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Geometric characterizations of the Radon–Nikodym property in Banach spaces

by

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Abstract. It is shown that a Banach space X has the Radon–Nikodym property (RNP) if and only if every closed bounded (not necessarily convex) subset of X has an extreme point. Other related characterizations of the RNP are also proved.

Introduction. A Banach space X is said to have the *Radon–Nikodym Property* (RNP) provided for every measure space $(\Omega, \Sigma, \lambda)$ with $\lambda(\Omega) < \infty$, and every λ-continuous measure $\mu\colon \Sigma \to X$ of finite variation, there exists a Bochner integrable function $f\colon \Omega \to X$ such that $\mu(E) = \int_E f d\lambda$ $(E \in \Sigma)$ (see the survey paper [5]). The space X is said to have the *Krein–Milman Property* (KMP) provided every closed bounded *convex* set in X has an extreme point

In the early 1940's, R. S. Phillips [13] showed that reflexive spaces have the RNP, and N. Dunford and B. J. Pettis [6] showed that separable dual spaces have the RNP. It is well known that c_0 and L^1 fail to have the RNP. Also, c_0 and L^1 fail to have KMP. Obviously reflexive spaces have the KMP, and in 1966 C. Bessaga and A. Pełczyński [1], extending work of J. Lindenstrauss [10], showed that separable dual spaces have the KMP. Based on these and other results, J. Diestel [4] in January 1973 posed the question of the relationship between the RNP and the KMP.

Recently, geometric characterizations of the RNP have been found (Theorems 2 and 3 below) which show that the RNP implies the KMP. In [9] it is shown that for dual spaces the KMP implies the RNP. In this paper we give further geometric characterizations of the RNP (Theorem 4). Specifically, if one defines X to have the *Strong Krein–Milman Property* (SKMP) provided every closed bounded (not necessarily convex) set in X has an extreme point (i.e., a point which is not a convex combination of other points in the set), then the SKMP and the RNP are equivalent. It remains an open question whether or not there exists a space with the KMP but not the SKMP.

1. A preliminary theorem on dentable sets. Throughout, let $(X, \|\cdot\|)$ denote a Banach space with continuous dual X^*. Let $\mathrm{co}(A)$ and $\overline{\mathrm{co}}(A)$ denote the convex hull and closed convex hull, respectively, for sets $A \subset X$. Let $B_r(x)$ denote the closed ball of radius $r > 0$ and center $x \in X$; let $B_r(A) = \bigcup\{B_r(x)\colon x \in A\}$ if $A \subset X$.

A subset A of X is said to be *dentable* if for every $r > 0$ there exists a point x in A such that $x \notin \overline{\mathrm{co}}(A \setminus B_r(x))$. M. A. Rieffel [14] introduced the notion of dentability and observed that A is dentable if $\overline{\mathrm{co}}(A)$ is dentable ([14], Prop. 2). An easy argument (see [3], Lemma 1) shows that a closed convex bounded set K is *not* dentable if and only if there exists $r > 0$ such that $K = \overline{\mathrm{co}}(K \setminus B_r(x))$ for every x in K. The separation theorem gives the following equivalent formulation [12]: K is not dentable if and only if there exists $r > 0$ such that for every continuous linear functional f and for every $a < \sup f(K)$ the set

$$S(f, a, K) = \{x \in K\colon f(x) \geqslant a\}$$

has diameter at least r. A set of the type $S(f, a, K)$ (with $f \in X^*$ and $a < \sup f(K)$) is called a *slice of* K.

An *r-net* for a set A is a set $N \subset A$ such that $A \subset B_r(N)$.

THEOREM 1. *If K is a closed bounded convex set in a Banach space X, then the following three statements are equivalent.*

(1) *K is not dentable.*

(2) *There exists $r > 0$ such that no slice of K has a finite r-net.*

(3) *There exists $r > 0$ such that $K = \overline{\mathrm{co}}(K \setminus B_r(x_1, \ldots, x_n))$ for every finite set $\{x_1, \ldots, x_n\} \subset K$.*

Proof. An application of the separation theorem easily shows that (2) implies (3); and it is clear that (3) implies (1). We prove that (1) implies (2).

Suppose that K is not dentable, and assume without loss of generality that K is contained in the unit ball of X. There exists $\delta > 0$ such that every slice of K has diameter larger than δ. Let $r = \frac{1}{3}\delta$.

Let f be in X^*, let $a < \sup f(K)$, let $S = S(f, a, K)$, and let

$$H = \{x \in K\colon f(x) = a\} \subset S.$$

Suppose there exists a finite r-net for S; i.e., suppose there exist points $x_1, \ldots, x_n$ in S such that $S \subset B_r(x_1, \ldots, x_n)$. Since $S \neq H = \overline{\mathrm{co}}(H)$, we may assume without loss of generality that for some $m \leqslant n$

$$S = \overline{\mathrm{co}}\big(H \cup (S \cap B_r(x_1, \ldots, x_m))\big)$$

but

$$S \neq K_1 = \overline{\mathrm{co}}\big(H \cup (S \cap B_r(x_2, \ldots, x_m))\big).$$

(If $m = 1$, take $K_1 = H$.) Let y_0 be in $[S \cap B_r(x_1)] \setminus K_1$ and choose g in X^* such that

$$c = \sup g(S) \geqslant g(y_0) > \sup g(K_1) = a.$$

Choose β such that $a < \beta < c$ and

$$\frac{\beta - a}{c - a} > 1 - \frac{\delta}{12}.$$

Let

$$S' = S(g, \beta, S) = \{x \in S\colon g(x) \geqslant \beta\}.$$

We show that the diameter of S' is at most δ. Let

$$L = \{x \in B_r(x_1) \cap S\colon g(x) \geqslant a\}$$

and

$$K_2 = \{x \in S\colon g(x) \leqslant a\}.$$

Note that $K_1 \subset K_2$ and $y_0 \in L$. Also,

$$L \cup K_2 \supset H \cup \big(S \cap (B_r(x_1, \ldots, x_m))\big),$$

so that

$$S = \overline{\mathrm{co}}(L \cup K_2).$$

Note that

$$[\mathrm{co}(L \cup K_2)] \cap S'$$

is dense in S'. (For, $\{x \in S\colon g(x) > \beta\}$ is dense in S' and is a relatively open subset of S. Since $\mathrm{co}(L \cup K_2)$ is dense in S, it is dense in this relatively open subset.) Let $\lambda x + (1-\lambda) y$ be in $[\mathrm{co}(L \cup K_2)] \cap S'$, where $0 \leqslant \lambda \leqslant 1$, $x \in L$ and $y \in K_2$. (Any element of $\mathrm{co}(L \cup K_2)$ must be of this form since L and K_2 are convex.) Then

$$\beta \leqslant g\big(\lambda x + (1-\lambda) y\big) \leqslant \lambda c + (1-\lambda) a = \lambda(c - a) + a,$$

so $\lambda \geqslant (\beta - a)/(c - a) > 1 - \frac{1}{12}\delta$, i.e., $(1-\lambda) < \frac{1}{12}\delta$. Let $\lambda' x' + (1-\lambda') y'$ be another point in $[\mathrm{co}(L \cup K_2)] \cap S'$ with $x' \in L$, $y' \in K_2$ and $0 \leqslant \lambda' \leqslant 1$. Again, $(1-\lambda') < \frac{\delta}{12}$, and we have

$$\begin{aligned}
\|[\lambda x + (1-\lambda) y] - [\lambda' x' + (1-\lambda') y']\| &\leqslant \|\lambda x - \lambda' x'\| + \|(1-\lambda) y\| + \|(1-\lambda') y'\| \\
&\leqslant \|x - (1-\lambda) x - x' + (1-\lambda') x'\| + \tfrac{\delta}{12} + \tfrac{\delta}{12} \\
&\leqslant \|x - x'\| + \tfrac{\delta}{12} + \tfrac{\delta}{12} + \tfrac{\delta}{12} + \tfrac{\delta}{12} \\
&\leqslant 2r + \tfrac{1}{3}\delta = \delta,
\end{aligned}$$

since K is in the unit ball of X and x, x' are in $B_r(x_1)$. Thus the diameter of S' is at most δ.

By the choice of δ, S' cannot be a slice of K. Clearly, $S' = S(g, \beta, S) \subset S(g, \beta, K)$, and so there must exist some z in $S(g, \beta, K) \setminus S'$. Then $g(z) \geqslant \beta$ and $z \notin S'$. Thus $z \notin S$ and so $f(z) < \alpha$. Let w be in S'. Then $f(w) > \alpha$ since $H \subset K_1$ and $K_1 \cap S' = \emptyset$. Hence there exists $0 < \mu < 1$ such that $\mu z + (1-\mu)w$ is in H. But $g(\mu z + (1-\mu)w) \geqslant \beta$ so $\mu z + (1-\mu)w$ is also in S', contradicting the fact that $S' \cap H$ is empty. This contradiction completes the proof.

The proof of the following corollary is only a slight modification of the proof of Lemma 2 in [3]. Let A° denote the interior of a set $A \subset X$.

COROLLARY. *Suppose that K is a closed convex non-dentable set with $K^\circ \neq \emptyset$. Then there exists* $r > 0$ such that

$$K^\circ = \mathrm{co}\big(K^\circ B_r \setminus (x_1, \ldots, x_n)\big)$$

for every finite set $\{x_1, \ldots, x_n)$ in K.

Proof. Choose $r > 0$ to satisfy (3) in Theorem 1. Let $\{x_1, \ldots, x_n\}$ be a finite set in K, and let $J = K \setminus B_r(x_1, \ldots, x_n)$. Then $K = \overline{\mathrm{co}}(J)$ and $J^\circ = K^\circ \setminus B_r(x_1, \ldots, x_n)$.

Note that $J \subset \overline{J^\circ}$. For, let y be any point in J. Then $y \in K$ and y is not in the closed set $B_r(x_1, \ldots, x_n)$. Let z be any point in K°. Then the half-open line segment $[z, y)$ is contained in K°, and for points w of $[z, y)$ sufficiently close to y, w is outside the closed set $B_r(x_1, \ldots, x_n)$. Hence y is the limit of points in $K^\circ \setminus B_r(x_1, \ldots, x_n) = J^\circ$.

It now follows that $\mathrm{co}\, J \subset \overline{\mathrm{co}}(J^\circ)$, and since the interior of a convex set (when non-empty) coincides with the interior of its closure,

$$K^\circ = (\overline{\mathrm{co}} J)^\circ = (\mathrm{co}\, J)^\circ \subset (\overline{\mathrm{co}}\, J^\circ)^\circ = \mathrm{co}\, J^\circ.$$

2. Geometric characterizations of the RNP. In 1967, M. A. Rieffel [14] showed that if every closed bounded subset of X is dentable, then X has the RNP. In 1972, H. Maynard gave a characterization of the RNP in terms of a modified version of dentability [11]. In 1973, W. J. Davis and R. R. Phelps [3] showed that Maynard's condition was actually equivalent to dentability, and independently R. Huff [8] showed that Maynard's proof could be modified so as to prove directly the converse of Rieffel's result. The papers above combine to give the following. (The equivalence of (3) with the others is in [3].)

THEOREM 2 ([14], [11], [3], [8]). *For a Banach space X, the following three statements are equivalent.*

(1) *X does not have the* RNP.

(2) *There exists a bounded non-dentable subset of X.*

(3) *There exists an equivalent norm for X such that the corresponding unit ball is not dentable.*

Using this characterization of the RNP, J. Lindenstrauss (see [12]) gave a simple, elegant proof that the RNP implies the KMP. (G. A. Edgar [7] has given an independent proof for separable spaces, actually establishing a Choquet theorem for separable closed, bounded convex subsets of spaces with the RNP.) In order to state a powerful extension of Lindenstrauss' result due to R. R. Phelps [12], we need a definition. If K is a closed convex set in X, a point x in K is said to be *strongly exposed* if there exists f in X^* such that $f(x) = \sup f(K)$ and such that

$$x_n \in K, \quad \lim_{n\to\infty} f(x_n) = f(x) \Rightarrow \lim_{n\to\infty} x_n = x.$$

As Rieffel observed, if x is a strongly exposed point of K, then x is a *denting point* of K, i.e., $x \notin \overline{\mathrm{co}}\big(K \setminus B_r(x)\big)$ for all $r > 0$. Trivially a strongly exposed point is an extreme point. Thus the following theorem extends the result that the RNP implies the KMP.

THEOREM 3 (R. R. Phelps [12]). *A Banach space X has the* RNP *if and only if every closed bounded convex subset of X is the closed convex hull of its strongly exposed points.*

We remark that if A is a closed subset of X, then any strongly exposed point of $\overline{\mathrm{co}}(A)$ must be in A since for any f in X^*, $\sup f\big(\overline{\mathrm{co}}(A)\big) = \sup f(A)$.

The main results of this paper are contained in the following.

THEOREM 4. *For a Banach space X, the following four statements are equivalent.*

(1) *X does not have the* RNP.

(2) *There exists a closed bounded set A in X which does not have an extreme point.*

(3) *There exist an equivalent norm $|||\cdot|||$ for X and a closed set A such that*

$$A \subset \{x \in X: |||x||| < 1\} \quad \text{and} \quad \overline{\mathrm{co}}(A) = \{x \in X: |||x||| \leqslant 1\}.$$

(4) *There exists a closed bounded set $A \subset X$ such that no non-trivial f in X^* attains its supremum on A.*

Proof. Trivially (3) implies (4). That (2) implies (1) and (4) implies (1) follows from the fact that if X has the RNP, then A must contain an exposed point for $\overline{\mathrm{co}}(A)$. It remains to prove that (1) implies (2) and (3), and these two implications will be proved simultaneously.

Suppose that X does not have the RNP. Then there exists a separable subspace Y which fails to have the RNP ([15], [11], and [8]). By Theorem 2 there exists an equivalent norm $|||\cdot|||_Y$ for Y such that $K = \{y \in Y: |||y|||_Y \leqslant 1\}$ is not dentable. By the corollary to Proposition 1, there exists $r > 0$ such that

$$K^\circ = \mathrm{co}\big(K^\circ \setminus B_r(x_1, \ldots, x_n)\big)$$

for every finite set $\{x_1, \ldots, x_n\}$ in K°. Let $\{y_n\}_{n=1}^{\infty}$ be a dense sequence in K°. Define a sequence of finite sets $F_1, F_2, \ldots \subset K^\circ$ by letting $F_1 = \{y_1\}$, and if $F_1, \ldots, F_{n-1}$ have been defined, let F_n be any finite subset of $K^\circ \setminus B_r(F_1 \cup \ldots \cup F_{n-1})$ such that $F_1 \cup \ldots \cup F_n \cup \{y_1, \ldots, y_n\} \subset \mathrm{co}(F_n)$. Let $F = \bigcup_{n=1}^{\infty} F_n$. Note that if $i < j$, then $F_i \subset \mathrm{co}(F_j)$ (so F has no extreme points). Clearly, if $i \neq j$ then

$$x \in F_i,\ y \in F_j \ \Rightarrow\ \|x-y\| > r,$$

and therefore F is closed and (2) is proved. Also, (3) is proved in the separable case since $\mathrm{co}(F)$ contains $\{y_n\}_1^\infty$ which is dense in K.

For each $n = 1, 2, \ldots$, let

$$G_n = F_n + \left\{x \in X:\ \|x\| \leqslant \left(1 - \frac{1}{n+1}\right)\frac{r}{6}\right\},$$

and let

$$A = \bigcup_{n=1}^{\infty} G_n.$$

Note that if $i < j$, then

$$G_i \subset \mathrm{co}(G_j)$$

and

$$x \in G_i,\ y \in G_j \ \Rightarrow\ \|x-y\| > \tfrac{2}{3} r.$$

(Clearly, A also has no extreme points.)

We next show that A is closed. Suppose that $\{x_n\}_{n=1}^{\infty}$ is in A and $\lim_{n\to\infty} x_n = x$. Then we must have that

$$\left(B_{\frac{r}{3}}(x)\right) \cap A$$

is entirely contained in some G_n. Since every G_n is closed, x must be in $G_n \subset A$.

Next, we claim that $\mathrm{co}(A)$ is dense in the set

$$B = K + \{x \in X:\ \|x\| < r/6\}.$$

For, let k be in K, let x be in X with $\|x\| < r/6$, and let $\varepsilon > 0$ be given. Choose an element y in $\mathrm{co}\left(\bigcup_{n=1}^{\infty} F_n\right)$ such that $\|k-y\| < \varepsilon$. Choose n so large that $\left(1 - \dfrac{1}{n+1}\right)\dfrac{r}{6} > \|x\|$ and $y \in \mathrm{co}(F_n)$. Then $(y+x)$ is in $\mathrm{co}(G_n)$ and $\|(k+x)-(y+x)\| = \|k-y\| < \varepsilon$.

Finally, let $|||\cdot|||$ be the gage of the set B.

Remarks. (i) Note that statement (2) in the theorem may be replaced by

(2′) *There exists a closed bounded set A in S such that $A \cap \mathrm{ext}(\overline{\mathrm{co}}(A)) = \emptyset$.*

(Here $\mathrm{ext}(K)$ denotes the set of extreme points of a set K.) The exactly opposite situation occurs in spaces with the RNP. For if X has the RNP, then for every closed bounded set A,

$$\overline{\mathrm{co}}\left(A \cap \mathrm{ext}(\overline{\mathrm{co}}(A)\right) \supset \overline{\mathrm{co}}\left(\text{strongly exposed points of } \overline{\mathrm{co}}(A)\right) = \overline{\mathrm{co}}(A).$$

(ii) Similarly, the exactly opposite situation from (4) occurs if X has the RNP. It is implicit in the proofs of Lemmas 6 and 7 of [12] that if X has the RNP and if K is a closed bounded convex subset of X, then the set

$$E(K) = \{f \in X^*:\ f \text{ strongly exposes some point of } K\}$$

is dense in X^*. [We remark also that the sets $U_n = \{f \in X^*:$ there exists a slice $S(f, \alpha, K)$ with diameter $< 1/n\}$ are open in X^*, and thus $E(K) = \bigcap U_n$ is a G_δ-subset of X^*.] If X has the RNP and if A is a closed bounded subset of X, then the set $\{f \in X^*:\ \sup f(K)$ is attained$\}$ contains the dense set $E(\overline{\mathrm{co}}(A))$. This should be compared with the Bishop–Phelps theorem [2]: for *any* Banach space X, if K is a closed bounded *convex* set, then the set $\{f \in X^*:\ \sup f(K)$ is attained$\}$ is dense in X^*.

(iii) Defining the Strong Krein–Milman Property (SKMP) as in the introduction, we have shown that the SKMP is equivalent to the RNP. It is well known [15] that the RNP is separably determined; i.e., X has the RNP if and only if every separable subspace of X has the RNP. Thus the SKMP is separably determined. It is an open question whether or not the KMP is separably determined.

(iv) In connection with property (3), note that if X is separable and if the unit ball of X is already non-dentable, then we do not have to renorm X. As is well known and easily proved, the unit ball in the space c of all convergent sequences (with norm $\|x\| = \sup_n |x_n|$) is not dentable and yet has many extreme points (in fact, it is the closed convex hull of its extreme points). Thus, in particular, there do exist closed bounded sets without extreme points, but whose closed convex hulls have extreme points. This indicates that if indeed the KMP and the SKMP are equivalent, the proof may be somewhat delicate.

(v) Finally, a concrete example of a set satisfying (3) may be of interest. The following example appears in [16], p. 9. Let the space c_0 of all sequences which converge to zero have its usual norm $\|x\| = \sup_n |x_n|$.

For every positive integer n, and every n-tuple $\sigma = (\varepsilon_1, \ldots, \varepsilon_n)$, where $\varepsilon_i = \pm 1$, let $x^{(n,\sigma)}$ denote the sequence whose ith term is given by

$$x_i^{(n,\sigma)} = \begin{cases} \dfrac{n}{n+1}\varepsilon_i & \text{if} \quad 1 \leqslant i \leqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

The norm of the difference of any two different such sequences is at least $\frac{2}{3}$, and thus the collection A of all such sequences is a closed set in the open unit ball. Since $\sup f(A) = \|f\|$ for every f in c_0^*, $\overline{\mathrm{co}}(A)$ is the closed unit ball of c_0.

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The distribution of energy in the Brownian motion in the Gaussian field and analytic-hypoellipticity of certain subelliptic operators on the Heisenberg group

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Abstract. Let a particle perform a Brownian motion $X(t)$ in $\boldsymbol{R}^n$ and $Y(t)$ be another Brownian motion in $\boldsymbol{R}^n$ independent of $X(t)$ interpreted as a random constant field of forces in $\boldsymbol{R}^n$. The joint distribution of X, Y, and the energy E produced by the motion is calculated by interpreting $(X(t), Y(t), E(t))$ as a process on the Heisenberg group G connected with a subelliptic operator on G.

In [4] and [2] a construction of a semigroup of functions p_t, $t > 0$, associated with a subelliptic operator on a Lie group was given. The aim of this note is to point out the role of the functions p_t in random walks on a Lie group which in turn may arise in a very elementary physical problem. The link of the p_t's to a subelliptic operator on the general Heisenberg group enables us to compute the p_t explicitly. This apart from an apparent physical interest shows that certain subelliptic operator on $\boldsymbol{R}^{2d+1}$ is analytic-hypoelliptic.

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1. Introduction. Let G be a connected Lie group, $U = U^{-1}$ an open neighbourhood of the identity in G such that the exponential map exp: $LG \to G$ is a local diffeomorphism onto U. Let $g_1, \ldots, g_k$ be a set of generators of G, i.e., the smallest analytic subgroup containing $g_1, \ldots, g_k$ is G, or, in other words, if $g_j = \exp X_j$, $j = 1, \ldots, k$, the smallest Lie subalgebra containing $X_1, \ldots, X_k$ is LG. Suppose that μ is a symmetric probability measure equally distributed on $g_1, \ldots, g_k, g_1^{-1}, \ldots, g_k^{-1}$ and let for $t > 0$ and $n = 1, 2, \ldots$ the probability measure $\mu_{t,n}$ of G be defined by

$$\mu_{t,n}(M) = s(2k)^{-1}, \quad \text{if } s \text{ of } \exp\left[\pm\sqrt{\frac{kt}{n}}\, X_j\right] \in M.$$

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