Proof. Expanding the exponential in power series and using Theorem 2 we have

$$\int_{\mathbf{R}^n} |Kf|^p e^{\lambda|Kf|} w dx = \sum_0^\infty \frac{\lambda^k}{k!} \int_{\mathbf{R}^n} |Kf|^{p+k} w\, dx$$

$$\leqslant \sum_0^\infty \frac{\lambda_k}{k!} c_1^{p+k}(p+k)^{p+k} \int_{\mathbf{R}^n} |f|^{p+k} w\, dx$$

$$\leqslant \Big[\sum_0^\infty \frac{\lambda^k}{k!} c_1^{p+k}(p+k)^{p+k} \|f\|_\infty^k \int_{\mathbf{R}^n} |f|^p w\, dx,$$

and using the ratio test and the fact that

$$\lim_{k\to\infty} \frac{1}{k} \frac{(p+k)^{p+k}}{(p+k-1)^{p+k-1}} = c,$$

we find that the series converges for $\lambda c_1 \|f\|_\infty c < 1$, which proves our assertion.

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Maximal smoothing operators and some Orlicz classes

by

C. P. CALDERÓN* and J. LEWIS* (Chicago, Ill.)

Abstract. The paper gives a characterization of the Orlicz classes of functions that are "near" $L_\alpha^{n/\alpha}(\mathbf{R}^n)$, $0 < \alpha < n$, for which the functions belonging to them have the property of possessing total differential of order α at almost all the points of $\mathbf{R}^n$. When α is not an integer, the finiteness of $M_\alpha^*(f)$ replaces the existence of the α-differential (see [5]).

0. Introduction, notation and definitions. In an earlier joint paper [5], one of the authors studied the differential properties of functions belonging to classes $L_\alpha^p(\mathbf{R}^n)$, $0<\alpha<n$, $p>n/\alpha$. The purpose of this paper is to extend those results to Orlicz classes of functions that are "near" $L_\alpha^{n/\alpha}(\mathbf{R}^n)$, $0<\alpha<n$. More precisely, we characterize those Orlicz classes that are "near" $L_\alpha^{n/\alpha}(\mathbf{R}^n)$, for which the functions belonging to them possess total differential of order α at almost all the points of $\mathbf{R}^n$. If α is not an integer, we replace the existence of the α-differential by the finiteness of $M_\alpha^*(f)$; see [5] or definition below.

Earlier results in this direction are due to A. P. Calderón [4] when $\alpha = 1$. Positive results go back to W. Stepanov [11]; see also [6], [7] and [9].

Throughout this paper we keep the notation and constructions used in [5] and our method is partially borrowed from [4] and [5].

Almost all the lemmas in this paper use results in [10] and [12], and we shall refer to them systematically.

0.1. Let $\psi(t)$ be a non-decreasing function of the variable $t \geqslant 0$, continuous and such that $\psi(0) = 0$. We say that $\psi(t)$ is *near* t^c if the following conditions are met:

(i) $\psi(t) = t^c\varphi(t)$, $t>0$ and $\varphi(t)>0$.

(ii) *$\varphi(t)$ is slowly varying, that is, for each positive δ, there exists a number $N>0$ such that for $t>N$, $\varphi(t)t^\delta$ is increasing, while $\varphi(t)t^{-\delta}$ is decreasing.*

* University of Illinois at Chicago Circle.

(iii) *There exists* $\eta > 0$ *such that if* $0 < t < \eta$, $\varphi(t) = K$, *where* $K > 0$.

0.2. $L^{\Psi}(\boldsymbol{R}^n)$ will denote the *class of measurable functions in* $\boldsymbol{R}^n$ *for which* $\Psi(|f|)$ *belongs to* $L(\boldsymbol{R}^n)$.

Here Ψ is near t^c for some $c \geqslant 1$. (See [12], p. 16.)

0.3. $L^{\Psi}_{\alpha}(\boldsymbol{R}^n)$ will denote the *class of functions that are represented by*

$$f(x) = \int_{\boldsymbol{R}^n} G_\alpha(x-y)g(y)\,dy, \quad \alpha > 0,$$

where $g \in L^{\Psi}(\boldsymbol{R}^n)$, $G_\alpha(x)$ denotes the Bessel kernel of order α (see [10], p. 132). (The Fourier transform of $G_\alpha(x)$ is $(1+|x|^2)^{-\alpha/2}$.)

0.4. Let h be a vector in $\boldsymbol{R}^n$; then $\Delta_h f(x) = f(x+h) - f(x)$ and $\Delta_h^{(k)} f(x) = \Delta_h \Delta_h^{(k-1)} f(x)$.

0.5. Let $f(x)$ be a function mapping $\boldsymbol{R}^n$ into $\boldsymbol{R}$. We say that f has a *total differential of order* k *at* x_0, if there exists a homogeneous polynomial $P(x)$ of degree k, $P(x)$: $\boldsymbol{R}^n \to \boldsymbol{R}$, such that

$$\lim_{|h|\to 0} \frac{1}{|h|^k} |\Delta_h^{(k)} f(x_0) - P(h)| = 0.$$

0.6. Let k be the smallest integer larger than or equal to $\alpha > 0$. Then $M^*_\alpha(f)(x)$ will denote the following supremum:

$$\sup_{h,|h|>0} \frac{|\Delta_h^k f(x)|}{|h|^\alpha},$$

where h takes all the values in $\boldsymbol{R}^n - \{0\}$ and f: $\boldsymbol{R}^n \to \boldsymbol{R}$.

1. Auxiliary lemmas. The following lemma is a version of Theorem 4.22 in [12] (see p. 116, vol. II).

1.1. Lemma a. *Let* T *be a sublinear operation mapping measurable functions in* $\boldsymbol{R}^n$ *into measurable functions in* $\boldsymbol{R}^n$ *and such that*

$$\big|E(|T(f)| > \lambda)\big| \leqslant \frac{C_i}{\lambda^{p_i}} \int_{\boldsymbol{R}^n} |f|^{p_i}\,dx, \quad i = 1, 2,$$

and $1 \leqslant p_1 < p_2 < \infty$, C_i *does not depend on* f. *Then if* ψ *is near* t^p *with* $p_1 < p < p_2$ *we have*

$$\int_{\boldsymbol{R}^n} \psi(|T(f)|)\,dx \leqslant C \int_{\boldsymbol{R}^n} \psi(|f|)\,dx.$$

The constant C *depends on* ψ, p_1 *and* p_2 *but not on* f.

Proof. The proof follows the pattern of that of Theorem 4.22 in [12] and the transition to infinite measure space relies on the following three easy to check inequalities.

Let us write $\psi(t) = t^p\varphi(t)$, $\varphi(t)$ slowly varying and constant in a neighborhood of the origin. Then

$$\int_a^\infty t^{p-1-r}\varphi(t)\,dt \leqslant K_1\varphi(a)\,a^{p-r}, \quad r > p, \tag{1.1.1}$$

$$\int_0^a t^{p-1-r}\varphi(t)\,dt \leqslant K_2\varphi(a)\,a^{p-r}, \quad r < p, \tag{1.1.2}$$

$$\int_0^\infty D_{\psi(|f|)}(\lambda)\,d\lambda < K_3 \int_0^\infty D_{|f|}(\lambda)\,\lambda^{p-1}\varphi(\lambda)\,d\lambda. \tag{1.1.3}$$

Here, $D_g(\lambda)$, $g \geqslant 0$, stands for the distribution function of g, K_1 and K_2 do not depend on a, and K_3 does not depend on f.

1.2. Remark. The Sobolev space $S^{\psi}_k(\boldsymbol{R}^n)$, where k is an integer larger than or equal to one, and ψ is near t^p, where $p > 1$, is the space of functions f such that

$$\int_{\boldsymbol{R}^n} \Psi(|D^\beta f|)\,dx \leqslant \infty$$

for $0 \leqslant |\beta| \leqslant k$. Here, the derivatives have been taken in the distribution sense and $D^0 f = f$.

We have the identity $S^{\psi}_k(\boldsymbol{R}^n) = L^{\psi}_k(\boldsymbol{R}^n)$; indeed, in the case when $\Psi(t)$ is near t^p, $p > 1$, k is an integer $\geqslant 1$, Lemma a yields that the proof of Theorem 3, Chapter V in [10] could, be carried out without change if norms are replaced by integral expressions of the form

$$\int_{\boldsymbol{R}^n} \psi(|D^\beta f|)\,dx.$$

1.3. Lemma b. *If* $\psi(t)$ *is near* t^p *and* $p > 1$, *then there exists a convex function* $\psi(t)$ *such that*

$$0 < M_0 \leqslant \frac{\psi(t)}{\psi(t)} \leqslant M_1 \quad \textit{for} \quad t > 0.$$

Proof. Write $\psi(t) = t^p\varphi(t)$ and define $g(t)$ in the following way

$$\begin{aligned} g(t) &= t^{p-1}\varphi(L) && \text{if} \quad 0 < t \leqslant L, \\ g(t) &= t^{p-1}\varphi(t) && \text{if} \quad t > L. \end{aligned} \tag{1.3.1}$$

L has been chosen, so that $t^{p-1}\varphi(t)$ is increasing for $t \geqslant L$. Now we define $\psi(t)$ in the following way:

$$\psi(t) = \int_0^t g(s)\,ds. \tag{1.3.2}$$

So, near the origin we have $\psi(t) \sim ct^p$. For $t > 4L$ we have

$$(1.3.3)\qquad \int_0^t g(s)\,ds \geqslant \int_{t/2}^t s^{p-1}\varphi(s)\,ds \geqslant C\left(\frac{t}{2}\right)^p \varphi\left(\frac{t}{2}\right)$$

and on the other hand, $\int_0^t g(s)\,ds \leqslant t^p\varphi(t)$. This finishes the proof.

1.4. Remark. Lemma b shows that $L^\psi(\boldsymbol{R}^n)$ is equivalent to $L^{\bar\psi}(\boldsymbol{R}^n)$.

1.5. Lemma c. *Let a be such that $0 < a < n$. Suppose that Ψ is near $t^{n/a}$. Let $\bar\psi(t)$ be any convex function equivalent to ψ in the sense of Lemma* b *Let us denote by $\theta(t)$ a conjugate of $\bar\psi(t)$ in the Orlicz sense. Then*

$$I_1 = \int_0^1 \theta\left(\frac{1}{r^{n-a}}\right) r^{n-1}\,dr$$

is finite if and only if

$$I_2 = \int_1^\infty \left(\frac{t}{\psi(t)}\right)^{a/(n-a)} dt$$

is finite.

Proof. According to Lemma b it is enough to consider $\bar\psi(t)$ in I_2 instead of $\psi(t)$.

On account of the construction of $\bar\psi(t)$ we see that $\bar\psi'(t)$ behaves as $t^{-1}\bar\psi(t)$ for large values of t. Therefore I_2 is equivalent to

$$(1.5.1)\qquad \int_1^\infty \left[\frac{1}{\bar\psi'(t)}\right]^{a/(n-a)} dt.$$

By introducing the change of variables $\bar\psi'(t) = s$ in (1.5.1) and taking into account that θ' and $\bar\psi'$ are the inverse of each other, out integral becomes

$$(1.5.2)\qquad \int_{s_0}^\infty \left[\frac{1}{s}\right]^{a/(n-a)} d\theta'(s),$$

where the integral should be understood in the Stieltjes sense. (1.5.2) behaves as

$$(1.5.3)\qquad \sum_{k>1}\left[\frac{1}{2^k}\right]^{a/(n-a)} [\theta'(2^{k+1}) - \theta'(2^k)].$$

Call $n/a = c+1$; according to Lemma b, $\bar\psi'(t) = t^c\eta(t)$ with $\eta(t)$ slowly varying.

Let $\{t_k\}$ be a sequence defined by the equations

$$(1.5.4)\qquad 2^k = t_k^c\eta(t_k), \quad k = 1, 2, \ldots,$$

and let

$$(1.5.5)\qquad \frac{1}{2} = \frac{t_k^{c+\varepsilon}}{t_{k+1}^{c+\varepsilon}}\cdot\frac{\eta(t_k)t_k^{-\varepsilon}}{\eta(t_{k+1})t_{k+1}^{-\varepsilon}},$$

where $\varepsilon > 0$ and small. For t large enough $\eta(t)t^{-\varepsilon}$ is decreasing, thus

$$(1.5.6)\qquad \left(\frac{t_k}{t_{k+1}}\right)^{c+\varepsilon} < \frac{1}{2} \quad \text{if} \quad k > k_0.$$

Consequently,

$$(1.5.7)\qquad \frac{\theta'(2^k)}{\theta'(2^{k+1})} < C_0 < 1 \quad \text{if} \quad k > k_0.$$

This inequality gives the fact that (1.4.3) behaves as

$$(1.5.8)\qquad \sum_{k\geqslant 1}\left[\frac{1}{2^k}\right]^{a/(n-a)} \theta'(2^{k+1}) \quad \text{or} \quad \sum_k \left[\frac{1}{2^k}\right]^{a/(n-a)} 2^{-k}\theta(2^k).$$

The second series (1.5.8) behaves as

$$(1.5.9)\qquad \int_1^\infty \left[\frac{1}{t}\right]^{a/(n-a)} \frac{1}{t^2}\theta(t)\,dt.$$

Setting $t = 1/r^{n-a}$ in the above integral we get

$$(1.5.10)\qquad c\int_0^1 \theta\left(\frac{1}{r^{n-a}}\right) r^{n-1}\,dr.$$

This finishes the proof.

1.6. Lemma d. *Let $\psi(t)$ be near $t^{n/a}$ and suppose that*

$$\int_1^\infty \left[\frac{t}{\psi(t)}\right]^{a/(n-a)} dt$$

is divergent. Then, there exists a function $g(r)$ non-negative, non-increasing and supported in the interval $[0,1]$ such that:

(i) $\int_0^1 \psi(g(r))r^{n-1}\,dr < \infty$;

(ii) $\int_0^1 \frac{1}{r^{n-a}} g(r)r^{n-1}\,dr = \infty$.

Proof. On account of the property $\psi(2t) \leqslant C\psi(t)$ and $\bar\psi(2t) \leqslant \bar C\bar\psi(t)$ (where $\bar\psi$ is the function of Lemma b) we have

$$\sup_{f\in\mathfrak{F}}\left|\int_0^1 \frac{1}{r^{n-a}} f(r)r^{n-1}\,dr\right| = \infty,$$

where $\mathfrak{F} = \{f;\ \int_0^1 \bar\psi(|f|)r^{n-1}\,dr \leqslant 1\}$.

See [12] (p. 170: 10.1, Theorem 10.4, 10.8, and line 27 on page 175) and Lemma c.

On account of the particular shape of $1/r^{n/\alpha}$, there exists a denumerable family $f_m(r)$ of functions in $\mathfrak{F}$ such that

$$(1.6.1)\qquad f_m(r) \geqslant 0, \quad \text{non-increasing and supported on } [0,1],$$

$$(1.6.2)\qquad \int_0^1 \frac{1}{r^{n-\alpha}} f_m(r) r^{n-1} dr \to \infty \quad \text{as} \quad m \to \infty,$$

$$(1.6.3)\qquad f_m(r) \leqslant A_r \quad \text{for} \quad r > 0.$$

On account of conditions (1.6.1) and (1.6.3), there exists a subsequence $f_{m_j}(r)$ converging to a non-increasing function $f(r)$ except in a set at most denumerable.

Now it is easy to verify that if we take $g(r)$ to be $f(r)$ conditions (i) and (ii) in the thesis are satisfied. This finishes the proof.

2. Statement of the main results.

2.1. THEOREM A. *If ψ is near $t^{n/\alpha}$, $0 < \alpha < n$, and*

$$\int_1^\infty \left[\frac{t}{\psi(t)}\right]^{\alpha/(n-\alpha)} dt = \infty,$$

then, there exists a function $f \in L_\alpha^\psi(\boldsymbol{R}^n)$ such that

(i) $\overline{\lim}_{|h|\to 0} |f(x+h) - f(x)| = \infty$ *for almost every x in $\boldsymbol{R}^n$. In particular, $f(x)$ fails to have total differential of any order at almost all the points in $\boldsymbol{R}^n$.*

Proof. Consider the function $g(r)$ of Lemma d and let us construct the following function $f(x)$:

$$(2.1.1)\qquad f(x) = \sum_1^\infty c_k g(|x - x_k|),$$

where $c_k > 0$ and $\sum_1^\infty c_k = 1$. The set $\{x_k\}$ is a denumerable set dense in $\boldsymbol{R}^n$. Define now the following function $F(x)$:

$$(2.1.2)\qquad F(x) = \int_{\boldsymbol{R}^n} G_\alpha(x-y) f(y)\, dy;$$

here G_α is the Bessel kernel of order α and $f(x)$ is the function (2.1.1).

Now, we are going to show that $F(x)$ has the desired properties.

In the first place $f \in L_\alpha^\psi(\boldsymbol{R}^n)$; this follows from Lemma d and the fact that ψ satisfies the conditions of Lemma b.

On the other hand, given x and a neighborhood E of x, it is possible to find for each $N > 0$ a point s belonging to E such that

$$(2.1.3)\qquad F(s) > N.$$

In fact, let x_{k_0} be a point of $\{x_k\}$ and $x_{k_0} \in E$. On account of the fact that $G_\alpha(y) > 0$ for all y we have

$$(2.1.4)\qquad F(s) > c_{k_0} \int_{\boldsymbol{R}^n} G_\alpha(s-y) g(|y - x_{k_0}|)\, dy.$$

On the other hand,

$$(2.1.5)\qquad G_\alpha(s-y) > C_\alpha \frac{1}{|s-y|^{n-\alpha}}, \quad |s-y| < B_0,$$

see [10], p. 132.

By selecting s near x_{k_0} we have

$$(2.1.6)\qquad F(s) > C_\alpha C_{k_0} \int_{\boldsymbol{R}^n} \frac{1}{|s-y|^{n-\alpha}} g(|y - x_{k_0}|)\, dy.$$

On account of Lemma d, if s is close enough to x_{k_0}, the expression on the right-hand side of (2.1.6) can be made larger than N.

On the other hand, since $G_\alpha \in L^1(\boldsymbol{R}^n)$, $0 < \alpha$, $F(x)$ is finite a.e. Therefore, for a.e. x the difference $|F(x) - F(s)|$ can be made arbitrarily large for points s in each neighborhood of x.

This finishes the proof.

2.2. COROLLARY. *There exists a function f belonging to $L_\alpha^{n/\alpha}(\boldsymbol{R}^n)$, $0 < \alpha < n$, that fails to have total differential of any order at almost all the points of $\boldsymbol{R}^n$.*

Proof. Let $\psi(t) = t^{n/\alpha}$; then

$$\int_1^\infty \left[\frac{t}{t^{n/\alpha}}\right]^{\alpha/(n-\alpha)} dt = \int_1^\infty \frac{dt}{t} = \infty.$$

More generally, we have

2.3. COROLLARY. *Suppose that $\psi(t)$ has the form $t^{n/\alpha}\varphi(t)^{s(n-\alpha)/\alpha}$, $0 < \alpha < n$, $0 \leqslant s \leqslant 1$, and $\varphi(t)$ has any of the forms:*

$$1 + \log^+ t, \quad 1 + (\log^+ t)(\log^+\log^+ t),$$
$$1 + (\log^+ t)(\log^+\log^+ t)(\log^+\log^+\log^+ t), \ \ldots$$

Then there exists a function f belonging to $L_\alpha^\psi(\boldsymbol{R}^n)$ that fails to have total differential of any order at almost all the points of $\boldsymbol{R}^n$, in particular, for that function $M_\alpha^*(f)$ is infinite a.e..

2.4. THEOREM B. *Let f belong to $L_\alpha^\psi(\boldsymbol{R}^n)$, $0<\alpha<n$, where ψ is near $t^{n/\alpha}$ and satisfies*

$$\int_1^\infty \left[\frac{t}{\psi(t)}\right]^{\alpha/(n-\alpha)} dt < \infty.$$

Then

(i) $\left|E\left(M_\alpha^*(f)>\lambda\right)\right| < \left(\frac{C_1}{\lambda^{n/\alpha}} + \frac{C_2}{\lambda^{n/\alpha+\varepsilon}} + \frac{C_3}{\psi(\lambda)}\right)\int_{\boldsymbol{R}} \psi(|g|)\,dy$ *where* $f = G_\alpha * g$ *and the constants C_1, C_2 and C_3 do not depend on f.*

(ii) *If α is an integer, then f has total differential of order α at almost all the points in $\boldsymbol{R}^n$.*

Proof. Consider $\Delta_h^{(k)} f(x)$ and its integral expression as a Bessel potential:

$$\int_{\boldsymbol{R}^n} \Delta_h^{(k)} G_\alpha(x-y)\,g(y)\,dy. \tag{2.4.1}$$

Here, k is the smallest integer bigger than or equal to α. Now let L be an integer larger than $3k$ and let us split (2.4.1) in the following way:

$$\int_{|x-y|<L|h|} \Delta_h^{(k)} G_\alpha(x-y)\,g(y)\,dy + \int_{|x-y|>L|h|} \Delta_h^{(k)} G_\alpha(x-y)\,g(y)\,dy. \tag{2.4.2}$$

Let $g_\alpha^*(x)$ be

$$\sup_{h\in\boldsymbol{R}^n, |h|>0} \frac{1}{|h|^\alpha}\left|\int_{|x-y|>L|h|} \Delta_h^{(k)} G_\alpha(x-y)\,g(y)\,dy\right|. \tag{2.4.3}$$

$g_\alpha^*(x)$ satisfies

$$\|g_\alpha^*\|_p \leqslant C_p\|g\|_p, \qquad 1<p<\infty \tag{2.4.4}$$

(see proof of Theorem I in [5]). On account of Lemma a we have

$$\int_{\boldsymbol{R}^n} \psi(g_\alpha^*)\,dx \leqslant C\int_{\boldsymbol{R}^n} \psi(|g|)\,dx. \tag{2.4.5}$$

Now, on account of the fact that ψ is non-decreasing, and increasing in a neighborhood of the origin and in a neighborhood of infinity; we have

$$|E(g_\alpha^*>\lambda)| < \frac{C}{\psi(\lambda)}\int_{\boldsymbol{R}^n} \psi(|g|)\,dx. \tag{2.4.6}$$

We estimate

$$\frac{1}{|h|^\alpha}\int_{|x-y|<L|h|} |\Delta_h^k G_\alpha(x-y)|\,|g(y)|\,dy.$$

Write $g = g_1+g_2$, where $g_1 = g$ if $|g|\leqslant 1$ and zero otherwise, and $g_2 = g$ if $|g|>1$ and zero otherwise. Choose $\delta>0$, and consider $|g_1|^{1/(1+\delta)}$; clearly, $|g_1|^{1/(1+\delta)} \in L^{n(1+\delta)/\alpha}(\boldsymbol{R}^n)$ since $|g_1|$ is bounded and $\psi(t) = Kt^{n/\alpha}$ for $0<t<\eta$. On the other hand, $|g_1| \leqslant |g_1|^{1/(1+\delta)}$; therefore

$$\begin{aligned}\sup_{h\in\boldsymbol{R}^n, |h|>0} \frac{1}{|h|^\alpha}\int_{|x-y|<L|h|} |\Delta_h^k G_\alpha(x-y)|\,|g_1(y)|\,dy \\ \leqslant \sup_{h\in\boldsymbol{R}^n, |h|>0} \frac{l}{|h|^\alpha}\int_{|x-y|<L|h|} |\Delta_h^k G_\alpha(x-y)|\,|g_1|^{1/(1+\delta)}\,dy.\end{aligned} \tag{2.4.7}$$

Calling $\bar{g}_\alpha(x)$ to the right-hand member of (2.4.7), we have

$$\begin{aligned}|E(\bar{g}_\alpha>\lambda)| &< \frac{C}{\lambda^{n(1+\delta)/\alpha}}\int_{\boldsymbol{R}^n} |g_1|^{n/\alpha}\,dy \\ &\leqslant \frac{\bar{C}}{\lambda^{n(1+\delta)/\alpha}}\int_{\boldsymbol{R}^n} \psi(|g|)\,dy.\end{aligned} \tag{2.4.8}$$

This estimate follows from the proof of Theorem I in [5] since

$$|g_1|^{1/(1+\delta)} \in L^{n(1+\delta)/\alpha}(\boldsymbol{R}^n), \quad \text{and} \quad \frac{n}{\alpha}(1+\delta) > \frac{n}{\alpha}.$$

Now, we shall deal with

$$\left|\frac{1}{|h|^\alpha}\int_{|x-y|<L|h|} \Delta_h^{(k)} G_\alpha(x-y)\,g_2(y)\,dy\right|.$$

Let us consider the following estimate for the Bessel kernel of order α, $\alpha>0$.

$$0 < G_\alpha(s) \leqslant \frac{C_\alpha}{|s|^{n-\alpha}}, \qquad 0<|s|<\infty \tag{2.4.9}$$

(see [10], p. 132 and 133).

On account of (2.4.9) we have the domination

$$|\Delta_h^k G(x-y)| \leqslant 2C_\alpha \sum_{j=0}^{k} \frac{1}{|x+jh-y|^{n-\alpha}}. \tag{2.4.10}$$

Take a typical term:

$$\int_{|x-y|<L|h|} \frac{1}{|x+lh-y|^{n-\alpha}}\,|g_2(y)|\,dy \tag{2.4.11}$$

where $0\leqslant l\leqslant k<L/3$.

Consider a ball B_1, centered at $x+lh$ and having radius $2L|h|$. Clearly,

$$(2.4.12)\qquad \int\limits_{|x-y|<L|h|} \frac{1}{|x+lh-y|^{n-a}}|g_2(y)|\,dy \leqslant \int\limits_{|x+lh-y|<2L|h|} \frac{1}{|x+lh-y|^{n-a}}|g_2(y)|\,dy.$$

Let E_j be the set in B_1, where

$$(2.4.13)\qquad 2^j \leqslant |g_2(y)| < 2^{j+1}.$$

Call $\beta_j(x)$ to be the characteristic function of E_j. Then we have

$$(2.4.14)\qquad \sum_0^\infty 2^j\beta_j(x) \leqslant |g_2(y)| \leqslant \sum_0^\infty 2^{j+1}\beta_j(x).$$

These estimates follow from the fact that either $|g_2|>1$ or $|g_2(y)|=0$.

The right-hand member of (2.4.12) is dominated by

$$(2.4.15)\qquad \sum_0^\infty 2^{j+1}\int\limits_{E_j}\frac{1}{|x+lh-y|^{n-a}}\,dy \leqslant \sum_0^\infty 2^{j+1}\int\limits_{|x+lh-y|<|E_j|^{1/n}}\frac{1}{|x+lh-y|^{n-a}}\,dy \leqslant \sum_0^\infty 2^j|E_j|^{a/n}.$$

On the other hand:

$$(2.4.16)\qquad 2\sum_0^\infty 2^j|E_j|^{a/n} = 2\sum_0^\infty \frac{2^j}{[\psi(2^j)]^{a/n}}[\psi(2^j)]^{a/n}|E_j|^{a/n}.$$

Applying Hölder's inequality with exponents n/a and $n/(n-a)$ to the right-hand member of (2.4.16), we get the domination

$$(2.4.17)\qquad 2\left(\sum_0^\infty \frac{2^{jn/(n-a)}}{[\psi(2^j)]^{a/(n-a)}}\right)^{(n-a)/n}\left(\sum_0^\infty \psi(2^j)|E_j|\right)^{a/n}.$$

In turn, (2.4.17) is dominated by

$$(2.4.18)\qquad C\left(\int_1^\infty\left[\frac{t}{\psi(t)}\right]^{a/(n-a)}dt\right)^{(n-a)/n}\left(\int\limits_{|x-y|<3L|h|}\psi(|g|)\,dy\right)^{a/n}.$$

Consequently,

$$\sup_{h\in \boldsymbol{R}^n, |h|>0}\frac{1}{|h|^a}\int\limits_{|x-y|<L|h|}|\Delta_h^k G_a(x-y)||g_2(y)|\,dy \leqslant \bar{C}\sup_{\varepsilon>0}\left(\frac{1}{\varepsilon^n}\int\limits_{|x-y|<3L\varepsilon}\psi(|g|)\,dy\right)^{a/n}.$$

This concludes the proof of part (i).

Part (ii) follows from part (i) by using a standard argument; see [5], Corollary I.

2.5. Theorem C. *All f belonging to $L_k^\psi(\boldsymbol{R}^n)$, where k is an integer, $1\leqslant k<n$, and ψ near $t^{n/k}$ has total differential of order k a.e. in $\boldsymbol{R}^n$ if and only if*

$$\int_1^\infty\left[\frac{t}{\psi(t)}\right]^{k/(n-k)}dt<\infty.$$

The proof follows from Theorems A and B.

2.6. Theorem D. *Let f be a real function defined on $\boldsymbol{R}^n$ and suppose that $D^\beta f\in L^\psi_{\mathrm{loc}}(\boldsymbol{R}^n)$, for $0\leqslant|\beta|\leqslant k$, $1\leqslant k<n$, ψ near n/k and satisfying*

$$\int_1^\infty\left[\frac{t}{\psi(t)}\right]^{k/(n-k)}dt<\infty.$$

Then f has the total differential of order k at almost all the points of $\boldsymbol{R}^n$. (Here D^β are derivatives in the distributions sense.)

Proof. Let $\gamma(x)$ be a C_0^∞ function such that $\gamma(x)=1$ if $|x|\leqslant 1$ and zero if $|x|>\frac{3}{2}$.

The functions $f(x)\cdot\gamma(\varepsilon x)\in L_k^\psi(\boldsymbol{R}^n)$ for all $\varepsilon>0$. Now the result follows from Theorem B and 1.2.

2.7. Remark. If $\Psi(t)=t^{n/a}[\varphi(t)]^{(n-a)/a}$, $0<a<n$, and $\Phi(t)$ is any of the functions

$$1+(\log^+t)^{1+\varepsilon},\quad 1+\log^+t\cdot(\log^+\log^+t)^{1+\varepsilon},\quad 1+(\log^+t)(\log^+\log^+t)(\log^+\log^+\log^+t)^{1+\varepsilon},\ \ldots$$

Then the conclusions of Theorem B hold.

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Inequalities for the maximal function relative to a metric

by

A. P. CALDERÓN* (Chicago, Ill.)

Abstract. Weighted L^p-norm inequalities for the maximal function relative to a family of spheres defined by a pseudo-metric are obtained.

The purpose of this note is to obtain weighted L^p-norm inequalities for the maximal function defined by the spheres of a certain pseudo-metric. These inequalities generalize those known to hold in Euclidean space with the ordinary metric (see [2]), and other metrics considered by D. Kurtz [3] but they do not cover his results about maximal functions defined by certain families of rectangles.

Let X be a metric space with a measure μ and assume that the space of continuous functions with bounded support is contained and is dense in the space of integrable functions. Further, suppose that there is given a real-valued function $\varrho(x, y)$ in $X\times X$ (it need not be the distance function) with the following properties

(i) $\varrho(x, x) = 0$;

(ii) $\varrho(x, y) = \varrho(y, x) > 0$ if $x \neq y$;

(iii) *there is a constant c such that $\varrho(x, z) \leqslant c[\varrho(x, y)+\varrho(y, z)]$ for all x, y, and z;*

(iv) *given a neighborhood N of a point x there is an ε, $\varepsilon > 0$, such that the sphere $B_\varepsilon(x) = \{y|\ \varrho(x, y)\leqslant \varepsilon\}$ with center at x is contained in N;*

(v) *the spheres $B_r(x) = \{y|\ \varrho(x, y)\leqslant r\}$ are measurable, the measure $|B_r(x)|$ of $B_r(x)$ is a continuous function of r for each x, and there is a constant c, $c > 1$, such that*

$$|B_{2r}(x)| \leqslant c\,|B_r(x)| < \infty$$

for all r and x. For convenience we shall assume that the constant here coincides with the one in (iii).

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