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On the summability of Fourier series of functions of Λ-bounded variation

by

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Abstract. For f in ΛBV with $\Lambda = \{n^{\beta+1}\}$, it is shown that the Fourier series of f, $S[f]$, is everywhere (C, β) bounded, $-1 \leqslant \beta < 0$, and everywhere (C, α) summable for $\alpha > \beta$. If f is continuous in $\{n^{\beta+1}\}$-variation, $-1 < \beta < 0$, $S[f]$ is everywhere (C, β) summable. These results hold uniformly on each closed interval of continuity of f. If $\Lambda\mathrm{BV} \supset \{n^{\beta+1}\}\mathrm{BV}$ properly, there is a continuous function f in ΛBV such that $S[f]$ is not (C, β) bounded at some point. A lemma on the continuity of the Λ-variation and a summability test similar to the Lebesgue convergence test are the principal tools. A new proof is indicated for the fact that functions of harmonic bounded variation satisfy the Lebesgue test.

It is well known that functions of various generalized bounded variation classes have Fourier series which converge everywhere and converge uniformly on each closed interval of continuity. Here we shall show that the Fourier series of functions of one such class, the definition of which depends on a parameter β, $-1 \leqslant \beta < 0$, are everywhere (C, β) bounded, uniformly (C, β) bounded on each closed interval of continuity and, for $\alpha > \beta$, are everywhere (C, α) summable and uniformly (C, α) summable on each closed interval of continuity. With an additional restriction, we may choose $\alpha = \beta$ for $\beta > -1$. We also show that, for $\beta > -1$, each larger class of ΛBV functions contains a continuous function whose Fourier series is not (C, β) bounded at some point.

In § 1 we present the basic concepts and state the principal results. In § 2 we prove the result which shows in what sense the main theorem is best possible. In § 3 we establish a lemma on the continuity of the Λ-variation which is the principal tool in the proof of the main theorem. In § 4 we present the proof of the main theorem and, finally, in § 5 we show that the lemma of § 3 can be employed to furnish a simpler demonstration of the convergence properties of functions of harmonic bounded variation.

1. Definitions and results. Let f be a real function of period 2π. Let Λ denote a non-decreasing sequence of positive numbers λ_n such that $\sum 1/\lambda_n$ diverges, $\{I_n\}$ a sequence of non-overlapping intervals $I_n = [a_n, b_n] \subset [0, 2\pi]$, and let $f(I_n) = f(b_n) - f(a_n)$. The function f is said to be of

Λ-bounded variation (ΛBV) if the *Λ-variation* of f,

$$V_\Lambda(f) = \sup\Big\{\sum |f(I_n)|/\lambda_n\colon \{I_n\}\Big\} < \infty.$$

The collection of ΛBV functions is a Banach space with norm $\|f\| = |f(0)| + V_\Lambda(f)$. Let $\Lambda^m = \{\lambda_{n+m}\}$, $m = 0, 1, 2, \ldots$ A function f of ΛBV is said to be *continuous in Λ-variation* if $V_{\Lambda^m}(f) \to 0$ as $m \to \infty$. When $\Lambda = \{n\}$, the class ΛBV is referred to as the functions of *harmonic bounded variation*. We have studied the convergence behavior of the Fourier series of functions of this class in a previous paper [3].

Here we shall consider the summability properties of Fourier series of functions of class $\{n^{\beta+1}\}$BV with $-1 \leqslant \beta < 0$.

For $\beta = -1$ it is clear that our class is exactly the functions of bounded variation in the usual sense. A numerical series $\sum a_n$ is said to be $(C, -1)$ summable if it converges and $a_n = o(n^{-1})$. It is said to be $(C, -1)$ bounded if its partial sums are bounded and $a_n = O(n^{-1})$. For a series of functions we may speak of *uniform* $(C, -1)$ summability or boundedness on a set if the convergence and the order condition are uniform on that set. Clearly, the Fourier series of a BV function is $(C, -1)$ bounded, convergent everywhere, uniformly convergent on each closed interval of continuity, and will, therefore, by a theorem of Hardy and Littlewood ([1], p. 121), be (C, α) summable for every $\alpha > -1$ and uniformly (C, α) summable on each closed interval of continuity.

It is known that functions of ΛBV have only simple discontinuities. We may assume, without loss of generality, that they are *regulated*, that is, $f(x) = \frac{1}{2}[f(x+)+f(x-)]$ for each x.

Our main result is the following

THEOREM 1. *The Fourier series, $S[f]$, of a function f of class $\{n^{\beta+1}\}BV$, $-1 \leqslant \beta < 0$, is everywhere (C, β) bounded and is uniformly (C, β) bounded on each closed interval of continuity. For $\alpha > \beta$, $S[f]$ is everywhere (C, α) summable to sum $f(x)$ and summability is uniform on each closed interval of continuity.*

If f is continuous in $\{n^{\beta+1}\}$-variation, $-1 < \beta < 0$, then $S[f]$ is everywhere (C, β) summable to sum $f(x)$ and summability is uniform on each closed interval of continuity.

This result is best possible in a certain sense, as is expressed in our second theorem.

THEOREM 2. *If $\Lambda BV \supset \{n^{\beta+1}\}BV$ properly, $-1 < \beta < 0$, then there is a continuous function in ΛBV whose Fourier series is not (C, β) bounded at some point.*

The proof of Theorem 2 is quite direct. Theorem 1 is based on a test for (C, β) summability obtained jointly with B.N. Sahney [2].

As is usual we denote the Fourier series of an integrable function f of period 2π by $S[f]$ and set

$$\varphi(t) = \varphi_x(t; f) = \tfrac{1}{2}\,[f(x+t)+f(x-t)-2f(x)],$$

$$\Phi(t) = \Phi_x(t; f) = \int_0^t |\varphi(u)|\,du.$$

Our test is embodied in the following

THEOREM. *$S[f]$ is (C, β) summable to $f(x)$, $-1 < \beta < 0$, at every point x at which*

(i) $\Phi(h) = o(h)$,

(ii) $\Phi(\pi) - \Phi(\pi - h) = o(h^{-\beta})$,

(iii) $\displaystyle\int_\eta^{\pi-\eta} \frac{|\varphi(t)-\varphi(t+\eta)|}{t^{\beta+1}}\,dt = o(n^\beta)$ *as* $\eta = \pi\Big/\Big(n+\dfrac{\beta+1}{2}\Big) \to 0$,

and summability is uniform over any closed interval of continuity where conditions (ii) *and* (iii) *are satisfied uniformly. If f is bounded, condition* (ii) *may be omitted; if f is regulated, conditions* (i) *and* (ii) *may be omitted.*

If (i′), (ii′), (iii′) *denote* (i), (ii), (iii) *with "o" replaced by "O", $S[f]$ will be (C, β) bounded at every point x at which* (i′), (ii′), (iii′) *hold, and uniformly (C, β) bounded on every set on which the conditions are satisfied uniformly. If f is bounded, conditions* (i′) *and* (ii′) *may be omitted.*

We turn next to the proof of Theorem 2.

2. Proof of Theorem 2. Since $\Lambda BV \supset \{n^{\beta+1}\}BV$ properly, there is a sequence, $\{a_n\}$, such that $a_n \searrow 0$, $\sum a_n/\lambda_n < \infty$ and $\sum a_n/n^{\beta+1}$ diverges.

For $x \in [0, 2\pi]$ let $\tau(i, n) = (2i+\beta/2)\eta$, $\tau'(i, n) = (2i+1+\beta/2)\eta$. Let χ_{in} be the characteristic function of the interval $I_{in} = (\tau(i, n), \tau'(i, n))$. Let

$$f_n(x) = \sum_{i=1}^{n-1} a_i \chi_{in} \quad \text{for} \quad x \in [0, 2\pi]$$

and be of period 2π on the real line. Clearly, in ΛBV,

$$\|f_n\| \leqslant 2\sum_1^{n-1} a_i/\lambda_i = b_n = O(1) \quad \text{as} \quad n \to \infty.$$

Letting $\sigma_n^\beta(x; f)$ denote the nth (C, β) mean of $S[f]$ at x, for some $\theta(t)$, $|\theta(t)| \leqslant 1$, we have ([4], p. 85)

$$\pi\sigma_n^\beta(0; f_n) = \int_0^{2\pi} f_n(t)\left(\frac{\sin\Big[\Big(n+\dfrac{\beta+1}{2}\Big)t - \dfrac{\pi\beta}{2}\Big]}{A_n^\beta\big(2\sin(t/2)\big)^2} + \frac{2\theta\beta}{n\big(2\sin(t/2)\big)^2}\right)dt$$
$$= \sum_{i=1}^{n-1}\int_{I_{in}} \ldots$$

$$\geqslant \frac{1}{A_n^\beta}\sum_{i=1}^{n-1}\frac{2}{\left(n+\frac{\beta+1}{2}\right)}\cdot\frac{a_i}{(\tau'(i,n))^{\beta+1}} - \frac{2}{n}\,\frac{\pi}{n+\frac{\beta+1}{2}}\sum_{i=1}^{n-1}\frac{a_i}{\left(\frac{2}{\pi}\tau(i,n)\right)}$$

$$\geqslant C\sum_{i=1}^{n-1}\frac{a_i}{i^{\beta+1}} - C'\sum_{i=1}^{n-1}\frac{a_i}{i^2},$$

where C and C' are independent of n. The last sum is $O(1)$ as $n\to\infty$; the first becomes unbounded. Thus

$$\|\sigma_n^\beta(0;\cdot)\| \geqslant |\sigma_n^\beta(0;f_n)|/b_n \to \infty$$

as $n\to\infty$, implying that there is an $f\in\Lambda\mathrm{BV}$ for which $\{\sigma_n^\beta(0;f)\}$ is unbounded. Since the continuous functions of $\Lambda\mathrm{BV}$ themselves form a Banach space and the f_n can be modified so as to be continuous without any substantial change in the argument, we have the desired result.

3. Preliminaries to the proof of Theorem 1. We have already defined the Λ-variation of a function f with domain $[0, 2\pi]$. If I is any closed interval in $[0, 2\pi]$ and $\{I_n\}$ denotes a collection of non-overlapping intervals, we may define the *Λ-variation of f on I* to be

$$V_\Lambda(f; I) = \sup\Big\{\sum f(I_n)/\lambda_n:\ \{I_n\}\ \text{such that}\ I_n\subset I\Big\}.$$

The following lemma on the continuity of the Λ-variation will be of considerable use to us.

LEMMA. *Let f be of class ΛBV on $I = [a, b]$. Then*

(i) *if f is right continuous at a,*

$$V_\Lambda(f; [a, x]) \to 0 \quad \text{as} \quad x\searrow a,$$

if f is left continuous at b,

$$V_\Lambda(f; [x, b]) \to 0 \quad \text{as} \quad x\nearrow b,$$

(ii) *if $[x, y]\subset I^0$, then*

$$V_\Lambda(f; [x, y]) \to 0$$

as x and y together approach either a or b.

Proof. It is clear that (i) implies (ii), for if we suppose $[x, y]\subset I^0$ and set $g(x) = f(x)$ for $x\in(a, b)$ and $g(a) = f(a+)$, $g(b) = g(b-)$, then g is right continuous at a and left continuous at b and

$$0\leqslant V_\Lambda(f; [x, y]) = V_\Lambda(g; [x, y]) \leqslant V_\Lambda(g; [a, y]) \to 0 \quad \text{as} \quad y\searrow a.$$

Similarly,

$$0\leqslant V_\Lambda(f; [x, y]) \leqslant V_\Lambda(g; [x, b]) \to 0 \quad \text{as} \quad x\nearrow b.$$

Thus we need only prove (i). We consider only the case in which f is right continuous at a.

There is a collection of non-overlapping intervals $\{I_n\}_1^{N_1}$, such that $|f(I_n)|\searrow$, $I_n\subset[a, b]$, $f(I_n)\neq 0$ for each n, and

$$\sum_1^{N_1}|f(I_n)|/\lambda_n > \tfrac{1}{2}V_\Lambda(f; [a, b]).$$

Since f is right continuous at a, we may assume that $I_n\subset(a, b]$ for each n.

Choose $y_1\in(a, b)$ such that

$$[a, y_1]\cap\bigcup_1^N I_n = \emptyset, \qquad y_1 < (a+b)/2,$$

and $I\subset(a, y_1]$ implies

$$|f(I)|\leqslant\min\{|f(I_n)|:\ n = 1,\ldots,N_1\}.$$

Then there is a collection of non-overlapping intervals $\{I_n\}_{N_1+1}^{N_2}$ such that $|f(I_n)|\searrow$, we have $I_n\subset(a, y_1]$ and $f(I_n)\neq 0$ for each n, and

$$\sum_{N_1+1}^{N_2}|f(I_n)|/\lambda_{n-N_1} > \tfrac{1}{2}V_\Lambda(f; [a, y_1]).$$

Continuing in this manner we can choose, for $k = 1, 2, \ldots$, $y_k\searrow a$ (by choosing $y_{k+1} < (a+y_k)/2$) and $I_{N_k+1},\ldots,I_{N_{k+1}}$, non-overlapping intervals contained in $[y_{k+1}, y_k]$ with

$$\sum_{N_k+1}^{N_{k+1}}|f(I_n)|/\lambda_{n-N_k} > \tfrac{1}{2}V_\Lambda(f; [a, y_k])$$

and $|f(I_n)|\searrow$.

Let $|f(I_n)| = a_n$, $1/\lambda_n = b_n$. Then $a_n\searrow 0$, $b_n = O(1)$. Now $V(f; [a, b]) \geqslant\sum a_nb_n$ implies that given $\varepsilon > 0$, there is an N such that

$$\sum_{N+1}^{\infty}a_nb_n < \varepsilon$$

and, therefore, since for $j > 0$, $a_{n+j}b_n\leqslant a_nb_n$,

$$\sum_{N+1}^{\infty}a_{n+j}b_n < \varepsilon.$$

Now there is a $J(\varepsilon)$ such that

$$\sum_1^N a_{n+j}b_n \leqslant a_{j+1}\sum_1^N b_n < \varepsilon \quad \text{if} \quad j > J(\varepsilon).$$

Writing $j = N_k$, we see that

$$\tfrac{1}{2}V_\Lambda(f;\,[a,y_k]) < \sum_{1}^{\infty} a_{n+N_k} b_n < 2\varepsilon$$

if k is sufficiently large. Hence

$$V_\Lambda(f;\,[a,y_k]) \to 0 \quad \text{as} \quad k\to\infty$$

implying that

$$V_\Lambda(f;\,[a,y]) \to 0 \quad \text{as} \quad y \searrow a$$

since $V_\Lambda(f;\,[a,y])$ is a monotone function of y.

4. Proof of Theorem 1. Clearly $S[f]$ is $(C,1)$ summable everywhere and is uniformly $(C,1)$ summable on each closed interval of continuity. Hence if we show that $S[f]$ is (C,β) bounded at a point, $-1<\beta<0$, it follows from a well-known convexity theorem ([1], p. 127), that $S[f]$ is (C,α) summable at that point for $\alpha>\beta$. The result on uniform (C,α) summability follows from uniform (C,β) boundedness in the same way.

From the statement of the summability test we see that we need only verify condition (iii) or (iii′).

Let us consider, for $\eta<\delta<\pi$, $\eta = \pi\Big/\left(n+\dfrac{\beta+1}{2}\right)$,

$$I(n,x,\delta) = \int_\eta^\delta \frac{|\varphi(t)-\varphi(t+\eta)|}{t^{\beta+1}}\,dt.$$

We have

$$2I(n,x,\delta) \leqslant \int_\eta^\delta |f(x+t)-f(x+t+\eta)|\,t^{-(\beta+1)}\,dt + {}$$
$$+\int_\eta^\delta |f(x-t)-f(x-t-\eta)|\,t^{-(\beta+1)}\,dt = I_1+I_2.$$

Now if $m=[\delta/\eta]$, letting $V([a,b])$ denote the $\{n^{\;+1}\}$-variation of f on $[a,b]$, we have

$$n^{-\beta}I_1 \leqslant n^{-\beta}\sum_{i=1}^{m}\int_{i\eta}^{(i+1)\eta}|f(x+t)-f(x+t+\eta)|\,t^{-(\beta+1)}\,dt$$
$$\leqslant n^{-\beta}\eta^{-\beta}\sum_{i=1}^{m}\operatorname{osc}(f;\,[x+i\eta,\,x+i+2\eta])/i^{\beta+1}$$
$$\leqslant 2\pi^{-\beta}V([x+\eta,\,x+\delta+2\eta]).$$

Applying the lemma of § 3 we see that

$$n^{-\beta}I_1(n,x,\delta) = o(1) \quad \text{as} \quad \delta\to 0,\ n\to\infty.$$

If we suppose that $x_n\to x_0$, a point of continuity of f, then

$$n^{-\beta}I_1(n,x_n,\delta) \leqslant 2\pi^{-\beta}V([x_n+\eta,\,x_n+\delta+2\eta]).$$

Adopting the convention that the variation of f on the empty interval is zero, we have

$$n^{-\beta}I_1(n,x_n,\delta) \leqslant 2\pi^{-\beta}\big\{V\big([x_n+\eta,\,x_n+\delta+2\eta]\cap(-\infty,x_0]\big)+$$
$$+V\big([x_n+\eta,\,x_n+\delta+2\eta]\cap[x_0,\infty)\big)\big\}$$
$$= o(1) \quad \text{as} \quad \delta\to 0,\ n\to\infty$$

by the lemma. Similar results hold for $n^{-\beta}I_2$. Thus we see that

$$n^{-\beta}I(n,x,\delta) = o(1) \quad \text{as} \quad \delta\to 0,\ n\to\infty$$

for each x, and the convergence is uniform at each point of continuity.

Let us now consider

$$J(n,x,\delta) = \int_\delta^{\pi-\eta}\frac{|\varphi(t)-\varphi(t+\eta)|}{t^{\beta+1}}\,dt \leqslant \tfrac{1}{2}\,(J_1+J_2),$$

where J_1 and J_2 are obtained in the same manner as I_1 and I_2. Then if $m=[\delta/\eta]$,

$$n^{-\beta}J_1 \leqslant \pi^{-\beta}\sum_{i=m}^{n-1}\operatorname{osc}(f;\,[x+i\eta,\,x+(i+2)\eta])/i^{\beta+1} \leqslant 2\pi^{-\beta}V_m,$$

where V_m denotes the $\{n^{\beta+1}\}^m$-variation of f on $[0,2\pi]$. The same inequality holds for J_2.

Thus, if f is continuous in $\{n^{\beta+1}\}$-variation,

$$n^{-\beta}J(n,x,\delta) = o(1) \quad \text{as} \quad n\delta\to\infty$$

uniformly in x. If our hypothesis is that f is of $\{n^{\beta+1}\}$BV, then V_m is dominated by the $\{n^{\beta+1}\}$-variation of f and so

$$n^{-\beta}J(n,x,\delta) = O(1) \quad \text{as} \quad n\delta\to\infty$$

uniformly in x.

Now choose $\delta_n\searrow 0$ so that $n\delta_n\to\infty$. If f is of $\{n^{\beta+1}\}$BV, then for each x

$$n^{-\beta}\big(I(n,x,\delta_n)+J(n,x,\delta_n)\big) = o(1)+O(1) = O(1) \quad \text{as} \quad n\to\infty.$$

If $x_n\to x_0$, a point of continuity of f, we see that

$$n^{-\beta}\big(I(n,x_n,\delta_n)+J(n,x_n,\delta_n)\big) = o(1)+O(1) = O(1) \quad \text{as} \quad n\to\infty.$$

Thus $S[f]$ is everywhere (C,β) bounded, uniformly (C,β) bounded at x_0 and, therefore, on each closed interval of continuity. If f is continuous in $\{n^{\beta+1}\}$-variation, then for each x

$$n^{-\beta}\big(I(n,x,\delta_n)+J(n,x,\delta_n)\big)=o(1) \quad \text{as} \quad n\to\infty,$$

implying that $S[f]$ is (C,β) summable to $f(x)$ for each x. If in addition $x_n\to x_0$, a point of continuity of f, then

$$n^{-\beta}\big(I(n,x_n,\delta_n)+J(n,x_n,\delta_n)\big)=o(1) \quad \text{as} \quad n\to\infty.$$

Thus $S[f]$ is uniformly (C,β) summable at x_0 and, therefore, on each closed interval of continuity.

5. Some remarks on Harmonic Bounded Variation. In our previous paper [3], we discussed the Fourier series of functions of class HBV, i.e., of ΛBV with $\Lambda=\{n\}$. We showed there that functions of that class satisfy the Lebesgue convergence test. Using the lemma of § 3, a more direct proof of this fact may be furnished.

We are required to show that

$$I=\int_{\eta}^{\pi}\frac{|\varphi(t)-\varphi(t+\eta)|}{t}\,dt=o(1) \quad \text{as} \quad n\to\infty,$$

where $\eta=\pi/n$. Suppose I_1 and I_2 are defined in the same fashion as in § 4. Then

$$I_1\leqslant\sum_{i=1}^{m}\int_{i\eta}^{(i+1)\eta}|f(x+t)-f(x+t+\eta)|t^{-1}dt\leqslant 2V([x+\eta,x+\delta+2\eta])$$

and, by the lemma, $I_1=o(1)$ as $\delta\to0$, $n\to\infty$. I_2 is treated similarly. Now consider

$$J=\int_{\delta}^{\pi}\frac{|\varphi(t)-\varphi(t+\eta)|}{t}\,dt\leqslant\frac{1}{\delta}\int_{0}^{\pi}|\varphi(t)-\varphi(t+\eta)|\,dt.$$

The last integral is $o(1)$ as $n\to\infty$ uniformly in x. Now choose $\delta_n\searrow 0$ so slowly that $n\delta_n\nearrow\infty$ and

$$\frac{1}{\delta_n}\int_{0}^{\pi}|\varphi(t)-\varphi(t+\eta)|\,dt=o(1) \quad \text{as} \quad n\to\infty.$$

We see at once that f satisfies the Lebesgue test at each point. That f satisfies the test for uniform convergence on a closed interval of continuity may be shown by considering $I_1(n,x_n,\delta_n)$ and $I_2(n,x_n,\delta_n)$ as in § 4.

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