(iii) *Let A be a semisimple regular commutative Banach algebra, and let $x, a, y_1, y_2, \ldots$ in A satisfy $x = a^n y_n$ for $n = 1, 2, \ldots$ If $\|y_n\|^{1/n}$ does not tend to infinity as n tends to infinity, then there is an f in A such that $fx = x$.*

For let F be the support of x in the carrier space Φ of A. The remark will follow if we show that the closure F^- of F is compact ([10], Corollary 3.7.3; Theorem 3.6.13).

If ψ is in F, then

$$1 = \liminf |\psi(x)|^{1/n} \leqslant |\psi(a)| \cdot \liminf \|y_n\|^{1/n}.$$

Thus there is a $\delta > 0$ such that $|\psi(a)| \geqslant \delta$ for all ψ in F, and so for all ψ in F^-. Therefore F^- is compact, completing the proof of (iii).

(iv) Corollary 1 and Remark (iii) lead to the following question. If x in a (commutative) Banach algebra A may be written as $x = a^n y_n$ for all n and if $\|y_n\|^{1/n}$ does not tend to infinity is there an f in A such that $fx = x$?

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On the minimum time control problem and continuous families of convex sets

by

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Abstract. Let a linear system with time variable be given

$$(X \xrightarrow{A} \square \xrightarrow{B_t} Y),$$

where X, $\square$, Y are Banach spaces, A and B_t, $0 \leqslant t \leqslant T$, are continuous linear operators. Let U be a convex closed set in X containing 0 in its interior. Let $\|x\| = \inf\{t > 0: x/t \in U\}$ be the Minkowski norm generated by U. By $\varphi(t)$ we denote

$$\varphi(t) = \inf\{\|x\|: B_t Ax \in Y(t)\},$$

where $Y(t)$ is a given continuous family of closed sets.

We prove that if $B_t A(U)$ is a continuous family of sets at t_0 and the set $B_{t_0}A(U)$ has an interior, the $\varphi(t)$ is a continuous function at t_0.

By a *time control linear system* we shall understand a system of three Banach spaces over reals, X being called the *space of input*, $\square$ the *space of trajectories*, and Y the *space of output*, of a continuous linear operator A: $X \to \square$, called the *operator of input*, and of a family of continuous linear operators B_t: $\square \to Y$, t being real, $0 \leqslant t \leqslant T$.

Let U be a convex closed bounded set in X. Let $Y(t)$, $0 \leqslant t \leqslant T$, be a family of sets in Y. In the minimum time control problem we are looking for

$$T_0 = \inf\{t > 0: B_t A(U) \cap Y(t) \neq 0\} \tag{1}$$

and we ask when

$$B_{T_0} A(U) \cap Y(T_0) \neq 0. \tag{2}$$

The problem has been investigated in papers [2], [3], [4], where the respective sufficient conditions for (2) were given. Those conditions were only of the existence type.

In many problems which appear in the theory of control another, more effective, approach to the problem is used.

Namely, we assume that the set U has an interior. Of course, without loss of generality, we may assume that $0 \in \operatorname{Int} U$. Let $\|\ \|$ be the Minkowski norm induced by U:

(3) $$\|x\| = \inf\left\{s > 0:\ \frac{x}{s} \in U\right\}.$$

Let

(4) $$\varphi(t) = \inf\{\|x\|:\ B_t A x \in Y(t)\}$$

and let

(5) $$\tilde{T} = \inf\{t > 0:\ \varphi(t) \leqslant 1\}.$$

Of course, $\tilde{T} \geqslant T_0$. If we assume that $B_t A(U)$ are closed, then $\tilde{T} = T_0$ ([3], [4]).

The continuity of $\varphi(t)$ is important for the calculation of $\tilde{T}$.

In order to formulate the theorem giving sufficient conditions for the continuity of $\varphi(t)$ it is necessary to introduce the notion of continuous families of sets.

Let E be a linear topological space. Let $H(t)$, $0 \leqslant t \leqslant T$, be a family of subsets of E. We say that the family $H(t)$ is *continuous at a point* t_0 if for an arbitrary neighbourhood of zero V there is a positive number δ such that for all t, $|t - t_0| < \delta$, we have

(6) $$H(t) \subset H(t_0) + V$$

and

(7) $$H(t_0) \subset H(t) + V.$$

If only (6) holds, we say that the family $H(t)$ is *upper semicontinuous.*

THEOREM. *Let $Y(t)$ and $B_t A(U)$ be families of sets continuous at a point t_0. If $B_{t_0} A(U)$ has an interior, then $\varphi(t)$ is continuous at the point t_0.*

The theorem has been proved in [4] (Theorem V.5.1) under the additional assumption that there is an open interval $(t_0 - \delta, t_0 + \delta)$ such that $B_t A(U)$ has an interior for each t belonging to the interval.

In [3] I asked about the necessity of this hypothesis. The present note shows that it is not necessary.

The proof is based on the following lemmas:

LEMMA 1. *Let X be a Banach space over reals. Let $\Gamma \subset X$ be a closed convex set without interior. If $0 \in \Gamma$, then for an arbitrary $M > 0$, there is a continuous linear functional f^M such that*

(8) $$\|f^M\| \geqslant M$$

and

(9) $$\Gamma \subset \{x:\ f^M(x) \leqslant 1\}.$$

Proof. Let x_0 be an arbitrary point of Γ. Since Γ does not have an interior, there is a sequence of points $\{x_n\}$, $x_n \notin \Gamma$, tending to x_0 and such that $\|x_n - x_0\| \leqslant 1/n$.

By separation theorems there are continuous linear functionals f_n separating x_n and Γ, i.e., such that

(10) $$f_n(x_n) \geqslant 1$$

and

(11) $$f_n(x) \leqslant 1 \quad \text{for} \quad x \in \Gamma.$$

Now there are two possibilities:

1) *the sequence $\{f_n\}$ is unbounded.* Then, for each $M > 0$, there is an f_{n_M} such that $\|f_{n_M}\| > M$. Putting $f^M = f_{n_M}$, we get (9) from (11).

2) *the sequence $\{f_n\}$ is bounded*, i.e., there is a constant $K > 0$ such that $\|f_n\| \leqslant K$, $n = 1, 2, \ldots$ In the second case, by the Alaoglu theorem there is a cluster point f_0 of the sequence $\{f_n\}$ in the weak-$*$-topology. Obviously, $\|f_0\| \leqslant K$. By the definition of weak-$*$-topology we can find a subsequence $\{f_{n_k}\}$ such that $\lim\limits_{k\to\infty} f_{n_k}(x_0) = f_0(x_0)$.

Since f_0 is a cluster point of the sequence f_n in the weak-$*$-topology, $f_0(x) \leqslant \limsup\limits_n f_n(x)$ for all $x \in X$. Therefore

(12) $$f_0(x) \leqslant 1 \quad \text{for} \quad x \in \Gamma.$$

On the other hand (compare [5]),

(13) $$\begin{aligned}|f_0(x_0) - f_{n_k}(x_{n_k})| &\leqslant |f_0(x_0) - f_{n_k}(x_0)| + |f_{n_k}(x_0) - f_{n_k}(x_{n_k})| \\ &\leqslant |f_0(x_0) - f_{n_k}(x_0)| + K\|x_{n_k} - x_0\| \to 0.\end{aligned}$$

Thus

(14) $$f_0(x_0) \geqslant \liminf_{k\to\infty} f_{n_k}(x_{n_k}) \geqslant 1.$$

Since $x_0 \in \Gamma$, by (13) and (14) we get

(15) $$f_0(x_0) = 1.$$

Suppose now that for each $x_0 \in \Gamma$ there is a continuous linear functional $f_{(x_0)}$ satisfying (12) and (15).

If

(16) $$\sup_{x_0 \in \Gamma} \|f_{(x_0)}\| = +\infty,$$

then it is easy to choose a linear continuous functional f^M, $\|f^M\| > M$, such that (9) holds.

If (16) does not hold, then there is a constant $C > 0$ such that

(17) $$\|f_{(x_0)}\| \leqslant C \quad \text{for all } x_0 \in \Gamma.$$

Formula (17) implies

$$|f_{(x_0)}(x_0)| \leqslant C\|x_0\|, \tag{18}$$

and we get a contradiction of formula (15) because $0 \in \Gamma$.

In the case where X is finite dimensional, Lemma 1 can be formulated in a stronger way. Namely, if Γ is a convex set without interior, then there is a continuous linear functional f such that

$$\Gamma \subset \{x: f(x) = 0\}. \tag{19}$$

For infinite-dimensional spaces, there are Γ such that there is no linear continuous functional satisfying (19). What is more, there are bounded convex sets Γ without interior such that for every continuous linear functional f we have

$$\sup_{x\in\Gamma} f(x) - \inf_{x\in\Gamma} f(x) \geqslant \|f\|, \tag{20}$$

as follows from the following example:

EXAMPLE. Let $X = L^2[0, 1]$. Let

$$\Gamma = \{x \in L^2[0,1]: \|x\| \leqslant 1,\ x(t) \geqslant 0\}.$$

Let f be an arbitrary continuous linear functional defined on X. Let $f(t)$ be the function belonging to $L^2[0,1]$ corresponding to f. Let

$$f_+(t) = \frac{f(t)+|f(t)|}{2}, \quad f_-(t) = \frac{f(t)-|f(t)|}{2}.$$

The linear continuous functional corresponding to $f_+(t)$ and $f_-(t)$ will be denoted by f_+ and f_-, respectively.

It is easy to verify that

$$f = f_+ + f_-, \tag{21}$$

$$f_+(t) \geqslant 0, \quad f_-(t) \leqslant 0, \tag{22}$$

$$\|f\|^2 = \|f_+\|^2 + \|f_-\|^2. \tag{23}$$

Let

$$x_+ = \begin{cases} f_+/\|f_+\| & \text{if } f_+ \neq 0, \\ 0 & \text{if } f_+ = 0, \end{cases} \tag{24}$$

$$x_- = \begin{cases} f_-/\|f_-\| & \text{if } f_- \neq 0, \\ 0 & \text{if } f_- = 0. \end{cases} \tag{25}$$

Of course, $x_+, x_- \in \Gamma$. This implies

$$\sup_{x\in\Gamma} f(x) \geqslant f(x_+) = f_+(x_+) = \|f_+\| \tag{26}$$

and

$$\inf_{x\in\Gamma} f(x) \leqslant f(x_-) = f_-(x_-) = -\|f_-\|. \tag{27}$$

Hence

$$\sup_{x\in\Gamma} f(x) - \inf_{x\in\Gamma} f(x) \geqslant \|f_+\| + \|f_-\| \geqslant \|f\|. \tag{28}$$

LEMMA 2. *Let X be a Banach space over reals. Let $K_r = \{x: \|x\| < r\}$ be the ball of radius r with the centre at 0. Let Γ be a closed convex set without interior. Then*

$$K_r \not\subset \Gamma + K_{r/4}. \tag{29}$$

Proof. We shall consider two cases:

Case I. $\inf\{\|x\|: x \in \Gamma\} \geqslant r/3$.

In this case, of course, $0 \notin \Gamma + K_{r/4}$ and (29) trivially holds.

Case II. $\inf\{\|x\|: x \in \Gamma\} < r/3$.

Let x_0 be an arbitrary point of Γ such that $\|x_0\| < r/3$. We shall write

$$\tilde{\Gamma} = \Gamma - x_0, \tag{30}$$

$$\tilde{K}_r = K_r - x_0. \tag{31}$$

Since $\|x_0\| < r/3$, $\tilde{K}_r \supset K_{2r/3}$. The set $\tilde{\Gamma}$ contains 0 and does not have an interior. Thus by Lemma 1 there is a continuous linear functional f such that

$$\|f\| > 3/r \tag{32}$$

and (9) holds. Since (32),

$$K_{2r/3} \not\subset \{x: f(x) \leqslant 1\} + K_{r/4}. \tag{33}$$

Therefore

$$\tilde{K}_r \not\subset \Gamma + K_{r/4}. \tag{34}$$

Formula (34) trivially implies (29).

LEMMA 3. *Let X be a Banach space over reals. Let A be an arbitrary subset of X with a non-empty interior. Then there is a number $r > 0$ such that for an arbitrary closed convex set Γ without interior, the set A is not contained in the set $\Gamma + K_{r/4}$.*

Proof. Since the set A has an interior, A contains a ball $K_r(x_0)$ of radius r with centre at x_0. Let $\tilde{A} = A - x_0$. Then $K_r \subset \tilde{A}$. Now applying Lemma 2, we trivially get Lemma 3.

LEMMA 4. *Let X be a Banach space over reals. Let Γ_t, $0 \leqslant t \leqslant T$, be a family of closed convex sets continuous at the point t_0. Suppose that Γ_{t_0} has an interior. Then there is a positive number δ such that for t, $|t - t_0| < \delta$, the set Γ_t has an interior.*

Proof. Since the set Γ_{t_0} has an interior, by Lemma 3 there is an $r > 0$ such that Γ_{t_0} is not contained in $\Gamma + K_{r/4}$ for any closed convex set Γ without interior.

On the other hand, the family Γ_t is continuous. Thus there is a positive number δ such that for t, $|t-t_0| < \delta$,

$$\Gamma_{t_0} \subset \Gamma_t + K_{r/4}. \tag{35}$$

Formula (35) implies that Γ_t has an interior.

LEMMA 5. *Let X be a Banach space over reals. Let Γ_t, $0 \leqslant t \leqslant T$, be a family of convex sets continuous at the point t_0. If the set Γ_{t_0}, has an interior, then there is a $\delta > 0$ such that for t, $|t-t_0| < \delta$, the sets $\overline{\Gamma}_t$ have an interior.*

Proof. It is an obvious consequence of the application of Lemma 4 to the family $\overline{\Gamma}_t$.

Of course, in the general case it is impossible to replace Lemma 5 by the assertion that Γ_t have an interior. As a counterexample we take as Γ_{t_0} the ball of radius r and as Γ_t, $t \neq t_0$, an arbitrary convex dense set without interior fixed for all $t \neq t_0$.

Fortunately, in the minimum time control problem we consider the family $\Gamma_t = B_t A(U)$ and for this family we have

LEMMA 6 ([1]). *The set $\Gamma_t = B_t A(U)$ either has an interior or is nowhere dense.*

Proof. Without loss of generality we may assume that $0 \in \operatorname{Int} U$. Using the second part of the proof of the Banach theorem on open maps, we get Lemma 6.

Of course, without the assumption that U is bounded Lemma 6 is not true. In fact, let $X = Y = c_0$. Let $U = X$ and let $B_t A(\{x_n\}) = \{n^{-t} x_n\}$. It is easy to verify that, for $t > 0$, $B_t AX$ is a dense subset of X different from the whole X and, for $t = 0$, $B_t A(U) = X$.

Proof of the theorem. By Lemma 6 and Lemma 5 there is a $\delta > 0$ such that for t, $|t-t_0| < \delta$, the set $\Gamma_t = B_t A(U)$ has an interior. Thus for arbitrary $\varepsilon > 0$

$$\overline{B_t A(U)} \subset \operatorname{Int}(1+\varepsilon) B_t A(U). \tag{36}$$

We fix now an arbitrary positive ε. Since the family $B_t A(U)$ is continuous, there is a δ_1, $0 < \delta_1 < \delta$, such that for t, $|t-t_0| < \delta_1$,

$$B_{t_0} A(U) \subset \operatorname{Int}(1+\varepsilon) B_t A(U). \tag{37}$$

The definition of $\varphi(t)$ implies

$$\overline{Y(t_0)} \cap \overline{\varphi(t_0) B_{t_0} A(U)} \neq 0. \tag{38}$$

Hence by (36)

$$Y(t_0) \cap \operatorname{Int}(1+\varepsilon)\varphi(t_0) B_t A(U) \neq 0. \tag{39}$$

Since $Y(t)$ is a continuous family of sets, there is a δ_2, $0 < \delta_2 < \delta_1$, such that for t, $|t-t_0| < \delta_2$,

$$Y(t) \cap \operatorname{Int}(1+\varepsilon)\varphi(t_0) B_t A(U) \neq 0. \tag{40}$$

Thus by the definition of $\varphi(t)$ we get

$$\varphi(t) \leqslant (1+\varepsilon)\varphi(t_0). \tag{41}$$

Exchanging the roles of t and t_0, we find that there is a $\delta_3 > 0$ such that, for t, $|t-t_0| < \delta_3$,

$$\varphi(t_0) \leqslant (1+\varepsilon)\varphi(t). \tag{42}$$

Formulas (41) and (42) imply the continuity of $\varphi(t)$.

We have proved Lemmas 1–5 for Banach spaces over reals, but the proofs can easily be extended to locally convex topological spaces either over reals or over complexes.

We do not know whether Lemmas 2–4 are valid for non-locally convex spaces?

There is also another interesting question. Without the assumption of the convexity of Γ_t Lemma 4 does not hold even in the finite-dimensional case. The examples are trivial.

We say that a set A contained in a linear space is *p-convex*, $0 < p \leqslant 1$, if x, $y \in A$ implies $tx + sy \in A$ for each s, $t > 0$ such that $s^p + t^p = 1$. Is Lemma 4 true if we replace the convexity of Γ_t by p-convexity?

It is so for finite-dimensional spaces.

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