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On the moduli of convexity and smoothness

by

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Abstract. In the paper the moduli of convexity and smoothness of general Banach spaces and products thereof are discussed. An attempt is made to give precise estimates where only qualitative results have been known. (E.g. it is proved that the moduli of $l_2(X)$ are equivalent to the corresponding ones of X.) The problem how far the modulus of convexity can be improved by a suitable renorming is studied for spaces with local unconditional structure.

In this paper we are concerned with general properties of the moduli of convexity and smoothness of Banach spaces and certain products thereof. Our purpose was to obtain some estimates, useful in the isomorphic theory of Banach spaces, in a precise form and with no redundant assumptions on the spaces involved. Renorming problems are considered only in the case of the existence of local unconditional structure, which may be regarded as elementary (cf. [5], [24]). Our terminology tends to be consistent with [16].

Section I is of an introductory nature. The main results are Propositions 3 and 10 and Corollary 11. The first two of them seem to have been implicit in the literature, but their role has not been recognized. For the sake of completeness, short proofs of some known results are also given.

The main result of Section II is that the moduli of convexity and smoothness of $l_2(X)$ are essentially the same as those of X. This completes the results of [7]. The method used to estimate $\varrho_{l_2(X)}$ can easily be adapted to the case of Orlicz spaces of vector valued functions, $L_M(X)$. The formulae obtained are analogous to those found in [18], where the case $X = \boldsymbol{R}$ is discussed. The results of Section I allow us to show that the latter formulae are the best possible. The corresponding results for the moduli of convexity are obtained by duality, with the use of some formulae for the Legendre transform.

In Section III we investigate the uniform convexifiability of a space E with an unconditional basis. The dual results are not formulated, their

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deduction being straightforward. We present some improved methods of estimating δ_E and then discuss how the norm can be modified so as to satisfy the conditions needed for those methods to work. We do not know whether the converse of Kadec's theorem (which would state "if E is uniformly convexifiable and a function f yields a lower estimate for unconditionally convergent series in E, then E admits an equivalent norm so that $f \leqslant C\delta_E$ for some $C < \infty$") is true, but our renorming of E is rather close to that.

I. Auxiliary results. Let $(X, \|\cdot\|)$ be a real Banach space with $\dim X \geqslant 2$ and let $S_X = \{x \in X\colon \|x\| = 1\}$ be the unit sphere. The *modulus of convexity* (resp. *of smoothness*) of X is defined by the formula

$$\delta_X(\varepsilon) = \inf\{1 - \|x+y\|/2\colon x, y \in S_X,\ \|x-y\| = \varepsilon\}$$

$$(\text{resp. } \varrho_X(\tau) = \sup\{\tfrac{1}{2}(\|x+\tau y\| + \|x-\tau y\| - 2)\colon x, y \in S_X\}).$$

Lindenstrauss observed in [15] that

$$\begin{aligned}
2\varrho_{X^*}(\tau) &= \sup_{f,g \in S_{X^*}} \{\|f+\tau g\| + \|f-\tau g\| - 2\} \\
&= \sup_{f,g \in S_{X^*}} \sup_{x,y \in S_X} \{(f+\tau g)(x) + (f-\tau g)(y) - 2\} \\
&= \sup_{x,y \in S_X} \{\|x+y\| + \tau\|x-y\| - 2\} \\
&= \sup_{\varepsilon \geqslant 0} \sup_{x,y \in S_X, \|x-y\| = \varepsilon} \{\tau\varepsilon - (2 - \|x+y\|)\} \\
&= \sup_{\varepsilon \geqslant 0} \{\tau\varepsilon - 2\delta_X(\varepsilon)\},
\end{aligned}$$

but it seems that there is no simple inverse relationship. The function $\tilde{\delta}_X$ defined by the dual formula

$$\tilde{\delta}_X(\varepsilon) = \sup_{\tau \geqslant 0}\{\tfrac{1}{2}\tau\varepsilon - \varrho_{X^*}(\tau)\},$$

which, by Lindenstrauss' formula, satisfies $\tilde{\delta}_X \leqslant \delta_X$, is convex, being the supremum of a family of convex functions. (In fact, it is simply the maximal convex function minorizing δ_X; observe that if for some a, b with $a > 0$ one has $\delta_X(\varepsilon) \geqslant a\varepsilon + b$ for all $\varepsilon \geqslant 0$, then for all $s \geqslant 0$

$$\tilde{\delta}_X(s) \geqslant as - \varrho_{X^*}(2a) = \inf_{\varepsilon \geqslant 0}\{as - a\varepsilon + \delta_X(\varepsilon)\} \geqslant as + b.)$$

Hence the relation "$\tilde{\delta}_X = \delta_X$" implies (in fact, is equivalent to) the convexity of δ_X. The example due to Liokoumovich [17] shows that the latter is not always the case.

The possibility of expressing ϱ_X in terms of δ_{X^*} allows one to deduce in a simple way many properties of moduli of smoothness from known facts about moduli of convexity. For instance, as noticed in [15], the result of Nordlander [23] stating that $\delta_{X^*}(\varepsilon) \leqslant \delta_{l_2}(\varepsilon)$ for all $\varepsilon \geqslant 0$, yields $\varrho_X(\tau) \geqslant \varrho_{l_2}(\tau)$ for all $\tau \geqslant 0$. (Let us recall, for future use, that $\varrho_{l_2}(\tau) = \sqrt{1+\tau^2} - 1$.)

Since we would like to be able to proceed in the other direction as well, it is important for us to know that δ_X and $\tilde{\delta}_X$ are in a sense equivalent. Namely, one has

PROPOSITION 1. *Let $0 < \gamma < 1$, $\varepsilon \geqslant 0$. Then*

$$\tilde{\delta}_X(\varepsilon) \geqslant (\gamma^{-1} - 1)\,\delta_X(\gamma\varepsilon).$$

This suggests the natural notion of *equivalence*. Given two non-negative functions f, g, each defined on a segment $[0, a]$, let us write $f \dashv g$ if there exist positive constants A, B, C such that $Af(Bt) \leqslant g(t)$ for $t \in [0, C]$; we shall consider f and g as equivalent iff $f \dashv g \dashv f$.

The estimates which we obtain in the sequel are usually of that form. We have made some effort, when it does not lead to complications, to get reasonable numerical bounds for the constants involved. This explains our frequent use of Nordlander's estimate in this section.

Proof of Proposition 1. It is a direct consequence of the following two facts.

LEMMA 2 (cf. [19]). *Let φ be a non-negative function defined on $[0, a]$ such that, for some $K \geqslant 1$, $0 \leqslant x < y \leqslant a$ implies $\varphi(x) \leqslant Kx\varphi(y)/y$. If $\tilde{\varphi}\colon [0, a] \to [0, \infty)$ is the maximal convex function minorizing φ, and $\gamma \in (0, 1)$, then*

$$\tilde{\varphi}(x) \geqslant \frac{1-\gamma}{K\gamma}\varphi(\gamma x) \quad \textit{for} \quad 0 \leqslant x \leqslant a.$$

PROPOSITION 3. *The function $\varepsilon \mapsto \delta_X(\varepsilon)/\varepsilon$ is non-decreasing.*

Proof of Lemma 2. We shall prove that, whenever $0 \leqslant y \leqslant x \leqslant z \leqslant a$ and $z - y > 0$, one has

$$L = \frac{z-x}{z-y}\varphi(y) + \frac{x-y}{z-y}\varphi(z) \geqslant \frac{1-\gamma}{K\gamma}\varphi(\gamma x) = \psi(x).$$

This will show that the convex envelope of the graph of φ lies "above" the graph of the function ψ, whence $\tilde{\varphi}(x) \geqslant \psi(x)$ as requested. Consider two cases. If $y \geqslant \gamma x$, then

$$L \geqslant \frac{z-x}{z-y}\cdot\frac{y}{K\gamma x}\varphi(\gamma x) + \frac{x-y}{z-y}\cdot\frac{z}{K\gamma x}\varphi(\gamma x) = \frac{1}{K\gamma}\varphi(\gamma x) \geqslant \psi(x).$$

If $y < \gamma x$, then

$$L \geqslant \frac{x-y}{z-y}\varphi(z) \geqslant \frac{x-\gamma x}{z}\cdot\frac{z}{K\gamma x}\varphi(\gamma x) = \varphi(x).$$

This completes the proof of the lemma.

It is convenient for us to assume in the proofs through the rest of this section that $\dim X < \infty$. The extension of the results to the general case is immediate, depending only on formulae like

$$\delta_X(\varepsilon) = \inf\{\delta_{X'}(\varepsilon)\colon X' \subseteq X,\ \dim X' < \infty\}.$$

LEMMA 4 (cf. [16], the proof of II.3.6). *Let $x, y \in B_X = \{z \in X\colon \|z\| \leqslant 1\}$. Then $\|x+y\| \leqslant 2\big(1-\delta_X(\|x-y\|)\big)$.*

Proof. Assume $\dim X < \infty$. Fix an $\varepsilon \in [0, 2]$ and pick vectors u, v in B_X so that $\|u+v\|$ be maximal subject to $\|u-v\| = \varepsilon$. It is enough to prove that $\|u\| = \|v\| = 1$. The case $\varepsilon = 0$ being trivial, assume $\varepsilon \neq 0$. Let $x^* \in X^*$ satisfy $\|x^*\| = 1$, $x^*(u+v) = \|u+v\|$.

It would suffice to prove that if, say, $\|v\| < 1$, then $x^*(v-u) = \varepsilon$ and $\|u\| < 1$. Indeed, an analogous reasoning would then yield $x^*(u-v) = \varepsilon$, and hence $\varepsilon = -\varepsilon$ a contradiction.

To this end let $A = \{w \in X\colon \|w-u\| = \varepsilon\}$. If $w \in A \cap B_X$, then, by the maximality of $\|u+v\|$,

$$x^*(u+w) \leqslant \|u+w\| \leqslant \|u+v\| = x^*(u+v).$$

Hence, if we had $\|v\| < 1$, then x^* would attain at v a local maximum on A. Consequently, x^* would norm the vector $v-u$, i.e., $x^*(v-u) = \|v-u\| = \varepsilon$ and also

$$\|u\| \leqslant \tfrac{1}{2}(\|u+v\| + \|u-v\|) = \tfrac{1}{2}[x^*(u+v) + x^*(v-u)] = x^*(v) < 1,$$

as promised. This completes the proof.

COROLLARY 5. *δ_X is a non-decreasing function on $[0, 2]$.*

Proof. Let $0 \leqslant \varepsilon_1 < \varepsilon_2 \leqslant 2$ and let $x, y \in S_X$ satisfy $\|x-y\| = \varepsilon_2$, $\|x+y\| = 2\big(1-\delta_X(\varepsilon_2)\big)$. Then, letting $c = (\varepsilon_2-\varepsilon_1)/2\varepsilon_2$, $x_1 = x + c(y-x)$, $y_1 = y - c(y-x)$, we have $x_1, y_1 \in B_X$, $\|x_1-y_1\| = \varepsilon_1$. It remains to apply Lemma 4 to get

$$\delta_X(\varepsilon_1) \leqslant 1 - \tfrac{1}{2}\|x_1+y_1\| = 1 - \tfrac{1}{2}\|x+y\| = \delta_X(\varepsilon_2).$$

Proof of Proposition 3. It follows from Corollary 5 and the next lemma.

LEMMA 6. *Let $\delta_1(s) = \inf\limits_{u,v\in S_X} \max\{\|u+sv\|, \|u-sv\|\} - 1$ and let, for $s > 0$, $f(s) = \delta_1(s)/s$. Then the function f is non-decreasing on $(0, \infty)$ and satisfies the identity*

$$\frac{\delta_X(\varepsilon)}{\varepsilon} = \frac{1}{2} f\left(\frac{\varepsilon}{2\big(1-\delta_X(\varepsilon)\big)}\right).$$

Proof. For any fixed $u, v \in S_X$ the function $g_{u,v}(s) = \max\{\|u+sv\|, \|u-sv\|\} - 1$ is convex and vanishes at 0; hence $g_{u,v}(s)/s$ is non-decreasing on $(0, \infty)$. Taking the infimum over all $u, v \in S_X$, we obtain the first statement.

Now let $\varepsilon \in [0, 2)$ be fixed. Assuming again $\dim X < \infty$, choose $x, y \in S_X$ so that $\|x-y\| = \varepsilon$, $\|x+y\| = 2\big(1-\delta_X(\varepsilon)\big)$, and let

$$u = (x+y)/\|x+y\|, \qquad v = (x-y)/\|x+y\|, \qquad s = \|v\| = \varepsilon\big[2\big(1-\delta_X(\varepsilon)\big)\big]^{-1}.$$

Then, since $\|u\| = 1$, and $\|u \pm v\| = 1/\big(1-\delta_X(\varepsilon)\big)$, we have

$$(*) \qquad \delta_1(s) \leqslant \|u+v\| - 1 = \delta_X(\varepsilon)/\big(1-\delta_X(\varepsilon)\big).$$

On the other hand, pick $u', v' \in X$ with $\|u'\| = 1$, $\|v'\| = s$, $\max\{\|u'+v'\|, \|u'-v'\|\} = 1 + \delta_1(s) = a^{-1}$. Writing $x' = a(u'+v')$, $y' = a(u'-v')$, we have $\|x'-y'\| = 2as$, $x', y' \in B_X$ (in fact $x', y' \in S_X$); hence, by Lemma 4,

$$\delta_X(2as) \leqslant 1 - \tfrac{1}{2}\|x'+y'\| = 1 - a = \delta_1(s)/\big(1+\delta_1(s)\big).$$

Using this and $(*)$, we get $\delta_X(2as) \leqslant \delta_X(\varepsilon)$; hence by Corollary 5 and the definitions

$$\delta_1(s) = a^{-1} - 1 \geqslant 2s\varepsilon^{-1} - 1 = \delta_X(\varepsilon)/\big(1-\delta_X(\varepsilon)\big).$$

This together with $(*)$ completes the proof of the lemma.

Remark. The identity stated in Lemma 6 can be recognized as Lemma 1.4 of Milman's [22]. We have given a complete proof since Milman's reasoning seems to yield only the inequality we denoted by $(*)$. It is unfortunate that we must not replace our proofs of Lemmas 8 and 12 below by simple deductions of much stronger estimates from the next lemma (Lemma 1.5) of [22] (as we did with our original proof of Proposition 3). A simple counterexample to (a very weak form of) Lemma 1.5 has been given by Professor Ebbe T. Poulsen and is reproduced here with his kind permission.

EXAMPLE. There is a norm $\|\cdot\|$ on $X = \mathbf{R}^2$ such that if $x, y \in S_X$ and $\varrho_X(1) = \frac{1}{2}(\|x+y\| + \|x-y\|) - 1$ then $\min\{\|x+y\|, \|x-y\|\} < 1$. For let $i = (1, 0)$, $j = (0, 1)$, $u = (\frac{1}{2}+\varepsilon)(1, -1)$, $v = (1-3\varepsilon)(1, 1)$, where ε is a small positive number. Define B_X to be the absolute convex hull of the set $\{i, j, u, v\}$. Let

$$C = \{\{x, y\} \subseteq S_X\colon \|x+y\| + \|x-y\| = 2 + 2\varrho_X(1)\}.$$

If a pair $\{x_0, y_0\} \in C$ consists of extreme points of B_X and ε is small enough, then $\{x_0, y_0\} \subseteq \{i, -i, v, -v\}$. Consequently, $C = \{\{i, v\}, \{i, -v\}, \{-i, v\}, \{-i, -v\}\}$. It remains to check that $\|i-v\| < 1$.

As an application of the previous results we modify a proof given in [16] so as to obtain quantitative versions (i.e., ones with uniform bounds for the constants) of the well-known results of [12] and [15]. Both facts will be needed in the sequel. We prove also a result on monotone basis sequences related to that of [8] (cf. also [24]).

THEOREM (Kadec). *If* $x_1, \ldots, x_n \in X$ *and* $\max_{\varepsilon_i=\pm 1} \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\| \leqslant 2$, *then* $\sum_{i=1}^{n} \delta_X(\|x_i\|) \leqslant 1$.

Proof. We may assume $x_1 \neq 0$. Let $S_k = \sum_{i=1}^{k} \varepsilon_i x_i$, where $\varepsilon_1 = 1$, and, for $k = 2, \ldots, n$, $\varepsilon_k = 1$ if $\|S_{k-1} + x_k\| \geqslant \|S_{k-1} - x_k\|$ and $\varepsilon_k = -1$ otherwise. Using Lemma 4 (with $x = S_k/\|S_k\|$, $y = (S_k - 2\varepsilon_k x_k)/\|S_k\|$) and Proposition 3, we get

$$\|S_k\| - \|S_{k-1}\| \geqslant \|S_k\| \delta_X(2\|x_k\|/\|S_k\|) \geqslant 2\delta_X(\|x_k\|).$$

The assertion follows by adding up all these inequalities.

THEOREM (Lindenstrauss). *If* $x_1, \ldots, x_n \in X$ *and* $\sum_{i=1}^{n} \varrho_X(\|x_i\|) = 1$, *then* $\min_{\varepsilon_i=\pm 1} \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\| < 1 + \sqrt{3}$.

Proof. Now the ε_i's must be defined so that, for $k = 2, \ldots, n$, one will have $\|S_{k-1} + \varepsilon_k x_k\| \leqslant \|S_{k-1} - \varepsilon_k x_k\|$. If $\|S_n\| \geqslant 1$, let m be the least index such that $\|S_k\| \geqslant 1$ for $k \geqslant m$. By the convexity of ϱ_X, if $k \geqslant m$, then

$$\begin{aligned}\|S_{k+1}\| - \|S_k\| &\leqslant \tfrac{1}{2}(\|S_k + x_{k+1}\| + \|S_k - x_{k+1}\| - 2\|S_k\|) \\ &\leqslant \|S_k\| \varrho_X(\|x_{k+1}\|/\|S_k\|) \leqslant \varrho_X(\|x_{k+1}\|).\end{aligned}$$

Let $A = \|x_m\|$. Since $1 \geqslant \varrho_X(\|x_m\|) \geqslant \varrho_{l_2}(A) = \sqrt{1+A^2} - 1$, we have $A \leqslant \sqrt{3}$. Hence, using the triangle inequality and elementary algebra we get

$$\begin{aligned}\|S_n\| &\leqslant \|S_{m-1}\| + \|x_m\| + \sum_{k=m}^{n-1} (\|S_{k+1}\| - \|S_k\|) \\ &< 1 + A + [1 - (\sqrt{1+A^2} - 1)] \leqslant 1 + \sqrt{3}.\end{aligned}$$

PROPOSITION 7. *Let* $(x_k)_{k=1}^{n}$ *be a monotone basic sequence in* X. *Then*

(a) $\left\| \sum_{k=1}^{n} x_k \right\| \leqslant 1$ *implies* $\sum_{k=1}^{n} \delta_X(\|x_k\|) \leqslant 1$,

(b) $\sum_{k=1}^{n} \varrho_X(\|x_k\|) \leqslant 1$ *implies* $\left\| \sum_{k=1}^{n} x_k \right\| < 5 - \sqrt{3}$.

Proof. Write, for $k = 1, \ldots, n$, $S_k = \sum_{i=1}^{k} x_i$. If $\|S_n\| \leqslant 1$, then, by monotonicity and the triangle inequality, one has, for $k = 1, \ldots, n$,

$$1 \geqslant \|S_k\| \geqslant \|S_{k-1} + \tfrac{1}{2} x_k\| \geqslant \|S_{k-1}\|;$$

hence, by Lemma 4 (with $x = S_k/\|S_k\|$, $y = S_{k-1}/\|S_k\|$) and Proposition 3, we get

$$\|S_k\| - \|S_{k-1}\| \geqslant \|S_k\| \cdot \delta_X(\|x_k\|/\|S_k\|) \geqslant \delta_X(\|x_k\|).$$

These estimates yield (a). To get (b) assume $\sum_{k=1}^{n} \varrho_X(\|x_k\|) \leqslant 1$. If $\|S_n\| \geqslant 1$, let m be the smallest index such that $\|S_m\| \geqslant 1$. Using the monotonicity of the basic sequence and the convexity of ϱ_X, we obtain, for $k = m, m+1, \ldots, n-1$,

$$\begin{aligned}\|S_{k+1}\| - \|S_k\| &\leqslant \|S_k + x_{k+1}\| + \|S_k - x_{k+1}\| - 2\|S_k\| \\ &\leqslant 2\|S_k\| \varrho_X(\|x_{k+1}\|/\|S_k\|) \leqslant 2\varrho_X(\|x_{k+1}\|).\end{aligned}$$

Letting again $A = \|x_m\|$, we get as before $A \leqslant \sqrt{3}$, and

$$\|S_n\| < 1 + A + 2(2 - \sqrt{1+A^2}) \leqslant 5 - \sqrt{3}.$$

This completes the proof of the proposition.

The next lemma appears in [15], however, the constant, which is now $\frac{1}{2}$, has been improved.

LEMMA 8. *For all* $\tau \geqslant 0$, *one has* $\varrho_X(2\tau) \leqslant 4(1 + \frac{1}{2}\tau)\varrho_X(\tau)$.

Proof. Let $\tau \geqslant 0$ be fixed. Pick vectors $x, y \in X$ so that $\|x\| = 1$, $\|y\| = \tau$, $\|x+2y\| + \|x-2y\| = 2 + 2\varrho_X(2\tau)$. Assume $a = \|x+y\| \geqslant \|x-y\| = b$. Clearly, $b \neq 0$. Following [15], we write

$$\begin{aligned}2\varrho_X(2\tau) &= \|x+2y\| + \|x-2y\| - 2 \\ &= (\|x+2y\| + \|x\| - 2a) + (\|x-2y\| + \|x\| - 2b) + 2(a+b-2) \\ &\leqslant 2[a\varrho_X(\tau/a) + b\varrho_X(\tau/b) + a + b - 2].\end{aligned}$$

Since $2 \leqslant a+b \leqslant 2\varrho_X(\tau) + 2$, we have in particular $a \geqslant 1$, and $\beta = 2\varrho_X(\tau) + 2 - a \geqslant b$. By the convexity of ϱ_X, we have $a\varrho_X(\tau/a) \leqslant \varrho_X(\tau)$. On the other hand,

$$b\varrho_X(\tau/b) + b \leqslant \beta\varrho_X(\tau/\beta) + \beta.$$

(For the function $f(s) = 1 + \varrho_X(s)$ is convex and satisfies $f(s) \geqslant s$ $\lim_{s\to\infty}(f(s)/s) = 1$, whence $f(s)/s$ is non-increasing on $(0, \infty)$.) Using those estimates, we get

$$(*) \qquad \varrho_X(2\tau) \leqslant 3\varrho_X(\tau) + \beta\varrho_X(\tau/\beta).$$

We may assume $\beta < 1$, since otherwise the right-hand side is $\leqslant 4\varrho_X(\tau)$. We also need the estimate $\beta \geqslant \max\{\frac{1}{2}, 1/(1+\tau)\}$. Since

$$\beta = 2 + 2\varrho_X(\tau) - a \geqslant 2\varrho_{l_2}(\tau) + 1 - \tau;$$

this reduces to checking that

$$\varrho_{l_2}(\tau) = \sqrt{1+\tau^2}-1 \geqslant \tfrac{1}{2}\tau-\tfrac{1}{4},$$
$$\varrho_{l_2}(\tau) = \tau^2/(1+\sqrt{1+\tau^2}) \geqslant \tfrac{1}{2}\tau^2/(1+\tau).$$

Now, by the convexity of ϱ_X,

$$\beta\varrho_X(\tau/\beta) \leqslant (2\beta-1)\varrho_X(\tau)+(1-\beta)\varrho_X(2\tau),$$

which together with (*) yields

$$\beta\varrho_X(2\tau) \leqslant 2+2\beta\varrho_X(\tau) \leqslant \beta(2+2\beta^{-1})\varrho_X(\tau).$$

Since $0<\beta^{-1}\leqslant 1+\tau$, it is equivalent to the statement of the lemma.

Lemma 9. *If $u\geqslant 1$, $v\geqslant 4/3$, then $\varrho_X(uv)\leqslant u^2\varrho_X(v)$.*

Proof. Observe that

$$\varrho_X(v)/v \geqslant \varrho_X(\tfrac{4}{3})/\tfrac{4}{3} \geqslant \tfrac{3}{4}\varrho_{l_2}(\tfrac{4}{3}) = \tfrac{1}{2}.$$

Hence, using the mean value theorem, we obtain the required estimate

$$\varrho_X(uv)-\varrho_X(v) \leqslant uv-v \leqslant (u-1)(u+1)\varrho_X(v).$$

Proposition 10. *If $0<\tau\leqslant\sigma$, then $\varrho_X(\sigma)/\sigma^2\leqslant L\varrho_X(\tau)/\tau^2$, where L is a constant $<2\prod_{n=0}^{\infty}(1+2^{-n}/3)$.*

Proof. By the previous lemma we may assume $\tau<\frac{4}{3}$. If $\sigma>v=\frac{4}{3}$, then, by the same lemma,

$$\tau^2\varrho_X(\sigma)/\sigma^2\varrho_X(\tau) \leqslant \tau^2\varrho_X(v)/v^2\varrho_X(\tau);$$

hence we may further assume $\sigma\leqslant\frac{4}{3}$. Write $\sigma_n=2^{-n}\sigma$, where $n=0,1,2,\ldots$ There is an m such that $\sigma_m\leqslant\tau<\sigma_{m-1}$. Since, by the convexity of ϱ_X,

$$\varrho_X(\sigma_m) \leqslant (\sigma_m/\tau)\varrho_X(\tau) \leqslant 2(\sigma_m/\tau)^2\varrho_X(\tau),$$

the desired inequality follows by applying Lemma 8

$$\begin{aligned}\varrho_X(\sigma) &= \varrho_X(\tau)\cdot[\varrho_X(\sigma_m)/\varrho_X(\tau)]\cdot\prod_{n=1}^{m}\big(\varrho_X(\sigma_{n-1})/\varrho_X(\sigma_n)\big)\\ &\leqslant \varrho_X(\tau)\cdot 2(\sigma_m/\tau)^2\cdot 4^m\prod_{n=1}^{\infty}(1+\sigma_n/2)\\ &\leqslant 2\prod_{n=1}^{\infty}(1+2^{1-n}/3)(\sigma/\tau)^2\varrho_X(\tau).\end{aligned}$$

Remark. More careful computations show that $L<3.18$.

Corollary 11. *If $0<\varepsilon\leqslant\eta$, then $\tilde{\delta}_X(\varepsilon)/\varepsilon^2\leqslant L\tilde{\delta}_X(\eta)/\eta^2$, $\delta_X(\varepsilon)/\varepsilon^2\leqslant 4L\delta_X(\eta)/\eta^2$, where L is that of Proposition 10.*

Proof. Let $c=\varepsilon/\eta$. If $cL\geqslant 1$, then, by the convexity of $\tilde{\delta}_X$,

$$\tilde{\delta}_X(\varepsilon) \leqslant c\tilde{\delta}_X(\eta) \leqslant Lc^2\tilde{\delta}_X(\eta);$$

hence we may assume that $m=(cL)^{-1}>1$. Then we have

$$\begin{aligned}\tilde{\delta}_X(\varepsilon) &= \sup_{\tau\geqslant 0}\{\tfrac{1}{2}\tau\varepsilon-\varrho_{X^*}(\tau)\}\\ &\leqslant \sup_{\tau\geqslant 0}\{\tfrac{1}{2}\tau c\eta-L^{-1}m^{-2}\varrho_{X^*}(m\tau)\}\\ &= L^{-1}m^{-2}\sup_{\tau\geqslant 0}\{\tfrac{1}{2}(m\tau)(c\eta Lm)-\varrho_{X^*}(m\tau)\}\\ &= Lc^2\tilde{\delta}_X(c\eta Lm) = Lc^2\tilde{\delta}_X(\eta).\end{aligned}$$

The proof in the case of δ_X is similar. If $c\geqslant\frac{1}{2}$, we can use Proposition 3; if $0<c<\frac{1}{2}$, then, by Proposition 1 and the previous part,

$$\delta_X(\varepsilon) \leqslant \tilde{\delta}_X(2\varepsilon) \leqslant L(2c)^2\tilde{\delta}_X(\eta) \leqslant 4Lc^2\delta_X(\eta).$$

This completes the proof.

The last lemma in this section shows that ϱ_X is essentially determined by pairs (x,y) such that y is "orthogonal" to x. More precisely, define

$$\begin{aligned}\bar{\varrho}(\tau) = \sup\{\tfrac{1}{2}(\|x+\tau y\|+\|x-\tau y\|-2):\ &x,y\in S_X\\ &\text{and there exists an } x^*\in S_{X^*} \text{ so that } x^*(x)=1,\ x^*(y)=0\}.\end{aligned}$$

Then we have the following

Lemma 12. *There exists a $K\leqslant 16$ such that $\varrho_X(\tau)\leqslant K\bar{\varrho}(\tau)$, for $\tau\geqslant 0$.*

Proof. Let $w\in[0,1)$, and let $x,y\in X$ be such that $\|x\|=1$, $\|y\|=w$, $\|x+y\|+\|x-y\|=2\varrho_X(w)+2$. Let $x^*\in X^*$ satisfy $\|x^*\|=x^*(x)=1$. We may assume, perhaps replacing y by $-y$, that $a=x^*(y)\geqslant 0$. Let $z=y-ax$; clearly, $\|z\|\leqslant 2w$ and $x^*(z)=0$. Define, for $v\in X$,

$$f(v) = \tfrac{1}{2}(\|x+v\|+\|x-v\|-2).$$

The functions $\bar{\varrho}$ and f being convex, we have

$$\begin{aligned}\varrho_X(w) = f(y) = f(ax+z) &\leqslant af(x)+(1-a)f\big((1-a)^{-1}z\big)\\ &\leqslant (1-a)\bar{\varrho}\big(2w/(1-a)\big) \leqslant (1-w)\bar{\varrho}\big(2w/(1-w)\big).\end{aligned}$$

Since w could be an arbitrary number in $[0,1)$, substituting $w=\tau/(\tau+2)$, we obtain for all $\tau\geqslant 0$

$$\varrho_X\big(\tau/(\tau+2)\big) \leqslant 2(2+\tau)^{-1}\bar{\varrho}(\tau).$$

Hence, if $\tau \leqslant 2$, then using the convexity of ϱ_X and Lemma 8, we get

$$\begin{aligned}\varrho_X(\tfrac{1}{2}\tau) &\leqslant (1-\tfrac{1}{2}\tau)\,\varrho_X\big(\tau/(\tau+2)\big)+\tfrac{1}{2}\tau\varrho_X\big(2\tau/(\tau+2)\big)\\ &\leqslant \varrho_X\big(\tau/(\tau+2)\big)\,\big[1-\tfrac{1}{2}\tau+4\big(1+\tfrac{1}{2}\tau/(\tau+2)\big)\cdot\tfrac{1}{2}\tau\big]\\ &\leqslant 2(2+\tau)^{-1}\bar{\varrho}(\tau)\,[1+\tfrac{3}{2}\tau+\tau^2/(\tau+2)].\end{aligned}$$

Since, by Lemma 8, $\varrho_X(\tau) \leqslant 4(1+\frac{1}{4}\tau)\varrho_X(\frac{1}{2}\tau)$, we see that

$$\varrho_X(\tau)/\bar{\varrho}(\tau) \leqslant (4+\tau)2(2+\tau)^{-1}\,[1+\tfrac{3}{2}\tau+\tau^2/(\tau+2)] = g(\tau),$$

for $\tau \in [0, 2]$. We need an estimate for the ratio which would work for large τ. Since $\varrho_X(\tau) \leqslant \tau$ and $\bar{\varrho}(\tau) \geqslant \tau-1$ (both estimates follow from the triangle inequality) we get, for $\tau > 1$, $\varrho_X(\tau)/\bar{\varrho}(\tau) \leqslant \tau/(\tau-1)$. Hence, having checked that $g(\tau) < 15$, for $\tau \leqslant 16/15$, we obtain $K < 16$.

One gets a better bound for K using for all $\tau > .56$ the estimate $\bar{\varrho}(\tau) \geqslant \varrho_{l_2}(\tau/\sqrt{2})$. The latter follows if we introduce on a 2-dimensional subspace $Y \subseteq X$ the norm $|\cdot|$ defined by an inner product on Y such that $|x| \leqslant \|x\| \leqslant \sqrt{2}\,|x|$, and $|x_0| = 1$ for some $x_0 \in S_Y$ (cf. [9], Th. 3). If $y_0 \in S_Y$ is $|\cdot|$ orthogonal to x_0, then, for $\tau \geqslant 0$,

$$\bar{\varrho}(\tau) \geqslant \tfrac{1}{2}(\|x_0+\tau y_0\|+\|x_0-\tau y_0\|-2) \geqslant \varrho_{l_2}(\tau/\sqrt{2}),$$

so that $\varrho_X(\tau)/\bar{\varrho}(\tau) \leqslant \tau/(\sqrt{1+\tau^2}-1)$. Combining this bound with that yielded by g, we infer that K may be taken to be < 8.

II. The moduli of Orlicz sums. It was shown in [7] that $\check{\delta}_X \dashv \delta_{l_2(X)}$ and $\varrho_{l_2(X)} \dashv \hat{\varrho}_X$, where $\check{\delta}_X$ is the maximal function minorizing δ_X and such that the function $f(t) = \check{\delta}_X(\sqrt{t})$ is convex, and $\hat{\varrho}_X$ is the minimal function majorizing ϱ_X and such that $\hat{\varrho}(\sqrt{t})$ is concave. The results of the previous section (viz. Lemma 2, Proposition 10 and Corollary 11) imply that $\delta_X \dashv \check{\delta}_X$ and $\hat{\varrho}_X \dashv \varrho_X$, thus establishing the equivalence of the moduli of $l_2(X)$ to those of X. The latter result is obtained again in the present section, when the moduli of more general products are dealt with. We find it more convenient to work mostly with the moduli of smoothness and use duality to obtain the corresponding results for the moduli of convexity. Let us remark that Proposition 17 is a consequence of Proposition 19. Also the equivalence of $\varrho_{l_p(X)}$ and ϱ_X, for $p > 2$, can be proved directly, by using the estimate

$$\begin{aligned}\|x+y\|^p+\|x-y\|^p \leqslant p\,\|x\|^{p-1}(\|x+y\|+\|x-y\|-2\,\|x\|)+\\ +2\,[(\|x\|+\|y\|)^p+|\,\|x\|-\|y\|\,|^p-\|x\|^p],\end{aligned}$$

valid for each $x, y \in X$ and $p \geqslant 2$, instead of Lemma 16.

Let p be a fixed number greater than 1. In the next lemma we shall use the following notation, Y being a Banach space:

$$\begin{aligned}A_Y &= \{(\|x-y\|^p, \|x+y\|^p)\colon\ x, y \in Y,\ \|x\| = \|y\| = 1\},\\ C_Y &= \{(\|x-y\|^p, \|x+y\|^p)\colon\ x, y \in Y,\ \|x\|^p+\|y\|^p = 2\},\end{aligned}$$

$\operatorname{conv} S$ will denote the convex hull of a subset S of $\boldsymbol{R}^2$.

LEMMA 13. (1) $\delta_X(\varepsilon) = \inf\{1-\frac{1}{2}r\colon (\varepsilon^p, r^p) \in A_X\}$.

(2) $A_{l_p(X)} = C_{l_p(X)} = \operatorname{conv} C_X$.

Proof. (1) is a direct consequence of the definitions. The second assertion will follow if we know that: (i) $A_{l_p(X)}$ is convex, (ii) $C_X \subseteq A_{l_p^2(X)}$, (iii) $C_{l_p(X)} \subseteq \operatorname{conv} C_X$.

For (i), let $x = (x_n)$, $y = (y_n)$, $z = (z_n)$, $w = (w_n)$ be arbitrary elements of the unit sphere of $l_p(X)$, and let $0 \leqslant t \leqslant 1$. Then one can write the identity

$$t(\|x-y\|^p, \|x+y\|^p)+(1-t)\,(\|z-w\|^p, \|z+w\|^p) = (\|\xi-\eta\|^p, \|\xi+\eta\|^p),$$

where $\xi = (\xi_n)$, $\eta = (\eta_n)$ are defined by the formulae

$$\begin{aligned}\xi_{2n-1} &= t^{1/p}x_n, & \eta_{2n-1} &= t^{1/p}y_n,\\ \xi_{2n} &= (1-t)^{1/p}z_n, & \eta_{2n} &= (1-t)^{1/p}w_n,\end{aligned}$$

for $n = 1, 2, \ldots$ Since ξ, η belong to the unit sphere of $l_p(X)$, and the left-hand side of the identity can represent any convex combination of any two elements of $A_{l_p(X)}$, (i) has been established.

To get (ii), observe that any element $(\|x-y\|^p, \|x+y\|^p) \in C_X$ can be represented as $(\|\xi-\eta\|^p, \|\xi+\eta\|^p)$, with $\xi = 2^{-1/p}(x, y)$, $\eta = 2^{-1/p}(y, x)$ belonging to the unit sphere of $l_p^2(X)$.

Finally, (iii) depends on the possibility of writing any $(\|x-y\|^p, \|x+y\|^p) \in C_{l_p(X)}$ in the form

$$\sum_{n=1}^{\infty} r_n(\|\xi_n-\eta_n\|^p, \|\xi_n+\eta_n\|^p),$$

where $x = (x_n)$, $y = (y_n)$, $r_n = \frac{1}{2}(\|x_n\|^p+\|y_n\|^p)$, and $\xi_n = r_n^{-1/p}x_n$, $\eta_n = r_n^{-1/p}y_n$, provided that $r_n \neq 0$; in the latter case ξ_n, η_n may be arbitrary elements of S_X. (Actually, any element of $C_{l_p(X)}$ is a convex combination of $\leqslant 3$ elements of C_X.) This completes the proof.

COROLLARY 14. *The function* $f(t) = \big(1-\delta_{l_p(X)}(t^{1/p})\big)^p$ *is concave, and hence* $g(\varepsilon) = \delta_{l_p(X)}(\varepsilon^{1/p})$ *is convex.*

Proof. The first statement is now obvious; the second follows from the first one, since $g(t) = 1-f(t)^{1/p}$.

COROLLARY 15. $\delta_{l_p(X)}(\varepsilon) \leqslant \inf\limits_{t\geqslant 1} t^{-p}\,\delta_X(t\varepsilon)$, *for* $\varepsilon \geqslant 0$;

$$\varrho_{l_p(X)}(\tau) \geqslant \sup_{t\geqslant 1} t^{-p}\,\varrho_X(t\tau), \quad \textit{for } \tau \geqslant 0.$$

Proof. The first inequality is a consequence of the previous corollary; for the second one observe that, if $t \geqslant 1$, then

$$t^{-p}\varrho_X(t\tau) = t^{-p}\sup_{\varepsilon\geqslant 0}\{\tfrac{1}{2}t\tau\varepsilon - \delta_{X^*}(\varepsilon)\} = \sup_{\varepsilon\geqslant 0}\{\tfrac{1}{2}t^{1-p}\tau\varepsilon - t^{-p}\delta_{X^*}(\varepsilon)\}$$
$$\leqslant \sup_{\varepsilon\geqslant 0}\{\tfrac{1}{2}(t^{1-p}\varepsilon)\tau - \delta_{l_q(X^*)}(t^{1-p}\varepsilon)\} = \varrho_{l_q(X)}(\tau).$$

LEMMA 16. *If $1 < p \leqslant 2$, then, for any $x, y \in X$, one has*

$$\|x+y\|^p + \|x-y\|^p - 2\|x\|^p - 2\|y\|^p \leqslant p\|x\|^{p-1}(\|x+y\| + \|x-y\| - 2\|x\|).$$

Proof. The elementary inequality

$$a^p + (p-1)b^p - pab^{p-1} \leqslant |a-b|^p,$$

valid whenever $a, b \geqslant 0$, $1 < p \leqslant 2$, yields, for any $x, z \in X$

$$\|x+z\|^p + (p-1)\|x\|^p - p\|x+z\|\,\|x\|^{p-1} \leqslant |\|x+z\| - \|x\||^p \leqslant \|z\|^p.$$

Substituting for z the vectors y and $-y$, and adding up the inequalities obtained, one gets the desired estimate.

PROPOSITION 17. *Let L be that of Proposition 10. Then*

$$\varrho_{l_2(X)}(\tau) \leqslant (1+\sqrt{L^2+1})\,\varrho_X(\tau).$$

More generally, if $1 < p \leqslant 2$ and $r(\tau) = \sup_{t\geqslant 1} t^{-p}\varrho_X(t\tau)$, then

$$r(\tau) \leqslant \varrho_{l_p(X)}(\tau) \leqslant 3r(\tau).$$

Proof. The first estimate has already been established in Corollary 15. Let $x = (x_n)$, $y = (y_n)$ be arbitrary norm one vectors in $l_p(X)$. It follows from Lemma 16 that, for any $\tau \geqslant 0$,

$$\|x+\tau y\|^p + \|x-\tau y\|^p - 2\|x\|^p - 2\|\tau y\|^p \leqslant 2p\sum_{n=1}^{\infty}\|x_n\|^p\varrho_X(\tau\|y_n\|/\|x_n\|).$$

Now, for n such that $\|y_n\| \leqslant \|x_n\|$, by the convexity of ϱ_X,

$$\varrho_X(\tau\|y_n\|/\|x_n\|) \leqslant \varrho_X(\tau)\,\|y_n\|/\|x_n\|.$$

For the remaining n's one has $\|y_n\| > \|x_n\|$, and hence, by the definition of $r(\tau)$,

$$\varrho_X(\tau\|y_n\|/\|x_n\|) \leqslant r(\tau)\,(\|y_n\|/\|x_n\|)^p.$$

Using Hölder's inequality, we obtain (letting $q = p/(p-1)$)

$$\sum_{n=1}^{\infty}\|x_n\|^p\varrho_X(\tau\|y_n\|/\|x_n\|) \leqslant \sum_{n=1}^{\infty}\|y_n\|\max\{\varrho_X(\tau)\|x_n\|^{p-1}, r(\tau)\|y_n\|^{p-1}\}$$
$$\leqslant \Big(\sum_{n=1}^{\infty}\|y_n\|\Big)^{1/p}\Big(\sum_{n=1}^{\infty}[\varrho_X(\tau)^q\|x_n\|^p + r(\tau)^q\|y_n\|^p]\Big)^{1/q} = (\varrho_X(\tau)^q + r(\tau)^q)^{1/q},$$

which, combined with our previous estimate and the mean value theorem, yields

$$\tfrac{1}{2}(\|x+\tau y\| + \|x-\tau y\|) \leqslant [\tfrac{1}{2}(\|x+\tau y\|^p + \|x-\tau y\|^p)]^{1/p}$$
$$\leqslant \big[1+\tau^p + p\big(\varrho_X(\tau)^q + r(\tau)^q\big)^{1/q}\big]^{1/p} \leqslant (1+\tau^p)^{1/p} + \big(\varrho_X(\tau)^q + r(\tau)^q\big)^{1/q}.$$

Hence, taking the supremum over all pairs x, y in the unit sphere, we get

$$\varrho_{l_p(X)}(\tau) \leqslant (1+\tau^p)^{1/p} - 1 + [\varrho_X(\tau)^q + r(\tau)^q]^{1/q}.$$

Now, if $p = 2$, then the result follows readily from the definition of L and Nordlander's estimate. From this moment we assume $p < 2$. Let $\tau_p = \sqrt{p(2-p)}/(p-1)$. We shall consider three cases: (i) $\tau \geqslant 3/4$, (ii) $\tau \leqslant \min\{3/4, \tau_p\}$, (iii) $\tau_p < \tau < 3/4$.

In case (i) we have

$$r(\tau) \geqslant \varrho_X(\tau) \geqslant \varrho_{l_2}(\tau) = \tau^2/(1+\sqrt{1+\tau^2}) \geqslant \tfrac{1}{3}\tau \geqslant \tfrac{1}{3}\varrho_{l_p(X)}(\tau).$$

In case (ii)

$$r(\tau) \geqslant \tau^p\tau_p^{-p}\varrho_{l_2}(\tau_p) \geqslant p(\sqrt[p]{1+\tau^p} - 1)\tau_p^{-p}\frac{2-p}{p-1};$$

hence, using our estimate for $\varrho_{l_p(X)}$, we get

$$\varrho_{l_p(X)}(\tau)/r(\tau) \leqslant \{(p-1)^{1-p}[p(2-p)]^{(p-2)/2} + 2^{1-1/p}\} = \varphi(p)$$
$$\leqslant \sup_{1<p<2}\varphi(p) < 2.8.$$

The last step involves some computations. Simply by estimating each term in the expression for φ separately, one gets $\varphi(p) \leqslant e^{1/e}\cdot 1\cdot e^{1/(2e)} + 2^{1/2} < 3.15$, which is slightly weaker than the fact stated in the proposition.

If (iii) occurs, then $\tau_p < 3/4$, whence $p > 9/5$. Observe that

$$(\sqrt[p]{1+\tau^p} - 1)/(\sqrt{1+\tau^2} - 1) \leqslant \frac{1}{p}\tau^p(1+\sqrt{1+\tau^2})/\tau^2 \leqslant \frac{5}{9}\tau^{p-2}(2+\tfrac{1}{2}\tau^2).$$

The right-hand side is increasing for $\tau \geqslant \sqrt{4(2-p)/p}$, and the latter number is $< \tau_p$, since $p \leqslant 2$. Therefore the ratio $\varrho_{l_p(X)}(\tau)/r(\tau)$ can now be estimated by

$$\frac{5}{9}\left(\frac{4}{3}\right)^{1/5}\left[2+\frac{1}{2}\left(\frac{3}{4}\right)^2\right] + 2^{1/2} < \frac{73}{54} + 2^{1/2} < 2.8.$$

This completes the proof of the proposition.

It is rather simple, especially in view of Corollary 14, to dualize the result of Proposition 17. We shall postpone that, however, until the general case has been settled; its application in the next proof could be avoided (cf. [7], Lemma 1).

PROPOSITION 18. *Given any sequence $(\|\cdot\|_n)$ of norms on X equivalent to $\|\cdot\|$, there exists another equivalent norm $|||\cdot|||$ such that, for $n = 1, 2, \ldots$,*

$$\delta_{(X,\|\cdot\|_n)} \dashv \delta_{(X,|||\cdot|||)}.$$

Proof. We may assume, by multiplying the $\|\cdot\|_n$'s by suitable constants, that $\|x\|_n \leqslant n^{-1}\|x\|$, for $x \in X$, $n = 1, 2, \ldots$ A norm with the required property can then be defined by the formula

$$|||x|||^2 = \sum_{n=1}^{\infty} \|x\|_n^2.$$

If $x, y \in X$, then $\|x+y\|_n^2 \leqslant (\|x\|_n + \|y\|_n)^2 \leqslant 2(\|x\|_n^2 + \|y\|_n^2)$, for each n, whence if $|||x||| = |||y||| = 1$, then

$$\begin{aligned} 2(\|x\|_n^2 + \|y\|_n^2) - \|x+y\|_n^2 &\leqslant 2(|||x|||^2 + |||y|||^2) - |||x+y|||^2 \\ &\leqslant 4(2 - |||x+y|||). \end{aligned}$$

Writing for brevity $\|x\|_n^2 + \|y\|_n^2 = 2a^{-2}$, $Y = l_2((X, \|\cdot\|_n))$, and using Corollary 14, Lemma 13 and the last estimate we get

$$\begin{aligned} \delta_Y(\|x-y\|_n) &\leqslant a^{-2}\delta_Y(\|ax-ay\|_n) \leqslant a^{-2}\big(1 - \|(ax+ay)/2\|_n\big) \\ &\leqslant a^{-2} - \|(x+y)/2\|_n^2 \leqslant 2 - |||x+y|||. \end{aligned}$$

Since there is a $c > 0$, depending only on n, and such that $\|u\|_n \geqslant c|||u|||$ for $u \in X$, we infer that

$$\delta_{(X,|||\cdot|||)}(\varepsilon) \geqslant \tfrac{1}{2}\delta_Y(c\varepsilon),$$

whence, n being arbitrary, the result follows from the relation $\delta_{(X,\|\cdot\|_n)} \dashv \delta_Y$.

Let us pass to the discussion of general Orlicz sums. To avoid measure-theoretic problems we shall concentrate on the case of spaces $l_M(X)$. This is only for convenience: the spaces $L_M(S, \Sigma, \mu, X)$ could be treated entirely analogously, the resulting formulae being similar. The reader may consult paper [18], from which the formulae for δ_{l_M} and ϱ_{l_M} originate, where the estimates are made over general measure spaces (S, Σ, μ). Our approach is more direct than that in [18] and allows us to consider Banach space valued functions.

Let M be a monotone, convex, continuous function satisfying $M(0) = 0$, $M(1) > 0$, which we shall refer to as an *Orlicz function*. M is said to satisfy the *Δ_2 condition* (at 0) *with constant* γ if $M(2t) \leqslant \gamma M(t)$ for $t \in [0, \frac{1}{2}]$. Given a sequence (t_n) of non-negative numbers, let

$$\|(t_n)\|_M = \inf\Big\{t > 0\colon \sum_n M(t_n/t) \leqslant M(1)\Big\}.$$

If (X_n) is a sequence of Banach spaces, we define

$$l_M\big((X_n)\big) = \big\{(x_n)\colon x_n \in X_n \ \&\ \|(x_n)\| = \big\|(\|x_n\|_n)\big\|_M < \infty\big\}.$$

The formula inside the braces defines a norm, since M is convex, and $l_M\big((X_n)\big)$ is a Banach space (cf. [16]). The spaces $l_p(X)$ which we have considered are a special case, where $M(t) = t^p$, and $X_n = X$ for each n.

PROPOSITION 19. *Assume that M is an Orlicz function (defined on $(0, 2]$) such that $M(1) = 1$ and, for each $t \in (0, 1]$, $s \in (-t, t)$,*

$$M(2t) \leqslant \gamma M(t), \qquad M'(t) \leqslant \beta M(t)/t,$$

$$M(t+s) = M(t) + sM'(t) + ds^2, \qquad \textit{where} \qquad d = d(t, s) \leqslant BM(t)/t^2.$$

Let (X_n) be a sequence of Banach spaces and let $\varrho(\tau) = \max\{\varrho_{l_2}(\tau), \sup_n \varrho_{X_n}(\tau)\}$. Writing $Y = l_M\big((X_n)\big)$ and, for $\tau \in (0, 1]$,

$$G(\tau) = \sup_{\substack{\tau \leqslant u \leqslant 1 \\ 0 < v \leqslant 1}} \varrho(\tau/u) M(uv)/M(v)$$

one has $\varrho_Y(\tau) \leqslant K_0 G(\tau)$, where K_0 depends only on γ, β and B.

Proof. Let $\tau \in (0, 1]$ be fixed, and let $x = (x_n)$, $y = (y_n)$ be, for the time being, elements of Y such that $\|x\| = 1$, $\|y\| = \tau$. We shall prove that

$$(*) \qquad \sum_n M(\|x_n + y_n\|_n) + M(\|x_n - y_n\|_n) - 2M(\|x_n\|_n) \leqslant 2K_1 G(\tau),$$

where $K_1 = K_1(\gamma, \beta, B)$. For n such that $\|y_n\|_n \geqslant \|x_n\|_n$, we have

$$\begin{aligned} M(\|x_n \pm y_n\|_n) &\leqslant M(2\|y_n\|_n) \leqslant \gamma M(\|y_n\|_n) \\ &\leqslant \gamma\varrho(1)^{-1} M(\|y_n\|_n/\tau) G(\tau), \end{aligned}$$

and so (recall that $\varrho(1) \geqslant \sqrt{2} - 1$)

$$M(\|x_n + y_n\|_n) + M(\|x_n - y_n\|_n) - 2M(\|x_n\|_n) \leqslant 2\gamma(\sqrt{2}+1) M(\|y_n\|_n/\tau) G(\tau).$$

For the remaining values of n we can estimate

$$\begin{aligned} &M(\|x_n + y_n\|_n) + M(\|x_n - y_n\|_n) - 2M(\|x_n\|_n) \\ &\quad\leqslant M'(\|x_n\|_n)\,(\|x_n + y_n\|_n + \|x_n - y_n\|_n - 2\|x_n\|_n) + (d' + d'')\|y_n\|_n^2 \\ &\quad\leqslant 2\beta M(\|x_n\|_n)\,\varrho(\|y_n\|_n/\|x_n\|_n) + 2BM(\|x_n\|_n)\,(\|y_n\|_n/\|x_n\|_n)^2 \\ &\quad\leqslant [2\beta + 2(\sqrt{2}+1)B]\,M(\|x_n\|_n)\,\varrho(\|y_n\|_n/\|x_n\|_n) \end{aligned}$$

(we have used the estimate $t^2/\varrho(t) \leqslant t^2/(\sqrt{1+t^2} - 1) = \sqrt{1+t^2} + 1 \leqslant \sqrt{2} + 1$, for $t \leqslant 1$).

If $\|y_n\|_n < \|x_n\|_n < \|y_n\|_n/\tau$, then, by the definition of G,

$$\varrho(\|y_n\|_n/\|x_n\|_n) M(\|x_n\|_n) \leqslant G(\tau) M(\|y_n\|_n/\tau).$$

In the remaining case we have $\|y_n\|_n \leqslant \tau \|x_n\|_n$, and hence

$$\varrho(\|y_n\|_n/\|x_n\|_n)M(\|x_n\|_n) \leqslant G(\tau)M(\|x_n\|_n).$$

Since $\sum_n M(\|x_n\|_n) = \sum_n M(\|y_n\|_n/\tau) = 1$, adding up the estimates obtained for all n, we see that $(*)$ holds with

$$K_1 = \gamma(\sqrt{2}+1)+2\beta+2(\sqrt{2}+1)B.$$

In the special case where $M(t) = t^p$ for some $p \geqslant 1$ $(*)$ is sufficient for us to complete the proof as in Proposition 17.

If M is arbitrary, let us assume that $\|x\pm y\| \geqslant 1$. Then, using the convexity of M and the assumption that $\|x\| = 1$, we get

$$\sum_n M(\|x_n \pm y_n\|_n) \geqslant \|x\pm y\|, \qquad \sum_n M(\|x_n\|_n) = 1,$$

whence, by $(*)$, $\|x+y\|+\|x-y\|-2\|x\| \leqslant 2K_1 G(\tau)$, and we are in a position to apply Lemma 12, which yields $K_0 \leqslant KK_1$.

The following well-known lemma shows that the conditions we have imposed on M are not too restrictive.

LEMMA 20. *Every Orlicz function satisfying the Δ_2 condition at* 0 *is equivalent to an Orlicz function satisfying the assumptions of Proposition* 19.

Proof. Assume first that N is convex on $[0, \infty)$, $N(0) = 0$, $N(1) > 0$ and $N(2x) \leqslant \gamma N(x)$ for $x \geqslant 0$. Consider the function M defined for $x \geqslant 0$ by the formula

$$M(x) = \int_0^x N(t)t^{-1}dt.$$

It is plain that M is convex and, for each $x > 0$,

$$M(2x) \leqslant \gamma M(x),$$
$$M(x) \leqslant N(x) \leqslant (\gamma-1)M(x),$$
$$M'(x) \leqslant \big(M(2x)-M(x)\big)/x \leqslant (\gamma-1)M(x)/x.$$

Observe that if $0 < u \leqslant 2x$ then

$$|N(u)/u - N(x)/x| \leqslant (\gamma-1)\,|u-x|\,N(x)/x^2.$$

Indeed, the latter inequality is obvious if either $u = x$ or $(\gamma-1)(x-u) \geqslant x$; otherwise, by the convexity of N, one has

$$\frac{N(u)-N(x)}{u-x} \leqslant \frac{N(2x)-N(x)}{2x-x} \leqslant (\gamma-1)N(x)/x,$$

and after simple transformations we get

$$|N(u)/u - N(x)/x| \leqslant (\gamma-2)\,|u-x|\,N(x)/xu \leqslant (\gamma-1)\,|u-x|\,N(x)/x^2.$$

Therefore, if $0 < |y| \leqslant x$, we can estimate

$$M(x+y)-M(x)-yM'(x) = \int_x^{x+y} \big(N(u)/u - N(x)/x\big)\,du$$
$$\leqslant \tfrac{1}{2}(\gamma-1)N(x)y^2/x^2 \leqslant \tfrac{1}{2}(\gamma-1)^2 M(x)y^2/x^2.$$

Hence the function M fulfils, after dividing by $M(1)$, the assumptions of Proposition 19.

Since every Orlicz function satisfying the Δ_2 condition at 0 is equivalent to an N described above, the proof is complete.

It is a well-known and easy fact that the equivalence of M and N implies the equivalence of the norms $\|\cdot\|_M$ and $\|\cdot\|_N$. Thus, whenever N satisfies the Δ_2 condition, Lemma 20 provides an equivalent renorming of $l_N\big((X_n)\big)$, after which the modulus of smoothness can be estimated by applying Proposition 19. (Incidentally, replacing in the definition of G the function M by an equivalent function M_1 leads to a function G_1 equivalent to G.) One may ask whether the modulus obtained after such renorming is in a sense the best possible and how precise is the estimate. So far we have discussed one special case (viz. that of Proposition 17), where the function G is the best possible estimate, up to equivalence, for the actual modulus of smoothness. The case of $l_p\big((X_n)\big)$, $p > 2$, is even easier, since G is readily seen (use Proposition 10) to be equivalent to ϱ. Another case, where an even stronger assertion can be made, is given in the following

PROPOSITION 21. *Let M be an Orlicz function satisfying the Δ_2 condition, let $|\cdot|$ be an equivalent norm on the space $l_M = l_M(\boldsymbol{R})$ and let $\varrho = \varrho_{(l_M,|\cdot|)}$. Then there is a $K > 0$ such that*

$$\varrho(t) \geqslant K \sup_{\substack{t\leqslant u\leqslant 1\\ 0<v\leqslant 1}} \frac{t^2M(uv)}{u^2M(v)} \quad \textit{for} \quad 0 < t \leqslant 1.$$

Proof. We may assume $M(1) = 1$. Let $\|\cdot\| = \|\cdot\|_M$; there are $a, b > 0$ such that $a|x| \leqslant \|x\| \leqslant b|x|$ for $x \in l_M$. Observe that, if $x_1, \ldots, x_n \in l_M$ and $\sum_{i=1}^n \varrho(\|x_i\|) \leqslant 1$, then $\sum_{i=1}^n \varrho(|ax_i|) \leqslant 1$; hence, by Lindenstrauss' theorem, there is a choice (ε_i) of signs such that $\big|\sum_{i=1}^n \varepsilon_i a x_i\big| \leqslant 1+\sqrt{3}$, whence

$$(*) \qquad \Big\|\sum_{i=1}^n \varepsilon_i x_i\Big\| \leqslant (1+\sqrt{3})ba^{-1} = d.$$

Let $[s]$ denote the greatest integer not exceeding s. Given $t, u, v \in (0, 1]$ such that $t \leqslant u$, let $m = [M(v)^{-1}]$, $n = [\varrho(u)^{-1}]$, $c = uM^{-1}(1/m)$. Let $x_1, \ldots, x_n \in l_M$ be disjointly supported vectors, each having exactly m nonzero coordinates, all of them equal to c. Then $\|x_i\| = u$, for each i, whence

$\sum_{i=1}^{n} \varrho(\|x_i\|) = n\varrho(u) \leqslant 1$. Using $(*)$ and the definition of $\| \ \|_M$, we obtain $mnM(c/d) \leqslant 1$, so that

$$mM(c/d) \leqslant 1/n \leqslant 2/(n+1) < 2\varrho(u). \tag{$**$}$$

Since $c \geqslant uv$, there is an a, depending only on d and the Δ_2 constant for M, and such that $M(uv) \leqslant aM(c/d)$. Hence, by $(**)$,

$$M(uv)/M(v) \leqslant 2amM(c/d) \leqslant 4a\varrho(u),$$

so that, applying Proposition 10, we get the desired estimate with $K \geqslant (4aL)^{-1}$.

Remark. It is easy to modify the proof of Proposition 21 to get analogous assertions for Orlicz function spaces and also for their moduli of convexity. (Of course, to deal with the moduli of convexity one uses the theorem of Kadec and Corollary 11.) The formulae obtained are the counterparts of those found in [18]. In each case the modulus of convexity or smoothness of a suitable (described in [18]) renorming of the Orlicz space happens to be the best possible modulus, up to equivalence, for any equivalent renorming of that space.

Another special class of Banach spaces, for which the best moduli of convexity have been found, is that of Lorentz sequence spaces $d(a, p)$, $p \geqslant 2$, (cf. [1]). It was noticed by Z. Altshuler that functions f known to be equivalent to the best modulus of convexity of a Banach space have a special property. Namely, such an f is equivalent to a function M which: (i) is supermultiplicative (i.e., $M(xy) \geqslant M(x)M(y)$, for $x, y \leqslant 1$), (ii) is of the form $M(t) = g(t^2)$, where g is convex. Conversely, for any such M_0 (vanishing at 0) he constructed a space $d(a, p)$ whose best modulus of convexity was equivalent to M_0. Now we see that a simpler example is furnished by the space l_{M_0}.

We know (by Corollary 11 and Lemma 2) that property (ii) is satisfied by any function δ_X. It is not the case with (i) (δ_X may even fail the Δ_2 condition) but it may be so provided that δ_X is the best possible modulus for X (if such exists).

Since the dual to the space l_M, where M satisfies the Δ_2 condition, is isomorphic to l_{M^*}, where

$$M^*(x) = \sup_{u \geqslant 0} xu - M(u), \quad \text{for} \quad x \geqslant 0, \tag{$*$}$$

is the complementary function, the duality of the moduli yields two ways of writing the formulae for the best moduli of l_M. Their equivalence suggests that the Legendre transform $(*)$ should commute with such operations on functions as those used to define the function G of Proposition 19. In fact, we shall prove that, if $\delta(\varepsilon) = \min\{\delta_{l_2}(\varepsilon), \inf_n \delta_{X_n^*}(\varepsilon)\}$, then

$(**)$ *the function G^* is equivalent to the function F, where*

$$F(\varepsilon) = \inf_{\varepsilon \leqslant u \leqslant 1 \leqslant v^{-1}} \delta(\varepsilon/u) M^*(uv)/M^*(v).$$

Since $M \leqslant N$ implies $M^* \geqslant N^*$, and $M(u) = cN(du)$, for $u \geqslant 0$, implies $M^*(x) = cN^*(x/cd)$, for $x \geqslant 0$, we conclude that $(\varrho_Y)^*$ is equivalent to δ_{Y^*}. Moreover, using Proposition 19 and Lemma 20, we get

COROLLARY 22. *Let N be an Orlicz function whose complementary function N^* satisfies the Δ_2 condition. Let (Y_n) be a sequence of Banach spaces. Then the space $l_N((Y_n))$ admits an equivalent renorming such that the modulus of convexity is $\succcurlyeq F$, where F is defined as in $(**)$ with $M^* = N$ and $X_n = Y_n^*$.*

Remark. In the case of l_p sums, i.e., where $N(t) = t^p$, $p > 1$, no renorming is necessary. Also the proof of $(**)$ is much easier in that case.

In the next lemma the following notation will be used, f, g being non-negative functions defined on $[0, 1]$, $t \in (0, 1]$, $x \geqslant 0$:

$$\begin{aligned} f^*(x) &= \sup_{0 \leqslant u \leqslant 1} \big(xu - f(u)\big), \\ (f \mathbin{\underline{*}} g)(t) &= \inf_{t \leqslant u \leqslant 1} f(t/u) g(u), \\ (f \mathbin{\overline{*}} g)(t) &= \sup_{t \leqslant u \leqslant 1} f(t/u) g(u), \\ f_{[g]}(t) &= \inf_{0 < v \leqslant 1} f(tv)/g(v), \\ f^{[g]}(t) &= \sup_{0 < v \leqslant 1} f(tv)/g(v). \end{aligned} \tag{$\overset{**}{*}$}$$

The formulae $(*)$ and $(\overset{**}{*})$ look different; however, if $f(u)/u \geqslant c > 0$, for $u > 1$ (which will be satisfied in our case), then the functions f^* resulting from both formulae coincide for $x \leqslant c$, whence they are equivalent, which is all that we need in the sequel.

LEMMA 23. *Let f be a non-negative function on $[0, 1]$ and let N be an Orlicz function such that $N(1) = 1$ and $N^*(2w) \leqslant cN^*(w)$, for $w \geqslant 0$. Then*

(a) *$N^*\big(N(v)/v\big) \leqslant N(v) \leqslant cN^*\big(N(v)/v\big)$, for $v \in (0, 1]$,*

(b) *if $f(u) < u$ for some u, then*

$$f^* \mathbin{\overline{*}} N^* \leqslant (f \mathbin{\underline{*}} N)^* \leqslant \big(c/f^*(1)\big)(f^* \mathbin{\overline{*}} N),$$

(c) *$(f_{[N]})^* \leqslant (f^*)^{[N^*]}$, if $N \leqslant f$, then $(f^*)^{[N^*]} \leqslant c(f_{[N]})^*$.*

Let us first deduce $(**)$ from Lemma 23 and known facts. Since $F = \delta_{\underline{*}}(N_{[N]})$, using (b), (c) we get ($\sim$ denoting the equivalence of functions),

$$F^* \sim \delta^* \mathbin{\overline{*}} (N_{[N]})^* \sim \delta^* \mathbin{\overline{*}} (N^*)^{[N^*]} \sim \varrho \mathbin{\overline{*}} M^{[M]} = G.$$

Consequently, $F^{**} \sim G^*$. On the other hand, using Proposition 3, we see that $\delta(\varepsilon)/\varepsilon$ is a non-decreasing function; hence so is $F(\varepsilon)/\varepsilon$. It follows, as in the case of δ_X, that F^{**} is equivalent to F. Since $F^{**} \sim G^*$, (**) has been established.

Proof of Lemma 23. (a) We may assume $N(v) > 0$. Then we have

$$N^*\big(N(v)/v\big)/N(v) = \sup_u \{u/v - N(u)/N(v)\}.$$

Let $u_0 = 2v/c$ (clearly $c \geqslant 2$, so that $u_0 \leqslant 1$); then

$$N(u_0) = \sup_w \{w(2v/c) - N^*(w)\} \leqslant c^{-1} \sup_w \{(2w)v - N^*(2w)\} = c^{-1}N(v),$$

which yields the upper estimate. The lower one is immediate, the expression on the right-hand side being $\leqslant 0$ for $u > v$, and $\leqslant 1$ for $u \leqslant v$.

(b) Fix an $x \leqslant 1$ and notice that

$$(f \underline{*} N)^*(x) = \sup_w \{xw - (f \underline{*} N)(w)\} = \sup_{0 \leqslant w \leqslant u \leqslant 1} \{xw - N(u) f(w/u)\}$$
$$= \sup_u N(u) \sup_{0 \leqslant w/u \leqslant 1} \{(w/u)\big(xu/N(u)\big) - f(w/u)\}$$
$$= \sup_u N(u) f^*\big(xu/N(u)\big).$$

Since, by (a), $N(u) \geqslant N^*\big(N(u)/u\big)$, the lower estimate is trivial. To get the upper one we use the other part of (a), which gives

$$(f \underline{*} N)^*(x) \leqslant c \sup_{0<u\leqslant 1} N^*\big(N(u)/u\big) f^*\big(xu/N(u)\big)$$
$$\leqslant c \max\{(f^* \bar{*} N^*)(x), \sup_{0<v<x} N^*(v) f^*(x/v)\}$$
$$\leqslant c \max\{(f^* \bar{*} N^*)(x), \sup_{0<v<x} (v/x) N^*(x) f^*(x/v)\}$$
$$\leqslant \big(c/f^*(1)\big)(f^* \bar{*} N^*)(x).$$

(c) The first estimate follows after simple transformations:

$$(f_{[N]})^*(x) = \sup_u \{xu - f_{[N]}(u)\} = \sup_{u,v} \{xu - f(uv)/N(v)\}$$
$$= \sup_v \big(N(v)\big)^{-1} \sup_{u \leqslant 1} (uv)\big(xN(v)/v\big) - f(uv)$$
$$\leqslant \sup_v f^*\big(xN(v)/v\big)/N(v) \leqslant \sup_v f^*\big(xN(v)/v\big)/N^*\big(N(v)/v\big)$$
$$= (f^*)^{[N^*]}(x).$$

Now observe that, if $f \geqslant N$, then the first inequality above becomes an equality (the maximized expression being negative, if $u \in (1, v^{-1}]$) and the second one can, by (a), be replaced by the estimate

$$\geqslant c^{-1} \sup_{0<v\leqslant 1} f^*\big(xN(v)/v\big)/N^*\big(N(v)/v\big) = c^{-1}(f^*)^{[N]^*}(x).$$

This completes the proof of the lemma.

We conclude this section with the following

EXAMPLE. Define, for $x \geqslant 0$, $n = 1, 2, \ldots$,

$$f(x) = \begin{cases} x/n, & \text{if} \quad (n-1)n^2 \leqslant x \leqslant n^2(n+2), \\ n(n+1)(n+2) - x, & \text{if} \quad n^2(n+2) \leqslant x \leqslant n(n+1)^2. \end{cases}$$

Clearly, the function $f(x)/x$ is non-increasing, whence $f(x+y) \leqslant f(x) + f(y)$, for $x, y \geqslant 0$. Notice also that $f(x+y) \geqslant f(x) - y$, and $\sup_{x,y\geqslant 0} \{f(x) - f(x+y)\} = \infty$.

Now let $p > 2$, $0 < c \leqslant p-2$. Define $M(0) = 0$, and for $t = 2^{-x}$, let $M(t) = 2^{-px-cf(x)}$. It is readily verified that, for $t, s \in (0, 1]$,

$$M(t) \leqslant t^p, \qquad M(ts) \geqslant M(t)M(s), \qquad M(ts)/M(s) \leqslant t^{p-c},$$

whence M and M^* satisfy the Δ_2 condition and M is equivalent to an Orlicz function M_1. Let $X = l_{M_1}$.

By Corollary 22, the remark following Proposition 21 and Lemma 23, the best modulus of convexity for X satisfies

$$\delta_X \sim F = \delta_{l_2} \underline{*} (M_{[M]}) = \delta_{l_2} \underline{*} M \sim M.$$

On the other hand, the modulus of convexity of the space $Y = l_p(X)$ satisfies, by Proposition 17 and Lemma 23,

$$\delta_Y \succeq \delta_X \underline{*} (t^p) \sim M \underline{*} (t^p) = \inf_{t \leqslant s \leqslant 1} M(s)(t/s)^p = N.$$

In fact, an argument similar to that of Proposition 21 shows that N is equivalent to the best modulus of convexity for the space Y.

It is no longer true, however, in contradistinction to the case of the moduli of convexity of the l_p sums with $p \in (1, 2]$, that $\delta_{l_p(X)} \sim \delta_X$, even though $\delta_X \dashv \delta_{l_p}$ (recall that $M(t) \leqslant t^p$). Indeed, M satisfying the Δ_2 condition, the non-equivalence of M and N will follow if we know that $\inf_{0<\varepsilon<1} N(\varepsilon)/M(\varepsilon) = 0$. The latter is, however, an immediate consequence of the relation $\sup_{x,y\geqslant 0} \{f(x) - f(x+y)\} = \infty$.

III. Uniform convexifiability in spaces with unconditional bases. In this section E will denote a Banach space with an unconditionally monotone basis, realized as a space of numerical sequences $x = (x_n)$ (i.e., we assume that, if $x = (x_n) \in E$ and $y = (y_n)$ satisfy $|y_n| \leqslant |x_n|$ for each n, then $y \in E$ and $\|y\| \leqslant \|x\|$). The extension of the results obtained here to the case of Banach lattices or, more generally, spaces with local unconditional structure (l.u.st.) is straightforward (cf. [5], [6], [10]). The corresponding facts for the moduli of smoothness can either be treated directly, or simply deduced by duality. We leave that to the reader.

Given a positive number p and a sequence $x = (x_n)$ such that $(|x_n|^p) \in E$, we write

$$\|x\|_{(p)} = \|(|x_n|^p)\|^{1/p}.$$

Following [5], we shall say that $(E, \|\cdot\|)$ is *p-convex* if $\|\cdot\|$ is of the form $|||\cdot|||_{(p)}$ for some unconditionally monotone norm $|||\cdot|||$ on a space F of sequences, or equivalently, if $\|\cdot\|_{(1/p)}$ is a norm, i.e., whenever $x, y \in E$, one has

$$\left\|\left((|x_n|^p + |y_n|^p)^{1/p}\right)\right\|^p \leqslant \|x\|^p + \|y\|^p.$$

Observe that the $\|\cdot\|$ is automatically 1-convex and it is p-convex for some $p > 1$ iff the canonical basis of the space $E(l_p)$ is block p-Hilbertian with constant 1 (cf. [10]); in particular, E is then p'-convex for every $p' \in [1, p]$. The dual notion is that of q-concavity; E is said to be *q-concave*, $1 < q < \infty$, if E^* is q^*-convex, where $q^* = q/(q-1)$. The interpretation using the spaces $E(l_q)$ and $E^*(l_{q^*})$ makes it clear that E is q-concave iff, for each $x, y \in E$,

$$\left\|\left((|x_n|^q + |y_n|^q)^{1/q}\right)\right\|^q \geqslant \|x\|^q + \|y\|^q.$$

Let us start with a very simple result to illustrate these notions. Even in the case of $E = l_p$, $1 < p < \infty$, the proof seems to be simpler than the existing ones and, if Lemma 25 is used, the constant obtained is the best possible one.

PROPOSITION 24. *Suppose E is p-convex and q-concave, where $1 < p \leqslant q < \infty$. Let $r = \max\{2, q\}$ and let $K = \max\{2, 2/\sqrt{p-1}\}$. Then $\delta_E(\varepsilon) \geqslant r^{-1}K^{-r}\varepsilon^r$, for $0 \leqslant \varepsilon \leqslant 2$.*

Proof. Let $t, s \in \mathbf{R}$ and assume, for the sake of normalization, that $|t|^p + |s|^p = 2$. There exists (cf. [5]) a $K = K(p)$ such that

$$(t-s)^2 \leqslant K^2\left(1 - ((t+s)/2)^2\right).$$

(Lemma 25, to be proved below, asserts that $K^2 \leqslant 4/(p-1)$, for $p \in (1, 2]$; if $p > 2$, one can use 2 instead of p in every place.) Therefore

$$\left|\frac{t-s}{K}\right|^r + \left|\frac{t+s}{2}\right|^r \leqslant \left|\frac{t-s}{K}\right|^r + 1 - \left(1 - \left(\frac{t+s}{2}\right)^2\right)^{r/2} \leqslant 1 = [\tfrac{1}{2}(|t|^p + |s|^p)]^{r/p}.$$

Let $z = (z_n)$, where $z_n^p = \frac{1}{2}(|x_n|^p + |y_n|^p)$. By the p-convexity,

$$\|z\|^p \leqslant \tfrac{1}{2}(\|x\|^p + \|y\|^p) = 1.$$

Using the q-concavity and the previous estimates, we obtain

$$\left\|\frac{x-y}{K}\right\|^r + \left\|\frac{x+y}{2}\right\|^r \leqslant \|z\|^r \leqslant 1;$$

hence

$$r\left(1 - \|(x+y)/2\|\right) \geqslant 1 - \|(x+y)/2\|^r \geqslant K^{-r}\|x-y\|^r,$$

which is exactly what was to be proved.

Proof of Lemma 25 (formulated above). Let $v \geqslant 1$; then $v \leqslant ((v+1)/2)^2$, whence

$$v^{-1-1/p} - v^{-2/p}\left((v+1)/2\right)^{-2+2/p} \geqslant 0.$$

Let $w \geqslant 1$; integrating from 1 to w and then multiplying by $2(2-p)p^{-2}w^{-1+2/p}$ one gets successively

$$-pw^{-1/p} + \frac{2p}{2-p}\left(\frac{w+1}{2w}\right)^{-1+2/p} - \frac{p^2}{2-p} \geqslant 0,$$

$$(4/p)\left((w+1)/2\right)^{-1+2/p} - 2w^{-1+2/p} - (2/p)(2-p)w^{-1+1/p} \geqslant 0.$$

Now we let $u \geqslant 1$ and integrate from 1 to u to obtain

$$4\left((u+1)/2\right)^{2/p} - pu^{2/p} - 2(2-p)u^{1/p} - p \geqslant 0.$$

Let t, s be such that $t \geqslant s > 0$, $t^p + s^p = 2$. We may substitute $u = (t/s)^p$. After multiplying the resulting expression by s^2, we get

$$4 - pt^2 - 2(2-p)ts - ps^2 \geqslant 0.$$

It is not hard to see that all other pairs (t, s) of reals, with $|t|^p + |s|^p = 2$, satisfy the latter inequality. This completes the proof of the lemma.

Remark. The notions of p-convexity and q-concavity have also been introduced (independently of [5], under other names and in the context of Banach lattices) in [13]. The existence of such renormings will be discussed later (cf. Corollary 28).

We shall need the following lemma.

LEMMA 26. *Let $0 < r < q \geqslant 2$ and let M be a non-negative function on $[0, 1]$ such that $M(0) = 0$ and $M(t)/t^q$ is non-decreasing on $(0, 1]$. Then, for any $t_i, c_i \in [0, 1]$ such that $\sum_{i=1}^n t_i = 1$, one has*

$$M\left(\left(\tfrac{1}{2}\sum_{i=1}^n t_i c_i\right)^{1/r}\right) \leqslant \sum_{i=1}^n t_i^{2/r} M(c_i^{1/r}) \left(\sum_{i=1}^n t_i^{(q-2)/(q-r)}\right)^{(q-r)/r}.$$

Proof. Let $\varphi(s) = M(s^{1/r})^{r/q}$, for $s \in [0, 1]$. Using Lemma 2 and Hölder's inequality, we obtain

$$\varphi\left(\tfrac{1}{2}\sum_{i=1}^n t_i c_i\right) \leqslant \tilde{\varphi}\left(\sum_{i=1}^n t_i c_i\right) \leqslant \sum_{i=1}^n t_i \varphi(c_i)$$

$$\leqslant \left(\sum_{i=1}^n t_i^{2/r}\varphi(c_i)^{q/r}\right)^{r/q} \left(\sum_{i=1}^n t_i^{(q-2)/(q-r)}\right)^{(q-r)/q},$$

which is equivalent to the statement of the lemma.

In the sequel, given an $x \in E$ and a set S of the indices of the basis, we shall denote by Sx the vector which agrees with x in coordinates in S and is zero in the other coordinates. The complement of the set S will be denoted by $\sim S$.

PROPOSITION 27. *Assume that E is p-convex, $p > 1$, and M is a non-negative function on $[0, 1]$ such that, for every $x \in S_E$ and every subset S of indices,*

$$M(\|Sx\|) \leqslant 1 - \|(\sim S)x\|.$$

Suppose that, for some $q \geqslant 2$, the function $M(t)/t^q$ is non-decreasing on $(0, 1]$. Then

(i) *if $p \geqslant 2$ or $q > 2$, one has $M \prec \delta_E$;*

(ii) *if $q = 2$ and $p < 2$, one has $M \prec \psi_p \circ \delta_E$, where*

$$\psi_p(0) = 0, \quad \psi_p(t) = t|\log t|^{(2-p)/p} \quad \textit{for} \quad 0 < t \leqslant 1.$$

The constants involved in "$\prec$" depend only on p and q.

Proof. Let $x = (x_n)$, $y = (y_n)$ be vectors in S_E such that $0 < \|x+y\| < 2$. Consider the sequences $u = (u_n)$, $v = (v_n)$, $w = (w_n)$, where $u_n = \frac{1}{2}|x_n + y_n|$, $v_n = \big(\frac{1}{2}(|x_n|^p + |y_n|^p)\big)^{1/p}$, $w_n = |x_n - y_n|$. It is clear that u, v, w belong to E; in fact, $\|u\| = \frac{1}{2}\|x+y\|$, $\|w\| = \|x-y\|$ and, by the p-convexity of E, $\|v\| \leqslant 1$.

Let $t = 1 - \|u\|$. We are to estimate t in terms of $\|w\|$. Since $t \neq 0$, $m = [\log t^{-1}]$ is a well-defined integer. Define the sets

$$S_0 = \{n\colon u_n < \tfrac{1}{2}v_n\},$$
$$S_i = \{n\colon e^{1-i}u_n \geqslant v_n - u_n > e^{-i}u_n\}, \quad \text{for} \quad i = 1, 2, \ldots, m,$$
$$S_{m+1} = \{n\colon e^{-m}u_n \geqslant v_n - u_n\}.$$

Pick an $x^* \in E^*$ such that $\|x^*\| = 1$ and $x^*(u) = 1 - t$, and write, for $i = 0, 1, \ldots, m$,

$$a_i = x^*(S_i u) = \sum_{n \in S_i} x_n^* u_n.$$

Clearly,

$$(1) \qquad t \geqslant x^*(v) - x^*(u) = \sum_n x_n^*(v_n - u_n) > \sum_{i=0}^{m} e^{-i} \sum_{n \in S_i} x_n^* u_n = \sum_{i=0}^{m} e^{-i} a_i.$$

Since $\|(1-t)^{-1}u\| = 1$, we obtain, for $i = 1, 2, \ldots, m$,

$$(2) \qquad \begin{aligned} M(\|S_i u\|) &\leqslant (1-t)^q M\big(\|S_i((1-t)^{-1}u)\|\big) \\ &\leqslant (1-t)\big(1 - x^*\big((\sim S_i)\,((1-t)^{-1}u)\big)\big) \\ &\leqslant x^*(u) - x^*\big((\sim S_i)u\big) = x^*(S_i u) = a_i. \end{aligned}$$

By Lemma 25, there is a $K = K(p)$ such that, for each n,

$$(3) \qquad w_n \leqslant K u_n \sqrt{(v_n/u_n)^2 - 1}.$$

It follows from (3), the definition of the S_i's and the monotonicity of the norm that, for $i = 1, 2, \ldots, m+1$,

$$(4) \qquad \|S_i w\| \leqslant K\sqrt{(e^{1-i}+1)^2 - 1}\,\|S_i u\| \leqslant 3Ke^{-i/2}\|S_i u\|.$$

The following estimates need no comments:

$$(5) \qquad \|S_{m+1}w\| \leqslant 3Ke^{-(m+1)/2} \leqslant 3Kt^{1/2},$$

$$(6) \qquad M(t^{1/2}) \leqslant (t^{1/2})^q M(1) \leqslant t^{q/2} \leqslant t,$$

$$(7) \qquad \|S_0 w\| \leqslant 2\|S_0 v\|,$$

$$(8) \qquad M(\|S_0 v\|) \leqslant 1 - \|(\sim S_0)v\| \leqslant 1 - x^*\big((\sim S_0)u\big) = t + a_0.$$

Let $r = \min\{2, p\}$ (if $q > 2$, then one can simply take $r = 1$). Since E is r-convex, using (7), (5) and (4), we get

$$\|w\|^r \leqslant \sum_{i=0}^{m+1} \|S_i w\|^r \leqslant 2^r\|S_0 v\|^r + (3K)^r t^{r/2} + (3K)^r \sum_{i=1}^{m} e^{-ir/2}\|S_i u\|^r.$$

Write, for any $z \geqslant 0$,

$$C_z = 2^z + (3K)^z + (3K)^z \sum_{i=1}^{n} e^{-iz/2}.$$

Unless $p \geqslant 2 = q$, using Lemma 26 with suitable numbers t_i, c_i (easily identifiable from (9)), we can now obtain

$$(9) \qquad M\big((2C_r)^{-1/w}\|w\|\big) \leqslant C_r^{-q/r} C_s^{(q-r)/r}\Big[2^2 M(\|S_0 v\|) + (3K)^2 M(t^{1/2}) + {} + (3K)^2 \sum_{i=1}^{m} e^{-i} M(\|S_i u\|)\Big],$$

where we have put $s = r(q-2)/(q-r)$. In the previously excluded case $p \geqslant 2 = q$ the estimate (9) is true with C_s replaced by 1. (Check what would be yielded by the proof of Lemma 26 if Hölder's inequality were not used.) Clearly, C_r can be estimated from above and from below by positive constants depending only on p and q. The same is true about C_s provided that $p \geqslant 2$ or $q > 2$. Otherwise $s = 0$ and $C_s = C_0 = m+2 \leqslant |\log e^{-2}t|$. The expression in brackets (in (9)) can, by (8), (6), (2) and (1), be estimated by

$$4(t + a_0) + 9K^2 t + 9K^2 \sum_{i=1}^{m} e^{-i} a_i \leqslant (4 + 9K^2)t + 9K^2 \sum_{i=0}^{m} e^{-i} a_i \leqslant (4 + 18K^2)t.$$

Thus we have proved that, if $\|x\| = \|y\| = 1$, $0 < \|x+y\| < 2$ and $t = 1 - \frac{1}{2}\|x+y\|$, then

$$M(C'\|x-y\|) \leqslant C'' e^{-2} t |\log e^{-2} t|^{\alpha}, \tag{10}$$

where C' and C'' are positive constants depending only on p and q, $\alpha = 0$, if $q > 2$ or $p \geqslant 2$ and $\alpha = (2-p)/p$ otherwise. The assumption that $\|x+y\| \neq 0$ is not essential for the final result. We would like to make sure that $\|x+y\| = 2$ is possible only if $\|x-y\| = 0$, which would complete the proof. This need not be true if $M(c) = 0$, for some $c > 0$, but in that case the proposition is trivial.

Suppose, therefore, that M vanishes only at zero and yet, for some $x, y \in S_E$ with $\|x-y\| \neq 0$, one has $\|x+y\| = 2$. The metric space $S_E \setminus \{x\}$ being connected, there exists a $z \in S \setminus \{x\}$ with $\|z+x\| = 2$ for which there is a sequence $z_1, z_2, \ldots$ in $S_E \setminus \{x\}$ such that $\|z_j + x\| < 2$ and $\|z_j - z\| \to 0$. Since $\|z_j + x\| < 2$, it follows from (10) that $M(C'\|x - z_j\|) \to 0$, whence $\|x - z_j\| \to 0$, so that $x = z$, which is the desired contradiction.

Remark. Consider the following modification of the modulus of convexity

$$d_E(\varepsilon) = \inf\{1 - \|v\| \colon u, v \in E,\ \|u\| = \varepsilon,\ \|u+v\| = 1,\ u, v \text{ are disjointly supported}\}.$$

Clearly, $\delta_E(2\varepsilon) \leqslant d_E(\varepsilon)$ for $\varepsilon \leqslant 1$, while, e.g., $\delta_{l_1}(2\varepsilon) = 0 = \varepsilon - d_{l_1}(\varepsilon)$ for $\varepsilon \leqslant 1$. Assume again that E is p-convex, $p > 1$, and define, for $q \geqslant 2$, $0 \leqslant \varepsilon \leqslant 1$,

$$M_q(\varepsilon) = \inf_{1 < t < \varepsilon^{-1}} d_E(t\varepsilon) t^{-q}.$$

It follows from Corollary 11 and Proposition 27 (i) that, for any $q > 2$, $M_q \dashv \delta_E \dashv M_2$, but, in general, it is not true that $M_2 \dashv \delta_E$. Let us discuss briefly an example (suggested by G. Pisier).

Example. Let $p \in (1, 2)$ be fixed and let E be the Lorentz sequence space $d(a, p)$, where $a_n = n^{-1+2/p}$, $n = 1, 2, \ldots$, i.e., if $(x_n) \in E$ and (y_n) is the non-increasing rearrangement of the sequence $(|x_n|)$, then

$$\|(x_n)\|_E^p = \sum_n n^{-1+2/p} y_n^p.$$

Clearly, E is p-convex. It is readily checked that if $x, y \in E$ are disjointly supported, then $\|x\|^2 + \|y\|^2 \leqslant \|x+y\|^2$, whence we have $M_2(\varepsilon) \geqslant \frac{1}{2}\varepsilon^2$. It follows from Proposition 27 (ii) that there is a $C > 0$ such that

$$\delta_E(\varepsilon) \geqslant C\varepsilon^2 (\log e/\varepsilon)^{(p-2)/p}.$$

On the other hand, we can show that (for any equivalent renorming of E)

$$\delta_E(\varepsilon) \leqslant C_1 \varepsilon^2 (\log e/\varepsilon)^{(p-2)/2p}, \tag{$*$}$$

for some $C_1 < \infty$. The latter estimate can be obtained by considering integrals of the form

$$I = \Big(\int_0^1 \Big\| \sum_{i=1}^m x_i r_i(t) \Big\|^2 dt\Big)^{1/2},$$

where $(x_1, \ldots, x_m)$ are suitable systems of vectors in E and $r_1, \ldots, r_m$ are the Rademacher functions. One can show, using some results of [20] and of [3], that

$$C_2 I \leqslant \Big\| \Big(\Big(\sum_{i=1}^m \big((x_i)_n\big)^2\Big)^{1/2} \Big) \Big\|_E \leqslant C_3 I,$$

where C_2, C_3 are positive constants depending only on E.

Another inequality, yielded by a result of [7] and Proposition 17, states that if δ is the modulus of convexity corresponding to an equivalent norm, $|||\cdot|||$, on E, then, for some $C_4 > 0$ depending only on that renorming, one has

$$I \geqslant C_4 \|(|||x_i|||)\|_\delta.$$

Let us indicate a suitable choice of the x_i's. For each positive integer m we consider the m-tuple $(x_1, \ldots, x_m)$, where the nth coordinate of the vector x_i is equal to

$$\begin{cases} (i+n-1)^{-1/2}, & \text{if } n \leqslant m-i+1; \\ (i+n-m-1)^{-1/2}, & \text{if } m-i+1 < n \leqslant m; \\ 0, & \text{if } n > m. \end{cases}$$

The proof of $(*)$ is now reduced to simple computations; hence, the result not being completely satisfactory, we omit further details.

Remark. Now let E be an arbitrary space $d(a, p)$ with $p > 1$, and let γ_E be the upper estimate for δ_E obtained by using Kadec's theorem in the same way as we did in the proof of Proposition 21 (without using Proposition 10). It was shown in [1] (without using the assumption that $p \geqslant 2$, which was needed for other purposes) that $d_E \geqslant c\gamma_E$, where $c = c(E) > 0$. Hence, if $p \geqslant 2$, then the main result of [1] can be deduced from this fact and Proposition 27 (i). To get an extension, let us remark that γ_E is equivalent to a supermultiplicative function. It is a well-known and easy fact that a bounded supermultiplicative function f on $[0, 1]$ which is not identically zero on $(0, 1)$ can be written as $x^q g(x)$, where $0 \leqslant g(x) \leqslant 1$ and $\lim_{x \to 0+} g(x) x^{-\varepsilon} = \infty$, for each $\varepsilon > 0$. The (uniquely determined) number q will be referred to as the characteristic exponent of f (and of any function equivalent to it).

Let q_0 be the characteristic exponent of γ_E. If $p < 2$ and $q_0 \geqslant 2$, then we can estimate δ_E only up to a power of $|\log \varepsilon|$. If, however, $q_0 < 2$, then it follows from the facts just mentioned that for any $q \in (q_0, 2)$ there is a $c = c(q, E) > 0$ such that $d_E(\varepsilon) \geqslant c\varepsilon^q$, whence the basis is block q-Besselian. Using Corollary 28 below and Proposition 24, we see that E can be equivalently renormed to have the modulus of convexity $\geqslant c'\varepsilon^2$ for some $c' > 0$.

Corollary 28 is at least partially known (cf. [5], [21]). Our proof depends on the following renorming result (used already in [5]):

If $\delta_E(\varepsilon) \geqslant K\varepsilon^q$, $K > 0$, then, for any $q' > q$, the formula

$$|||(t_n)||| = \sup\Big\{\Big(\sum_{i=1}^{m} \|(t_n a_{i,n})\|^{q'}\Big)^{1/q'} \colon \sum_{i=1}^{m} |a_{i,n}|^{q'} = 1, \text{ for } \forall n,\ m = 1, 2, \ldots\Big\}$$

defines on E a q'-concave norm equivalent to the original one.

It is easy to see that if $\|\cdot\|$ is p-convex for some $p > 1$, then so is $|||\cdot|||$.

Corollary 28. *If the basis of E is block q-Besselian, then, for any $q' > q$, E admits an equivalent q'-concave norm. The dual statement for block p-Hilbertian bases and the one involving both properties are also true.*

Proof. Assume that the basis of E is block q-Besselian. Fix an $r > \max\{1, 2/q\}$ and consider the space $E_{(r)}$ consisting of numerical sequences (x_n) such that $\|(x_n)\|_{(r)} < \infty$. $E_{(r)}$ is r-convex and the natural basis is block qr-Besselian. A renorming result used in [10] (proved again in a more general setting in Proposition 29 below) yields an equivalent r-convex norm, say $|\cdot|$, for which one can take in Proposition 27 $M(t) = ct^{qr}$, $c > 0$, whence, using that proposition and the renorming result mentioned before, we get another equivalent norm on $E_{(r)}$, say $|||\cdot|||$, which is r-convex and $q'r$-concave. The norm $|||\cdot|||_{(1/r)}$ on E has the required property.

If the basis of E is block p-Hilbertian and $1 < p' < p$, then the natural basis of E^* is block p^*-Besselian, whence E^* admits an equivalent $(p')^*$-concave renorming, say $|||\cdot|||$. Moreover, tracing the formulae used to define $|||\cdot|||$, we see that $|||\cdot|||$ is lower w^*-continuous, and hence dual to a p'-convex norm on E.

If the basis is both block p-Hilbertian and block q-Besselian, then we can first renorm E to be p'-convex and then make it q'-concave, while preserving the p'-convexity. This completes the proof.

Remark. One could modify the reasoning in Proposition 27 (mainly be using, instead of (3), an analogue of Lemme 1 of [7], with the exponent 2 replaced by p) to get estimates for $\delta_{E(X)}$, X being an arbitrary Banach space. In particular, if $\delta_X(\varepsilon) \geqslant c\varepsilon^2$ for some $c > 0$, then it can be found in [4] or deduced from the results of Section II that there is a $c_1 = c_1(p, c) > 0$ such that if $x, y \in X$, $\|x\|^p + \|y\|^p = 2$, then $\|(x+y)/2\| \leqslant 1 - c_1\|x-y\|^2$.

After some transformations we get the estimate replacing (3)

$$\|x-y\|^2 \leqslant c_2\Big(\big(\tfrac{1}{2}(\|x\|^p + \|y\|^p)\big)^{2/p} - \big(\tfrac{1}{2}\|x+y\|\big)^2\Big).$$

Since $d_E(\varepsilon) \geqslant \delta_E(2\varepsilon)$ and $\delta_E(\varepsilon)/\varepsilon^2$ is equivalent to an increasing function, we can now repeat the proof of Proposition 27 and find that $\delta_{E(X)}$ is not much worse than δ_X (the exact formulation is left to the reader). The latter result shows that the modulus of convexity of the space Y constructed in [2], which has a symmetric basis and contains a complemented subspace isomorphic to E, is almost as good as that of E.

In the next proposition we show, starting from some isomorphic properties of E, that E admits an equivalent norm, which satisfies the assumptions of Proposition 27 with a function M essentially equivalent to that involved in the isomorphic property.

We shall denote by $\mathscr{A}$ the set of non-negative functions on $[0, 1]$ and by $\mathscr{A}_0$ the subset of $\mathscr{A}$ consisting of non-decreasing functions. Given $f \in \mathscr{A}$ and a norm $|||\cdot|||$ on E equivalent to $\|\cdot\|$, $\omega(|||\cdot|||, f)$ will denote the least upper bound of the numbers $\sum_{i=1}^{m} f(|||S_i x|||)$, where $x \in E$ and $\mathscr{S} = (S_1, \ldots, S_m)$ is a collection of finite and mutually disjoint sets of indices such that $\big|\big|\big|\sum_{i=1}^{m} \varepsilon_i S_i x\big|\big|\big| \leqslant 1$ for each choice of signs (ε_i). Let us remark that if $f \in \mathscr{A}$ and $g(x) = \sup_{0 \leqslant u \leqslant x} f(u)$, then $g \in \mathscr{A}_0$, $g \geqslant f$ and $\omega(|||\cdot|||, f) = \omega(|||\cdot|||, g)$. Furthermore, if $|||\cdot|||_1$ is another equivalent norm on E, $|||\cdot|||_1 \leqslant |||\cdot|||$, and $\omega(|||\cdot|||, f) < \infty$, then $\omega(|||\cdot|||_1, f) < \infty$. If $f \in \mathscr{A}$ satisfies the Δ_2 condition and $c > 1$, then $\omega(|||\cdot|||, f) < \infty$ implies $\omega(c|||\cdot|||, f) < \infty$. These facts together with Lemma 31 and some remarks below imply that the set $\{f \in \mathscr{A} \colon \omega(|||\cdot|||, f) < \infty\}$ is the same for all norms $|||\cdot|||$ on E equivalent to $\|\cdot\|$, i.e., it depends only on the given unconditional basis. In the sequel we shall write $\omega(f)$ instead of $\omega(\|\cdot\|, f)$. Observe that if δ is the modulus of convexity of E equipped with an equivalent norm, then $\omega(\delta) < \infty$. We shall study a converse problem.

Proposition 29. *Assume that E is p-convex and let $g \in \mathscr{A}$ be a function such that $g(t^{1/p})$ is convex, $\omega(g) < \infty$, $g(1) = 1$ and $g(t)/t^r$ is non-increasing, for some $r > 0$. Then E admits an equivalent p-convex renorming, $|||\cdot|||$, so that, whenever $x, y \in E$ are disjointly supported and $|||x+y||| = 1$, one has*

$$g(|||y|||) \leqslant r\omega(g)^{r/p}(1 - |||x|||).$$

Proof. Define, for $x \in E$,

$$|||x||| = \inf\Big\{t > 0 \colon \sum_{i=1}^{m} g(\|S_i x\|/t) \leqslant g(1), \text{ for } \forall \mathscr{S}\Big\}.$$

Clearly, $|||x||| \geqslant \|x\|$ (consider $\mathscr{S} = (S_1)$, where $\|x - S_1 x\| < \varepsilon$) and, if $K = \omega(g)^{1/p}$, then, for any $\mathscr{S}$ and $x \neq 0$,

$$\sum_{i=1}^{m} g(\|S_i x\|/K\|x\|) \leqslant K^{-p} \sum_{i=1}^{m} g(\|S_i x\|/\|x\|) \leqslant g(1),$$

so that $|||x||| \leqslant K\|x\|$.

It is obvious that $|||\cdot|||$ is unconditionally monotone. Now suppose $|||(x_n)||| \leqslant 1$, $|||(y_n)||| \leqslant 1$, $z = (z_n)$, where $z_n^p = \frac{1}{2}(|x_n|^p + |y_n|^p)$. Then, for any $\mathscr{S} = (S_1, \ldots, S_m)$,

$$\sum_{i=1}^{m} g(\|S_i z\|) \leqslant \sum_{i=1}^{m} g\Big(\big(\tfrac{1}{2}(\|S_i x\|^p + \|S_i y\|^p)\big)^{1/p}\Big) \leqslant \sum_{i=1}^{m} \tfrac{1}{2}\big(g(\|S_i x\|) + g(\|S_i y\|)\big) \leqslant 1.$$

This proves that the unit ball of $(E, |||\cdot|||)_{(1/p)}$ is convex, whence $(E, |||\cdot|||)$ is p-convex.

Finally, let $x, y \in E$ be disjointly supported vectors with $|||x+y||| = 1$; we may assume $x \neq 0 \neq y$. Given any $\mathscr{S} = (S_1, \ldots, S_m)$, one has $\sum_{i=1}^{m} g(\|S_i x\|) + g(\|y\|) \leqslant 1$; hence

$$\sum_{i=1}^{m} g\Big(\|S_i x\|/\big(1 - g(\|y\|)\big)^{1/r}\Big) \leqslant \big(1 - g(\|y\|)\big)^{-1} \sum_{i=1}^{m} g(\|S_i x\|) \leqslant 1.$$

Consequently, $|||x||| \leqslant \big(1 - g(\|y\|)\big)^{1/r} \leqslant 1 - r^{-1} g(\|y\|)$ (it is clear that $r \geqslant 1$), so that (recall the definition of K)

$$g(|||y|||) \leqslant g(\|y\|)\,(|||y|||/\|y\|)^r \leqslant r\omega(g)^{r/p}(1 - |||x|||),$$

which completes the proof.

Let us write, for $t \in [0, 1]$,

$$\varphi(t) = \varphi_E(t) = \sup\{f(t) : f \in \mathscr{A},\ \omega(f) \leqslant 1\}.$$

The function φ_E is closely related to some numerical characteristics of E considered already in [14]. Indeed, since in order to compute $\varphi_E(t)$ it is enough to consider those $f \in \mathscr{A}$ which vanish off $\{t\}$, we see that $\varphi_E(t) = 0$ iff there exist arbitrarily long sequences of vectors $x_1, \ldots, x_m \in E$, with finite and mutually disjoint supports, such that $\|x_i\| = t$, $\big\|\sum_{i=1}^{m} x_i\big\| \leqslant 1$. Moreover, if $m < \infty$ is the maximal length of such a sequence, then $\varphi_E(t) = 1/m$. It is well known that the functions defined as φ_E are (equivalent to) supermultiplicative functions. For, if $t, s \in (0, 1]$, $\varphi(t) = 1/(m-1)$, $\varphi(s) = 1/(n-1)$, then, given any mn vectors $x_1, \ldots, x_{mn} \in E$ with finite, disjoint supports and $\|x_i\| = ts$, one has $\big\|\sum_{i=(j-1)m+1}^{jm} x_i/s\big\| > 1$, for $j = 1, \ldots, n$, whence $\big\|\sum_{i=1}^{mn} x_i\big\| > 1$. This proves that $\varphi(ts) > 1/mn$, so that

$$\varphi(t)\varphi(s) = 1/(m-1)(n-1) \leqslant 3/(mn-1) \leqslant 3\varphi(ts).$$

The latter inequality is obvious if $\varphi(t)\varphi(s) = 0$.

Since φ_E is non-decreasing, we infer that two cases are possible: either $\varphi_E(t) > 0$ for each $t > 0$, or $\varphi_E(t) = 0$ for each $t < 1$. The second case occurs iff E contains subspaces arbitrarily close to l_∞^m for $m = 1, 2, \ldots$ (the non-trivial implication follows from results of Maurey's [20]).

From now on we shall be assuming that $\varphi(t) \neq 0$, for $t > 0$. (Let us mention that under this assumption E admits a p-convex norm, $p > 1$, iff it does not contain subspaces close to l_1^m for large m; cf. [10].) Let q denote the characteristic exponent of φ_E, i.e., $\varphi(t) = O(t^q)$ and $\lim_{t\to 0+} \varphi(t)t^{-p} = \infty$ whenever $p < q$.

In common examples, like Lorentz and Orlicz spaces, one simply has $\omega(\varphi_E) < \infty$. It can be shown that the latter property implies that $\varphi_E(t^a)$ is equivalent to a convex function, provided that the basis is block p-Hilbertian and $a < 1/p$. On the other hand, W. B. Johnson has constructed in [11] a reflexive Y for which $\lim_{t\to 0+} \varphi_Y(t)t^{-1} = 1$. If it were true that $\omega(\varphi_Y) < \infty$, then Y would be isomorphic to l_1, which is absurd. Consequently, if $r > 1$ and $E = Y_{(r)}$, then $\lim_{t\to 0+} \varphi_E(t)t^{-r} = 1$ and $\omega(\varphi_E) = \infty$.

Let us give examples of functions f such that $\omega(f) < \infty$.

Lemma 30. *If $\psi \in \mathscr{A}_0$, then $\omega(\psi\circ\varphi) \leqslant \sum_{k=1}^{\infty} \psi(1/k)$.*

Proof. Let $x \in E$, $\|x\| \leqslant 1$, $\mathscr{S} = (S_1, \ldots, S_m)$. Using the definitions and Abel's transform, we get

$$\begin{aligned}\sum_{i=1}^{\infty} \psi\big(\varphi(\|S_i x\|)\big) &= \sum_{k=1}^{\infty} \psi(1/k)\,\mathrm{Card}\{i : \varphi(\|S_i x\|) = 1/k\}\\ &= \sum_{k=1}^{\infty} \big(\psi(1/k) - \psi(1/(k+1))\big)\,\mathrm{Card}\{i : \varphi(\|S_i x\|) \geqslant 1/k\}\\ &\leqslant \sum_{k=1}^{\infty} \big(\psi(1/k) - \psi(1/(k+1))\big)k = \sum_{k=1}^{\infty} \psi(1/k).\end{aligned}$$

Remark. We do not know whether for every $\psi \in \mathscr{A}_0$ with $\sum_{k=1}^{\infty} \psi(1/k) = \infty$ there is an E such that $\omega(\psi\circ\varphi_E) = \infty$. It is so if, for instance, $\psi(t) = \big(t\log\log(1/t)\big)/\big(\log(1/t)\big)$ for small t.

Also, it is not clear whether there is always an $f \in \mathscr{A}$ such that $\omega(f) < \infty$ and $\psi\circ\varphi_E \dashv f$ for each $\psi \in \mathscr{A}_0$ with $\sum_{k=1}^{\infty} \psi(1/k) < \infty$.

At this moment we can conclude that, if $q < 2$ and the basis of E is block p-Hilbertian for some $p > 1$, then, by Lemma 30, Corollary 28 and Proposition 24, E admits an equivalent norm such that $\delta_E(\varepsilon) \geqslant C\varepsilon^2$ for some $C > 0$. If $q \geqslant 2$, then the conclusion is not so satisfactory. Assuming again the basis to be block p-Hilbertian, we can infer in a similar manner, using now also Proposition 18, that E can be renormed so that $\lim_{\varepsilon\to 0+}\delta_E(\varepsilon)/\varepsilon^r = \infty$ for each $r > q$. The results of this section yield of course more specific estimates for the moduli that can be realized by a suitable renorming described above. They do not coincide with the upper estimates for possible moduli (even those estimates obtained by considering $l_2(E)$ or $E(l_2)$ instead of E), the ratio being, roughly, a power of $|\log\varepsilon|$; we shall therefore withhold an explicit formulation of either of them.

We close this section with an example, where φ_E is far from being convex. Then we give two lemmas, which allow one to replace an $f\in\mathscr{A}$ such that $\omega(f) < \infty$ by a greater function, which is closer to satisfying the assumptions of Proposition 29.

Example. There is a space E such that for some sequence (t_k) tending to zero one has $\lim_{k\to\infty}\varphi_E(t_k/k)/\varphi_E(t_k) = 1$. Indeed, let A be a fixed number greater than 1 and let, for $k = 1, 2, \ldots$,

$$M_k(t) = \begin{cases} t^2(\log A/t)^{-1/k}, & \text{if} \quad 0 < t \leqslant 1,\\ 0, & \text{if} \quad t = 0.\end{cases}$$

We shall prove that the space $E = l_2\big((l_{M_k}^{m_k})\big)$, where m_k is a suitably chosen sequence of positive integers, has the required property.

It can be deduced from the results of Section II and Kadec's theorem that there is a $C > 0$ such that if $E_{(k)} = \big(\sum_{i=k}^{\infty} l_{M_i}\big)_{l_2}$ then $\varphi_{E_{(k)}} \geqslant CM_k$ for $k = 1, 2, \ldots$ Hence, writing $t_k = (M_k)^{-1}(1/m_k)$, we can estimate

$$\varphi_E(t_k) \leqslant 1/m_k = M_k(t_k),$$

$$\varphi_E(t_k/k)^{-1} \leqslant \varphi_{E_{(k+1)}}(t_k/k)^{-1} + \sum_{i\leqslant k} m_i \leqslant C^{-1}M_{k+1}(t_k/k)^{-1} + \sum_{i\leqslant k} m_i.$$

Let the sequence m_k be chosen so that

$$\alpha_k = m_k^{-1}\sum_{i<k} m_i \to 0, \qquad \beta_k = M_k(t_k)/M_{k+1}(t_k/k) \to 0.$$

Then we easily obtain

$$1 \leqslant \varphi_E(t_k)/\varphi_E(t_k/k) \leqslant C^{-1}\beta_k + 1 + \alpha_k.$$

This concludes the proof.

Lemma 31. *Let $f\in\mathscr{A}$, $\omega(f) < \infty$ and $r > q$. Then there exists an $F\in\mathscr{A}_0$ such that $\omega(F) < \infty$, $F \geqslant f$, and the function $F(t)/t^r$ is non-increasing.*

Proof. We may assume $f\in\mathscr{A}_0$ and write $f(t) = f(1)$, for $t \geqslant 1$. Define, for $x\in[0,1]$,

$$f_1(x) = \sum_{n=0}^{\infty} 2^{-rn} f(2^n x), \qquad F(x) = \sup_{x<t\leqslant 1} f(t)\,(x/t)^r.$$

Clearly, $F \geqslant f$ and $f_1(2x) = 2^r\big(f_1(x) - f(x)\big) \leqslant 2^r f_1(x)$, for $x\in[0,\frac12]$; hence the argument used in Proposition 10 yields a $C < \infty$ such that $F \leqslant Cf_1$. We shall prove that, if $A > 1$ and $g(x) = f(Ax)$, then

$$(*) \qquad \omega(g) \leqslant 3\varphi(1/A)^{-1}\omega(f).$$

This will yield the desired estimate, since, if $q < s < r$, then $\lim_{t\to 0+} t^s/\varphi(t) = 0$; hence there is a $C_1 < \infty$ such that

$$\omega(f_1) \leqslant \sum_{n=0}^{\infty} 2^{-rn}\omega\big(f(2^n\cdot)\big) \leqslant 3\sum_{n=0}^{\infty} 2^{-rn}\varphi(2^{-n})^{-1}\omega(f)$$
$$\leqslant 3\sum_{n=0}^{\infty} 2^{-rn} C_1 2^{sn}\omega(f) < \infty.$$

To prove $(*)$ consider an arbitrary sequence of vectors $x_1, \ldots, x_m \in E$ with finite and mutually disjoint supports and $\big\|\sum_{i=1}^{m} x_i\big\| \leqslant 1$. Let $k_0 = 0$ and let k_j, $j = 1, 2, \ldots$, be the first index such that $\big\|\sum_{i=k_{j-1}+1}^{k_j} x_i\big\| > 1/A$. If k_l is the last index to be defined by this procedure, then $\big\|\sum_{i>k_l} x_i\big\| \leqslant 1/A$ and, by the definition of φ, $l\varphi(1/A) \leqslant 1$. Therefore

$$\sum_{i=1}^{m} g(\|x_i\|) = \sum_{i=1}^{m} f(\|Ax_i\|) = \sum_{j=1}^{l}\sum_{k_{j-1}<i<k_j} f(\|Ax_i\|) + \sum_{j=1}^{l} f(\|Ax_{k_j}\|) + \sum_{i>k_l} f(\|Ax_i\|)$$
$$\leqslant (l + l + 1)\,\omega(f) \leqslant 3\varphi(1/A)^{-1}\omega(f).$$

This completes the proof of the lemma.

The restriction imposed on E in the next lemma is not unnatural in view of the last example. Namely, we shall say that E satisfies the *reproducibility condition with exponent p* if there exists a sequence (λ_k) such that

(i) $\lambda_k = O(k^{1/p})$,

(ii) given any positive integer k and an abritrary sequence of vectors $x_1, \ldots, x_m \in E$ with finite and mutually disjoint supports, there is a similar system $(x_{i,j})$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant k$, such that $\|x_{i,j}\| = \|x_i\|$ for $i = 1, \ldots, m$, and

$$\Big\|\sum_{i,j} x_{i,j}\Big\| \leqslant \lambda_k \Big\|\sum_i x_i\Big\|.$$

(In fact, a weaker version of this property would be sufficient for our purposes.)

For instance, E may be an arbitrary space with a subsymmetric, block p-Hilbertian basis, or F may be an arbitrary space with an unconditionally monotone basis and $E = l_p(F)$ or $E = F(l_p)$. Let us remark that in our discussion of uniform convexifiability we might have replaced E by $l_2(E)$ or $E(l_2)$, which admit the same moduli of convexity; hence the reproducibility with exponent 2 is not a restrictive condition.

The proof of the following lemma is now a simple exercise left to the patient reader. (It is analogous to the previous one and besides the listed ingredients it involves Lemma 2.)

LEMMA 32. *Assume that E satisfies the reproducibility condition with exponent p_0. Then, given any $f \in \mathscr{A}$, with $\omega(f) < \infty$, and numbers $p \in (0, p_0)$, $r > q$, there is a $g \in \mathscr{A}$ such that $g \geqslant f$, $\omega(g) < \infty$, $g(t^{1/p})$ is a convex function and $g(t)/t^r$ is decreasing.*

Added in proof. Some problems discussed in Section III have been solved after this paper was submitted. Two facts should be mentioned.

If E is superreflexive and has local unconditional structure, then the converse of Kadec's theorem (stated for the Rademacher averages as in [7]) holds. It is no longer true for general superreflexive spaces.

The renorming result appears in the author's Exposé N° XXIV of the Séminaire Maurey-Schwartz 1974-1975; the corresponding example is due to G. Pisier and can be found in Annexe 2 (to the same publication). Let us remark that the techniques used in that exposé are based on final lemmas of the present paper (modified to allow vectors with not necessarily disjoint supports). The proofs will appear elsewhere.

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