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Norm-decreasing isomorphism on hermitian elements and the group of isometric and invertible multipliers of a Banach algebra

by

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Abstract. Let A_i $(i = 1, 2)$ be a complex Banach algebra; T, a norm decreasing algebra isomorphism of A_1 onto A_2; $H(A_i)$, the Banach space of hermitian elements in A_i and $G(A_i)$, the group of isometric and invertible multipliers in A_i. We show that

(i) If A_i is unital, $TH(A_1) \subseteq H(A_2)$, and $TG(A_1) \subseteq G(A_2)$. But if $A_1 = A_2 = A$ and $TG(A) = G(A)$, then $TH(A) = H(A)$.

(ii) If A_i is a B^* algebra, then T is a *-isomorphism.

(iii) If A_i has a minimal approximate identity; T^m, the induced map of the multiplier algebra A_1^m onto A_2^m is a norm decreasing extension of T and $T^m G(A_1^m) \subseteq G(A_2^m)$.

We finally construct an example to show that T does not in general preserve $G(A)$ and $H(A)$.

1. Introduction. We shall investigate, in this paper, the effect of norm-decreasing algebra isomorphism on hermitian elements $H(A)$ and the group of isometric and invertible multipliers $G(A)$ of a Banach algebra A. The motivation for this work is Wendel's paper in [7] on the preservation of $G(A)$ by a norm-decreasing T when A is a group algebra, Rigelhof's in [5] when A is a measure algebra on a locally compact group, and Wood's in [8] where A is $L^p(G)$ for compact group. We shall indicate that $G(A)$ and $H(A)$ are not, in general, preserved by a norm-decreasing T. But when $G(A)$ is preserved, $H(A)$ is also preserved.

This work is a part of the author's Ph.D. thesis and I wish to express my thanks to Dr G. V. Wood of the University College of Swansea for his help and advice as my Supervisor throughout my three years stay in Swansea.

2. Notations and definitions. We shall always consider Banach algebras A over the complex field $\boldsymbol{C}$ [A, assumed to be without order (i.e. $\forall\, x \in A$, $xA = (0) \Rightarrow x = 0$ or $Ax = (0) \Rightarrow x = 0$)]. We shall denote by $\boldsymbol{R}$ the real scalars and by 1, the identity in A if it has one. $\varrho(x)$ denotes the spectral radius of $x \in A$.

2.1. Definition: *Hermitian Elements* (see [1]). Let A be a complex unital Banach algebra (i.e. $1 \in A$ and $\|1\| = 1$). We denote by A^* the *dual* *space* of A and by $S(A)$ the *unit sphere* of A. Given $x \in S(A)$, we define

$$D(A, x) = \{f \in A^*: f(x) = 1 = \|f\|\}.$$

Given $a \in A$ and $x \in S(A)$, let

$$V(A, a, x) = \{f(ax): f \in D(A, x)\}$$

and

$$V(A, a) = \bigcup\{V(A, a, x): x \in S(A)\}.$$

$V(A, a)$ is called the *numerical range* of a. $h \in A$ is *hermitian* if $V(A, h) \subset \boldsymbol{R}$ and we shall denote by $H(A)$ the *set of all hermitian elements* of A.

$B(A)$ denotes the *Banach algebra* of all bounded linear operators in A. L_x (defined by $L_x a = xa\ \forall\, a \in A\ x \in A$) denotes the *left multiplication operator*. The *right multiplication operator* R_x is similarly defined.

If A has no identity, then $h \in A$ is hermitian if $L_h \in B(A)$ is hermitian.

We shall need the following results on hermitian elements.

2.2. Proposition. *Let A be a complex unital Banach algebra. Given $h \in A$, the following statements are equivalent*:

(i) $h \in H(A)$;

(ii) $\lim_{a\to 0} \frac{1}{a}\{\|1+iah\|-1\} = 0$;

(iii) $\|\exp iah\| = 1$ $(a \in \boldsymbol{R})$ (see Lemma 2, § 5 of [1]).

2.3. Proposition. *$H(A)$ is a Banach space* (see Lemma 4, § 5 of [1]).

2.4. Definition: *Multiplier algebras* (see [2]). The bounded linear operator φ on A is a *multiplier* of A if $\varphi(xy) = (\varphi x)y = x(\varphi y)\ \forall x, y \in A$. If $\varphi(xy) = (\varphi x)y$, φ is said to be a *left multiplier*. φ is a *right multiplier* if $\varphi(xy) = x(\varphi y)$. To save repetition, we shall deal with Banach algebras of left multipliers only in this paper; we denote this by A^m. Clearly, $L_x \in A^m$. If A has an identity, then $A = A^m$. $\varphi \in G(A)$ if $\varphi \in A^m$ and $\|\varphi^{-1}x\| = \|\varphi x\| = \|x\|\ \forall x \in A$. If A has an identity, then

$$G(A) = \{x: \|x\| = \|x^{-1}\| = 1\}.$$

In fact, $G(A)$ is a topological group in the strong operator topology (SOT). (see Lemma 1.6.1 of [2]). A net $\{\varphi_a\}$ *converges* to φ in the SOT iff $\lim_a \|\varphi_a x - \varphi x\| = 0\ \forall x \in A$. It converges in the weak operator topology (WOT) iff for each $x \in A$ and $y^* \in A^*$ we have

$$\lim_a |y^*(\varphi_a x) - y^*(\varphi x)| = 0.$$

2.5. Definition. A net $\{x_a\}$ in a Banach algebra A is a *left approximate identity* if $\lim_a \|x_a x - x\| = 0$ for each $x \in A$. It is a *right approximate identity* if $\lim_a \|x x_a - x\| = 0$ for each $x \in A$. It is an *approximate identity* if it is both left and right approximate identity. It is *minimal* if $\lim_a \|x_a\| = 1$ in addition.

$\{L_x: x \in A\}$ is dense in A^m in the SOT iff A has an approximate identity (see Theorem 1.1.6 of [2]).

3. Isomorphism of $H(A)$ and $G(A)$.

3.1. Theorem. *Let T be a norm decreasing algebra isomorphism of a complex unital Banach algebra A_1 onto another A_2. Then*

(i) *$TH(A_1) \subseteq H(A_2)$ and*

(ii) *$TG(A_1) \subseteq G(A_2)$.*

Proof. (i) Let $h \in H(A_1)$; then

$$\|\exp iah\| = 1, \quad a \in \boldsymbol{R} \quad \text{by 2.2.}$$

Since T is algebraic, $T\exp iah = \exp iaTh$. Hence

$$\|\exp iaTh\| = \|T\exp iah\| \leqslant \|\exp iah\| = 1 \quad \forall a \in \boldsymbol{R},$$

i.e.

$$\|\exp iaTh\| = 1 \quad \text{and} \quad Th \in H(A_2).$$

This proves (i).

(ii) Let $x \in G(A_1)$. Then $\|x\| = \|x^{-1}\| = 1$ since $A = A^m$. Hence $\|Tx\| \leqslant \|x\| = 1$ and $\|Tx^{-1}\| \leqslant \|x^{-1}\| = 1$. But $Tx^{-1} = (Tx)^{-1}$. Therefore $\|Tx\| = \|(Tx)^{-1}\| = 1$ and (ii) is proved. ■

Before proceeding with our investigation, we shall use 3.1 to show that norm decreasing is sufficient for Corollary 4.8.19 of [3] to hold.

3.2. Theorem. *Any norm-decreasing isomorphism T between two B^* algebras A_1 and A_2 is a* *.

Proof. It is known that if A is a Banach algebra with an approximate identity and $\tilde{A}$ is the unitization of A, then

$$H(A) = A \cap H(\tilde{A}) \quad \text{(see 1.3.6 of [3]).} \tag{1}$$

Let T be an algebra isomorphism of a Banach algebra A_1 onto another A_2 and $\tilde{T}$ be defined thus:

$$\tilde{T}(x, a) = (Tx, a), \quad x \in A_1,\ a \in C \text{ and } (x, a) \in \tilde{A}_1.$$

Clearly, $\tilde{T}$ is an algebra isomorphism of $\tilde{A}_1$, $\tilde{A}_2$ and it is norm-decreasing if T is norm-decreasing. The norm in $\tilde{A}_1$ is defined by

$$\|(x, a)\| = \|x\| + |a|.$$

Hence, if A_i $(i=1,2)$ has an approximate identity, we have

$$\begin{aligned} TH(A_1) = \tilde{T}\big(H(A_1)\big) &= \tilde{T}\big(A_1 \cap H(\tilde{A}_1)\big) && \text{from (1)} \\ &\subseteq T(A_1) \cap \tilde{T}H(\tilde{A}_1) \\ &\subseteq A_2 \cap H(\tilde{A}_2) && \text{by 3.1} \\ &= H(A_2) && \text{from (1).} \end{aligned}$$

Since a B^* algebra has an approximate identity, T then maps a hermitian element to a hermitian element. But an element of a B^* algebra is hermitian iff it is self-adjoint (see 2.1.3 of [3]). Hence T is a $*$.

We shall now show that under certain conditions 3.1 holds for non-unital algebras. But we first prove a result which could be of independent interest.

3.3. Lemma. *Let A_i $(i=1,2)$ be a Banach algebra with a minimal approximate identity. Suppose that T is a norm decreasing algebra isomorphism of A_1 onto A_2. Then T^m, the induced map of A_1^m onto A_2^m is a norm-decreasing extension of T.*

Proof. Let $\varphi \in A_1^m$, T^m is given by

$$T^m\varphi = T\varphi T^{-1} \quad \text{(see [6]).} \tag{2}$$

We shall only show that T^m is a norm-decreasing extension of T as other properties of an algebra isomorphism can easily be verified. Let $\{x_a\}$ be a minimal approximate identity in A_1. Then $\{Tx_a\}$ is an approximate identity in A_2 and $\|Tx_a\| \leqslant 1$. Let $x' \in A_2$; we have

$$(T^m\varphi)(Tx_a\cdot x') = (T\varphi T^{-1})(Tx_a\cdot x') = (T\varphi T^{-1}Tx_a)\cdot x' = (T\varphi x_a)x'.$$

Hence

$$\begin{aligned} \|(T^m\varphi)(Tx_a\cdot x')\| &= \|(T\varphi x_a)\cdot x'\| \\ &\leqslant \|T\|\,\|\varphi\|\,\|x_a\|\,\|x'\| \\ &\leqslant \|\varphi\|\,\|x'\|. \end{aligned}$$

Also

$$\liminf \|(T^m\varphi)(Tx_a\cdot x')\| = \|(T^m\varphi)x'\| \leqslant \|\varphi\|\,\|x'\|.$$

Therefore $\|T^m\varphi\| \leqslant \|\varphi\|$.

Since $\{L_x\colon x \in A\}$ is strong operator dense in A_1^m (see 2.5) and $\|x\| = \|L_x\|$ $\forall x \in A_1$, T^m is a norm-decreasing extension of T.

3.4. Theorem. *Suppose T is a norm-decreasing isomorphism of A_1 onto A_2 (A_i as in Lemma 3.3 above). Then*

(i) $TH(A_1) \subseteq H(A_2)$ *and*

(ii) $T^mG(A_1^m) \subseteq G(A_2^m)$.

Proof. Since A_i^m $(i=1,2)$ is a complex unital Banach algebra, (ii) is clear from 3.1 and 3.3. (i) Let $h \in H(A_1)$. Then $L_h \in H(A_1^m)$ (see 2.1). Since T^m is norm-decreasing (by 3.3), T^mL_h is in $H(A_1^m)$ by 3.1 (i).

Now, let $x_1 \in A_1$ and $x_2 \in A_2$; then

$$(T^mL_{x_1})x_2 = (TL_{x_1}T^{-1})x_2 = T\big(x_1(T^{-1}x_2)\big) = (Tx_1)x_2 = L_{Tx_1}x_2.$$

Hence

$$T^mL_h = L_{Th} \in H(A_2^m), \quad \text{i.e.} \quad Th \in H(A_2). \ \blacksquare$$

3.5. Remark. Lemma 3.3 is not known to be true if A_i $(i=1,2)$ contains neither an identity nor a minimal approximate identity. We also have no counterexample. But if A_i has the operator norm, and T is norm-decreasing, then T^m is clearly norm-decreasing on $\{L_x\colon x\in A\}$. Hence $TH(A_1) \subseteq H(A_2)$ in this case. It is not clear, however, whether T^m maps $G(A_1^m)$ onto $G(A_2^m)$ or not.

3.6. Remark. Norm-decreasing always implies isometry on $H(A)$ and $G(A)$ if A is a unital Banach algebra: The case for $G(A)$ is clear from the proof of 3.1. Let $h \in H$. Then $\varrho(h) = \|h\|$ (see [6]). Since T is algebraic, $\varrho(h) = \varrho(Th)$ and Th is a hermitian element of A by 3.1. Hence $\|h\| = \varrho(h) = \varrho(Th) = \|Th\|$. However, if T is an isometry, then it preserves both $H(A)$ and $G(A)$. For $L^p(G)$ and $M(G)$ algebras, norm decreasing of T was sufficient for the preservation of $G(A^m)$. It is easy to generalise the $L'(G)$ case by assuming

(1) that A has a minimal approximate identity;

(2) that $G(A^m)$ is strong operator dense in A^m and

(3) that no closed proper subgroup of $G(A^m)$ is dense in A^m.

Assumption (3) is not necessary in general for let $A = C([0,1])$. Then $G(A) = \{f\in A\colon |f(x)| = 1\ \forall x \in [0,1]\}$. $F = \{f \in G(A)\colon f(0) = 1\}$ is a proper subgroup of $G(A)$. But both $G(A)$ and F generate A by Stone–Weierstrass' theorem. However, since A is a B^* algebra, T is an isometry (by an earlier paper). Hence $TG(A) = G(A)$. The following assumption on A also generalises the preservation of $G(A)$ by a norm-decreasing T when $A = M(G)$ for discrete G:

(i) A has an identity.

(ii) Every element $x \in A$ is the norm limit of the linear span of the group $G(A)$ of isometric and invertible elements of A.

(iii) No subgroup of $G(A)$ spans A.

Rigelhof in [5] proves the case for locally compact group G but the topologies involved are the strong operator and the weak* topologies.

If A is a finite-dimensional Banach algebra, then $H(A)$ is a finite-dimensional Banach space (see 2.3) and since isomorphism preserves dimensionality, $TH(A) = H(A)$.

Preservation of $G(A)$ by a norm-decreasing T implies the preservation of $H(A)$ as we now show.

3.7. THEOREM. *Let T be a norm-decreasing algebra automorphism of a complex unital Banach algebra. Then $TG(A) = G(A)$ implies $TH(A) = H(A)$.*

Proof. Let $G_1(A)$ be the group generated by $E = \{\exp ih\colon h \in H(A)\}$. Then $G_1(A) \subset G(A)$ since $\|\exp ih\| = 1$. In fact, $TG_1(A) \subset G_1(A)$: for let $x = \exp ih$. $Tx = T\exp ih = \exp iTh$. But $Th \in H(A)$ by 3.1. Hence $Tx \in G_1(A)$. Since $G(A) \subseteq TG(A)$ by hypothesis; $T^{-1}x \in G(A)$ and $T^{-1}x = T^{-1}\exp ih = \exp iT^{-1}h$. Therefore, $\|\exp iT^{-1}h\| = \|T^{-1}x\| = 1$, i.e. $T^{-1}h \in H(A)$ by 2.2. Combining this with 3.1, we have $TH(A) = H(A)$ and $TG_1(A) = G_1(A)$.

3.8. Remark. The converse of 3.7 is not known. It is, however, clear that $G_1(A)$ is not necessarily dense in $G(A)$: For, let $A = C(\pi)$, (π the unit circle in the complex plane). $G(A) = \{f \in C(\pi)\colon \|f\| = \|f^{-1}\| = 1\}$ and $E = \{\exp ig\colon g$ is a real-valued function in $C(\pi)\}$. Let $z \in \pi$. Then $zG(A) \subset G(A)$ but $zE \not\subset E$. It is clear, however, that $TH(A) = H(A)$ implies $TG_1(A) = G_1(A)$.

We shall conclude this paper with an example to show that norm decreasing isomorphism does not, in general, preserve $G(A)$ and $H(A)$.

3.9. EXAMPLE. Let C' be the Banach space of all sequences of bounded variation with norm defined by

$$\|x'\| = |x_1'| + \sum_{n=1}^{\infty} |x_{n+1}' - x_n'| \quad \text{for} \quad x' \in C', \tag{3}$$

and C'', the Banach space of all convergent sequences with supremum norm

$$\|x''\| = \sup |x_n''| \quad \text{for} \quad x'' \in C'', \tag{4}$$

C'' is clearly a Banach algebra under pointwise multiplication. C' is also a Banach algebra under pointwise multiplication, for it can be shown by induction that

$$\Big(|x_1'| + \sum_{n=1}^{k} |x_{n+1}' - x_n'|\Big)\Big(|y_1'| + \sum_{n=1}^{k} |y_{n+1}' - y_n'|\Big)$$
$$\geqslant |x_1'y_1'| + \sum_{n=1}^{k} |x_{n+1}'y_{n+1}' - x_n'y_n'| \quad \forall \text{ positive integers } k.$$

If we allow k to tend to infinity, then

$$\|x'y'\| \leqslant \|x'\|\,\|y'\|, \quad x', y' \in C'.$$

All other properties are easily verified. C' and C'' are then complex unital Banach algebras with sequence (1) as the unit element and e_n as the basic elements in each of them (e_n is the sequence with 0 in every entry but the nth which is 1 and (1) is the sequence with 1 in every entry).

We now define the Banach algebra A as the direct sum of C' and C'' ($A = C' \oplus C''$) with norm defined by

$$\|(x', x'')\| = \max\{\|x'\|, \|x''\|\}, \quad (x', x'') \in A, \; x' \in C', \; x'' \in C''.$$

We define a map T on A thus:

$$T(x', x'') = (U', U''),$$

where

$$U_n' = x_{n+1}', \quad n \geqslant 1,$$
$$U_1'' = x_1', \quad U_n'' = x_{n-1}'', \quad n > 1.$$

Clearly, T is linear, multiplicative and one-to-one. It is also easy to show that T is onto. T is norm-decreasing since

$$\begin{aligned}\|(x', x'')\| &= \max\Big\{|x_1'| + \sum_{n=1}^{\infty} |x_{n+1}' - x_n'|,\ \sup |x_n''|\Big\}\\ &\geqslant \max\Big\{|x_2'| + \sum_{n=2}^{\infty} |x_{n+1}' - x_n'|,\ \max[|x_1'|, \sup |x_n''|]\Big\}\\ &= \max\{\|U'\|, \|U''\|\}\\ &= \|T(x', x'')\|.\end{aligned}$$

Using the fact that $x \in H(C')$ implies $\lim_{\alpha\to 0} \frac{1}{\alpha}\{\|1 + i\alpha x\| - 1\} = 0$, it is easy to show that $H(C')$ is the set of all real scalar multiples of the identity in C' and $H(C'')$ is the set of all real convergent sequences in C''.

Using the fact that $x \in H(C) \Rightarrow \|\exp i\alpha x\| = 1$, it is also easy to show that

$$H(C') \oplus H(C'') = H(A).$$

Suppose

$$(\lambda) \in H(C') \text{ and } (\mu_n) \in H(C'').$$

Then $\big((\lambda), (\mu_n)\big) \in H(A)$ and, by definition,

$$T\big((\lambda), (\mu_n)\big) = \big((\lambda), (v_n)\big)$$

where

$$v_1 = \lambda \quad \text{and} \quad v_n = \mu_{n-1}, \quad n > 1.$$

But not every element of $H(A)$ is of the form $\big((\lambda), (v_n)\big)$. Hence $TH(A) \neq H(A)$.

It can also be shown, by calculation, that

$$G(C') = \{x' \in C' \colon x' = (e^{i\alpha}),\ \alpha \in \mathbf{R}\}$$

and

$$G(C'') = \{x'' \in C'' \colon x'' = (e^{i\beta n})\}.$$

It is easy to show that

$$\big(G(C'),\ G(C'')\big) = G(A).$$

Hence

$$T\big((e^{i\alpha}),\ (e^{i\beta n})\big) = \big((e^{i\alpha}),\ (k_n)\big),$$

where

$$k_1 = e^{i\alpha} \quad \text{and} \quad k_n = e^{i\beta n} \quad \text{for} \quad n > 1.$$

Since not every element of $G(A)$ is of the form $\big((e^{i\alpha}),\ (k_n)\big)$, $TG \neq G$.

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A converse to some inequalities and approximations in the theory of Stieltjes and stochastic integrals, and for nth derivatives

by

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Abstract. The object of this report is to establish by counter-examples the best possible character of theorems recently obtained about stochastic integrals and Stieltjes integrals, and about nth derivatives and finite differences. The hypotheses involve a pair of estimate functions subject to the convergence of a corresponding integral or Y-series, and it is shown that the divergence of this integral or series render in each case the conclusion false.

1. Our notation will be largely that of [6], [7]. Let n be a positive integer, and let $\varphi(u)$, $\psi(u)$ be functions defined for $0 \leqslant u \leqslant 1$, such that φ is non-negative and Borel measurable, while ψ is continuous and monotone increasing, and takes the value $\psi(u) = 0$ only at $u = 0$; further suppose that, for $0 < \lambda < 1$,

$$\varphi(\lambda u) \geqslant (\tfrac{1}{2}\lambda)^n \varphi(u), \qquad \psi(\lambda u) \geqslant \tfrac{1}{2}\lambda\psi(u). \tag{1.1}$$

We denote by $\{h\}$ a decreasing sequence h_ν $(\nu = 0, 1, \ldots)$ with limit 0 and with initial term $h_0 \leqslant 1$. We write

$$Y = \sum_{\nu=0}^{\infty} (h_\nu)^{-n} \varphi(h_\nu)\psi(h_\nu), \tag{1.2}$$

and we denote by Y_N its partial sum for $0 \leqslant \nu < N$. For $n = 1$, $h_\nu = 2^{-\nu}$, the series (1.2) occurs in [2] and it then converges or diverges with the sum $\sum \varphi(1/\nu)\psi(1/\nu)$ previously introduced in [5]. This last sum has been termed Y-series by Leśniewicz and Orlicz [1]. We prefer here to term *Y-series* the series (1.2): it was itself introduced, for $n = 1$, in [3].

We say that the sequence $\{h\}$, or the Y-series Y, satisfies the condition $C(1)$, if for each ν the ratio $h_{\nu-1}/h_\nu$ is an integer expressible as an integer power of 2, and satisfies the condition $C(2)$ if

$$2\psi(h_\nu) \leqslant \psi(h_{\nu-1}) \leqslant 8\psi(h_\nu). \tag{1.3}$$

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