)–(7.19) the measure $\mu$ is $f$-homogeneous (put $c(E)=1$). ∎

Evidently the above theorem is also valid for a right-hand action of a group.

Now we shall show the simplest examples of transformations with finite entropy for which the assumptions of Theorem 7.2 are fulfilled.

Example 7.1. Let $G$ be a compact group and $f\colon G\to G$ a continuous homomorphism, $h(f)<\infty$. Put in Theorem 7.2 $X=G$, $T=f$, $\mu$ the Haar measure of $G$, and let the action be right side multiplication. Then the assumptions are fulfilled. Hence $f$ is asymptotically $h$-expansive. If we take $fx=Tx\cdot g_0$ ($g_0$ is a fixed element of $G$), i.e., an affine transformation, then the assumptions of Theorem 7.2 are also fulfilled. ∎

Example 7.2. Let $G$ be a locally compact group with a closed subgroup $H$ such that $X=G/H$ is compact and there exists a $G$-invariant normed measure on $X$. Then a continuous homomorphism $T\colon G\to G$ which preserves $H$ induces a continuous transformation $f\colon X\to X$. The action of $G$ is natural. We assume also $h(f)<\infty$. All the assumptions of Theorem 7.2 are very easy to check, except, maybe, the equicontinuity of the action. But the natural projection $G\to G/H$ is uniformly continuous and the superposition of a uniformly continuous mapping with an equicontinuous family of mappings gives an equicontinuous family of mappings. Hence the transformation $f$ defined above is asymptotically $h$-expansive. ∎

## § 8. Connection with pressure.

Let $(X,f)$ be a cascade and let $\varphi\colon X\to \boldsymbol{R}$ be a continuous function. The notion of topological entropy may be generalized to the notion of pressure $P(f,\varphi)$ (see [17]).



The following facts are stated in [17]:

$$P(f,0)=h(f) \quad\text{and}\quad P(f,\varphi)=\sup_{\mu\in\mathfrak{M}(X,f)}\,[h_\mu(f)+\textstyle\int\varphi\,d\mu].$$

If for some $\mu_0\in\mathfrak{M}(X,f)$, the equality $P(f,\varphi)=h_{\mu_0}(f)+\int\varphi\,d\mu_0$ holds, then $\mu_0$ is called an *equilibrium state* for $(f,\varphi)$. If $h_{.}(f)$ is an upper semicontinuous function, then

$$h_\mu(f)=\inf_{\varphi\in C(X,\boldsymbol{R})}\,[P(f,\varphi)-\textstyle\int\varphi\,d\mu]. \tag{8.1}$$

Thus Theorem 4.2 yields the following

Corollary 8.1. *If $f$ is asymptotically $h$-expansive, then there exists an equilibrium state for $(f,\varphi)$ for every $\varphi\in C(X,\boldsymbol{R})$ and the formula* (8.1) *is valid for every measure $\mu\in\mathfrak{M}(X,f)$.* ∎

In Section 7 we have given some important examples in which the assumptions of this corollary are fulfilled.

We may remark that it is easy to answer the question, raised in [17], whether the formula (8.1) is always valid. The right-hand side of (8.1) regarded as a function of $\mu$ is an infimum of a family of continuous functions, and therefore it is upper semicontinuous. Hence (8.1) is valid for every $\mu\in\mathfrak{M}(X,f)$ iff $h_{.}(f)$ is an upper semicontinuous function.

### References


[1] R. L. Adler, A. G. Konheim, M. H. McAndrew, *Topological entropy*, Trans. Amer. Math. Soc. 114 (1965), pp. 309–319.

[2] P. Billingsley, *Ergodic theory and information*, Wiley, New York 1965.

[3] R. Bowen, *Topological entropy and Axiom A*, Proc. Symp. Pure Math., Vol. 14, pp. 23–41.

[4] — *Entropy for group endomorphisms and homogeneous spaces*, Trans. Amer. Math. Soc. 153 (1971), pp. 401–414.

[5] — *Entropy-expansive maps*, ibid. 164 (1972), pp. 323–333.

[6] — *Topological entropy for noncompact sets*, ibid. 184 (1973), pp. 125–136.

[7] M. Denker, *Une démonstration nouvelle du théorème de Goodwyn*, C. R. Acad. Sci. Paris 275 (1972), pp. 735–738.

[8] E. I. Dinaburg, *A connection between various entropy characterizations of dynamical systems* (Russian), Izv. Akad. Nauk SSSR, Ser. Mat. 35 (1971), pp. 324–366.

[9] T. N. T. Goodman, *Relating topological entropy and measure entropy*, Bull. London Math. Soc. 3 (1971), pp. 176–180.

[10] L. W. Goodwyn, *Topological entropy bounds measure-theoretic entropy*, Proc. Amer. Math. Soc. 23 (1969), pp. 679–688.

[11] — *The product theorem for topological entropy*, Trans. Amer. Math. Soc., 158 (1971), pp. 445–452.

[12] B. M. Gurevič, *Topological entropy of denumerable Markov chains*, Dokl. Akad. Nauk SSSR, 187 (1969), pp. 715–718. (Soviet Math. Dokl. 10 (1969), pp. 911–915)

[13] F. Hahn, Y. Katznelson, *On the entropy of uniquely ergodic transformations*, Trans. Amer. Math. Soc., 126 (1967), pp. 335–360.

[14] J. L. Kelley, *General topology*, Van Nostrand, Princeton 1955.

[15] M. Misiurewicz, *Diffeomorphism without any measure with maximal entropy*, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys. 21 (1973), pp. 903–910.

[16] M. Smorodinsky, *Ergodic theory, entropy*, Lect. Notes in Math. 214, Springer, Berlin 1971.

[17] P. Walters, *A variational principle for the pressure of continuous transformations*, preprint.


WARSAW UNIVERSITY
INSTITUTE OF MATHEMATICS

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# A reflexive Banach space which is not sufficiently Euclidean

by

WILLIAM B. JOHNSON* (Columbus)

**Abstract.** An example is given of a reflexive Banach space with unconditional basis which is not sufficiently Euclidean.

**I. Introduction.** In [8], Stegall and Retherford ask whether every reflexive Banach space $Y$ is sufficiently Euclidean; i.e., whether $Y$ contains a sequence $(E_n)$ of subspaces with $\sup d(E_n, l_2^n) < \infty$ for which there are projections $P_n$ of $Y$ onto $E_n$ satisfying $\sup \|P_n\| < \infty$. ($d(E, F)$ is the Banach–Mazur distance coefficient $\inf\{\|T\|\,\|T^{-1}\|: T$ is an isomorphism from $E$ onto $F\}$.) This problem has a negative answer. In fact, we construct a reflexive Banach space $Y$ with unconditionally monotone basis for which $\|P\| \geqslant 2^{-8} d(W, l_2^n)^{-2} n^{1/2}$ for any projection $P$ from $Y$ onto an $n$-dimensional subspace $W$.

We use standard Banach space theory notation as may be found e.g. in [6]. We would like to thank Professor T. Figiel for simplifying the proof that the example constructed in Section II is reflexive.

**II. The example.** We work with the space $X$ of sequences of scalars which have only finitely many non-zero coordinates. Given a set $E$ of integers and $x \in X$, $Ex$ is the sequence which agrees with $x$ in coordinates in $E$ and is zero in the other coordinates. A sequence $(E_i)_{i=1}^n$ of sets of positive integers is called *allowable* provided $E_i \cap E_j = \emptyset$ for $i \neq j$ and $E_i \subseteq [n+1, \infty)$ for $1 \leqslant i \leqslant n$. We will construct a norm $\|\cdot\|$ on $X$ for which the unit vectors $(e_n)$ form an unconditionally monotone basis so that the completion of $(X, \|\cdot\|)$ is reflexive and $X$ satisfies

$$(*) \qquad \|x\| = \max\Big(\|x\|_{c_0}, \tfrac{1}{2}\sup\Big\{\sum_{i=1}^{n} \|E_i x\|: (E_i)_{i=1}^n \text{ is allowable}\Big\}\Big).$$

All Euclidean subspaces of $Y = [e_{p(n)}]$ are badly complemented if $X$ satisfies $(*)$ and $(p(n))$ grows fast enough. The proof of this assertion makes use of the following proposition, whose proof is omitted because it involves only a nonessential modification of the argument for Theorem A in [7] and a standard perturbation argument.

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