alors l'opération $F_0 \overset{\mathrm{df}}{=} \Phi^{-1}G_0(\Phi^*)^{-1}$ *satisfait à*

(ii) $$\|F_0Cx - x\|_2 \leqslant \gamma_2\|x\|_2 \quad \textit{pour} \quad x \in D(C),$$

où $\gamma_2 = a^{-1}(\gamma, \mu+\delta)$, $\mu = \max(\lambda_1, 1)$, $a = \min(\lambda_2^{-1}, 1)$, $\gamma_1 = \sqrt{\lambda_1\lambda_2}\gamma_0$, $\delta = \max(\lambda_1 - 1, \lambda_2^{-1} - 1)$ *et* λ_i, μ_i *sont les constantes définies dans* 5.2.

Démonstration. Prouvons d'abord que

(a) $$\|F_0Ux - x\|_2 \leqslant \sqrt{\lambda_1\lambda_2}\gamma_0\|x\|_2 \quad \text{pour} \quad x \in D(U).$$

En effet, en vertu de (5.3.7) on a

$$\|F_0Ux - x\|_2 = \|\Phi^{-1}G_0(\Phi^*)^{-1}\Phi^*A\Phi x - x\|_2 = \|\Phi^{-1}[G_0A\Phi x - \Phi x]\|_2$$
$$\leqslant \sqrt{\lambda_2}\|G_0A\Phi x - \Phi x\|_1 \leqslant \sqrt{\lambda_2}\gamma_0\|\Phi x\|_1 \leqslant \sqrt{\lambda_1\lambda_2}\gamma_0\|x\|_2,$$

en tenant aussi compte de (5.2.7).

D'autre part, les opérations U, C sont engendrées par $\Psi(x, y) = (\Phi x, \Phi y)_1$ et $(x, y)_2$, respectivement; posons dans le lemme 3 point 3.1: $\Psi(x, y)$ pour $\Psi_1(x, y)$, $(x, y)_2$ pour $\Psi_2(x, y)$, $(x, y)_2$ pour $(x, y)_1$, $\max(\lambda_1, 1)$ pour μ, $\min(\lambda_2^{-1}, 1)$ pour a, $\max(\lambda_1 - 1, \lambda_2^{-1} - 1)$ pour δ, γ_1 pour γ_0, F_0 pour F_0. Les hypothèses du lemme 3 étant ainsi vérifiées, on obtient:

(b) $$\|F_0Cx - x\|_2 \leqslant a^{-1}(\gamma, \mu+\delta)\|x\|_2. \ \blacksquare$$

COROLLAIRE. *Si le nombre* $a^{-1}(\gamma, \mu+\delta) < 1$, F_0 *est une presque-inverse de* $C = -\Delta$. *Sinon, on peut obtenir une presque inverse de* C *en utilisant la remarque du point* 3.1 (faisant suite au théorème 2).

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Bernoulli convolutions in LCA groups

by

CHUNG LIN and SADAHIRO SAEKI (Tokyo)

Abstract. Let G be a nondiscrete metrizable LCA group with character group Γ. Choose a local base $\{U_n\}_{n=1}^{\infty}$ at 0 consisting of compact sets satisfying $U_{n+1} + U_{n+1} \subset U_n$ for all n, and let $U = \prod_{n=1}^{\infty} U_n$. Take a sequence $\{(a_n, b_n, c_n)\}_{n=1}^{\infty}$ of triples of non-negative real numbers such that $a_n + b_n + c_n = 1$ for all n. Given $x \in U$, let ν_x denote the Bernoulli convolution

$$\mathop{\ast}_{n=1}^{\infty} \left(a_n\delta(0) + b_n\delta(x_n) + c_n\delta(-x_n)\right)$$

and let $\overline{\Gamma}_x$ denote the weak* closure of Γ in $L^\infty(\nu_x)$. Let S_x consist of those complex numbers in the closed unit disk D for which the corresponding constant function belongs to $\overline{\Gamma}_x$. Among other things, this is shown: If G is an I-group, then for quasi-all $x \in U$, S_x contains the multiplicative compact semigroup in D generated by all the complex numbers of the form $a + bz + c\bar{z}$, where (a, b, c) is an arbitrary limit point of $\{(a_n, b_n, c_n)\}_{n=1}^{\infty}$ and $|z| = 1$. It is also shown that in many cases $S_x = D$ for quasi-all $x \in U$.

§ 1. Introduction. We adhere to the notation established above. In addition, $M(G)$ will denote the usual convolution measure algebra of G. For $x \in G$, $\delta_x = \delta(x)$ denotes the unit point mass measure at x. The circle group and the group of r-adic integers are denoted by $\boldsymbol{T}$ and Δ_r, respectively. The set of all integers is denoted by $\boldsymbol{Z}$.

For $G = \boldsymbol{T}$ and Δ_r, Hewitt and Kakutani [5] proved in 1964 that there is a measure $\nu_x = \mathop{\ast}_{n=1}^{\infty} 2^{-1}\left(\delta(0) + \delta(x_n)\right) \in M(G)$ such that the weak* closure of Γ in $L^\infty(\nu_x)$ contains all constant functions with values in the unit disk D. Brown and Moran [1] proved later that if $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive integers $\geqslant 2$, $x_n = (a_1, \ldots, a_n)^{-1}$ and $\nu_x = \mathop{\ast}_{n=1}^{\infty} 2^{-1}\left(\delta(0) + \delta(x_n)\right) \in M(\boldsymbol{T})$, then the weak* closure of $\hat{\boldsymbol{T}} = \boldsymbol{Z}$ in $L^\infty(\nu_x)$ contains all constant functions with values in D if and only if $\sup a_n = \infty$. This result generalizes one of Hewitt and Kakutani's results in [5], since in [5] they only showed that ν_x has the required property if $\sum_{n=1}^{\infty} a_n^{-1} < \infty$. Brown and Moran

[2] recently proved the following more interesting result. Let

$$B = \left\{b = (b_n)_{n=1}^{\infty}\colon\ b_n \geqslant 0 \text{ and } \sum_{n=1}^{\infty} b_n^2 \leqslant 1\right\}$$

and ν_b be the measure

$$\mathop{*}_{n=1}^{\infty} 2^{-1}\big(\delta(-b_n)+\delta(b_n)\big)\in M(\boldsymbol{T}).$$

We may regard B as a subspace of the compact space $[0,1]^{\aleph_0}$. Then for quasi-all $b\in B$, the weak* closure of $\hat{\boldsymbol{T}}$ in $L^\infty(\nu_b)$ contains all constant functions with values in $[-1,1]$. The reason they used $[-1,1]$ instead of D is that the measure ν_b is hermitian in this case.

Brown and Moran [2] were only concerned with the circle group $\boldsymbol{T}$. In this note we shall first prove some analogs to their main result in [2] for nondiscrete, metrizable LCA groups G, and then generalize their result for $G=\boldsymbol{T}$. We would like to give our thanks to Professor K. A. Ross for his thoughtful reading of this note.

DEFINITION. The LCA group G is called an *I-group* if every neighborhood of the identity contains an element of infinite order.

DEFINITION. A local base $\{U_n\}_{n=1}^{\infty}$ at $0\in G$ is called *admissible* if each U_n is a compact neighborhood of 0 and $U_{n+1}+U_{n+1}\subset U_n$ for all n. A sequence $\{(a_n, b_n, c_n)\}_{n=1}^{\infty}$ of triples of non-negative real numbers is called *admissible* if $a_n+b_n+c_n=1$ for all n.

Let $\Delta[M(G)]$ denote the maximal ideal space of $M(G)$. We may regard it as a topological subspace of $\prod\{L^\infty(\mu)\colon \mu\in M(G)\}$, where each $L^\infty(\mu)$ carries the $\sigma\big(L^\infty(\mu), L^1(\mu)\big)$ topology [9]. For $f\in\Delta[M(G)]$ and $\mu\in M(G)$, let f_μ denote the function in $L^\infty(\mu)$ which is the restriction of f to $L^1(\mu)$.

In the sequel, we shall fix an arbitrary admissible local base $\{U_n\}_{n=1}^{\infty}$ at $0\in G$ and an arbitrary admissible sequence $\{(a_n, b_n, c_n)\}_{n=1}^{\infty}$. Let $U=\prod_{n=1}^{\infty} U_n$, and let L denote the set of all limit points of $\{(a_n, b_n, c_n)\}_{n=1}^{\infty}$ in $[0,1]^3$. For each $x=(x_1, x_2, \dots)\in U$, the convolution

$$\nu_x = \nu(x) = \mathop{*}_{n=1}^{\infty}\{a_n\delta(0)+b_n\delta(x_n)+c_n\delta(-x_n)\}$$

converges in the weak* topology of $M(G)$, as will be shown in § 2. We define $\bar{\Gamma}_x$ to be the weak* closure of Γ in $L^\infty(\nu_x)$. The set of all constant functions in $\bar{\Gamma}_x$ is denoted by S_x.

THEOREM 1. *If G is an I-group, then, for quasi-all $x\in U$, S_x contains the multiplicative compact semigroup in D generated by the set $\{a+bz+c\bar{z}\colon (a,b,c)\in L$ and $|z|=1\}$. If G is not an I-group, this conclusion fails to hold for some $\{U_n\}_{n=1}^{\infty}$ and some $\{(a_n, b_n, c_n)\}_{n=1}^{\infty}$.*

THEOREM 2. *Suppose G is a nondiscrete metrizable LCA group which is not an I-group, and define $q=q(G)$ to be the largest natural number such that every neighborhood of $0\in G$ contains an element of order q. Then, for quasi-all $x\in U$, S_x contains the compact semigroup in D generated by all complex numbers of the form $a+bz+c\bar{z}$, where $(a,b,c)\in L$ and $z^q=1$.*

COROLLARY 1. *Let G be a metrizable LCA group. Suppose the sequence $\{(a_n, b_n, c_n)\}_{n=1}^{\infty}$ has a limit point (a,b,c) such that $\max\{a,b,c\}<1$.*

(i) *If every neighborhood of $0\in G$ contains an element of order $\geqslant 4$, then quasi-all $x\in U$ have the property that*

$$\delta_y*\nu_x^m\perp\nu_x^n \quad (y\in G;\ m,n\geqslant 0;\ m\neq n).$$

(ii) *The same conclusion holds if $q(G)=2$ and $0\neq a\neq b+c$, or if $q(G)=3$ and $2a\neq b+c$.*

THEOREM 3. *Suppose the admissible sequence $\{(a_n, b_n, c_n)\}_{n=1}^{\infty}$ satisfies $\sum_1^{\infty}(b_n-c_n)^2<\infty$. Let $(d_n)_{n=1}^{\infty}$ be a given sequence of real positive numbers, and*

$$B=\left\{x\in\prod_1^{\infty}[0,d_n]\colon \sum_1^{\infty} x_n^2\leqslant C\right\},$$

where C is an arbitrary real positive number. Setting $b=\limsup_{n\to\infty} b_n$, we then have

(i) *For each $x\in B$, the convolution*

$$\nu_x=\nu(x)=\mathop{*}_{n=1}^{\infty}\{a_n\delta(0)+b_n\delta(x_n)+c_n\delta(-x_n)\}$$

converges in the weak topology of $M(\boldsymbol{T})$.*

(ii) *Quasi-all $x\in B$ have the property that the weak* closure of $\hat{\boldsymbol{T}}=\boldsymbol{Z}$ in $L^\infty(\nu_x)$ contains all the constants in $[1-4b, 1]$ if $b\geqslant 1/4$, in $[0,1]$ if $0<b<1/4$, and $\{1\}$ if $b=0$.*

(iii) *Let x_0 be a given element of $\boldsymbol{T}$ having infinite order and $\mu_x=\delta(x_0)*\nu_x$ $(x\in B)$. Then, for quasi-all $x\in B$, the weak* closure of $\hat{\boldsymbol{T}}$ in $L^\infty(\mu_x)$ contains all the constants in D if $b>0$, and in $\{|z|=1\}$ if $b=0$.*

In the case $a_n=0$, $b_n=c_n=1/2$, and $d_n=1$ for all n, part (ii) of the above theorem is due to Brown and Moran [2].

COROLLARY 2. *Under the hypotheses of Theorem 3, quasi-all $x\in B$ have the property that*

$$\delta_t*\nu_x^m\perp\nu_x^n \quad (t\in\boldsymbol{T};\ m,n\geqslant 0;\ m\neq n),$$

provided that $b>0$.

§ 2. Proofs of results. We shall preserve all the notation established in § 1. In particular, G denotes a metrizable nondiscrete LCA group. For each $x \in U$, we write

$$\nu_n = \nu_{x'n} = a_n \delta(0) + b_n \delta(x_n) + c_n \delta(-x_n)$$

for $n = 1, 2, \ldots$

LEMMA 1. *Given $x \in U$, the convolution product $\overset{\infty}{\underset{n=1}{*}} \nu_n$ converges to some $\nu_x \in M(G)$ in the weak* topology of $M(G)$. Moreover, the correspondence $(x, \chi) \to \hat{\nu}_x(\chi)$ is a continuous function on $U \times \Gamma$.*

Proof. Let $x \in U$, and $\chi \in \Gamma$. Given natural numbers $r > p$, we have

$$\begin{aligned} &|(\overset{r}{\underset{n=1}{*}} \nu_n)\hat{\ }(\chi) - (\overset{p}{\underset{n=1}{*}} \nu_n)\hat{\ }(\chi)| \\ &= \Big|\prod_{n=1}^{r} \hat{\nu}_n(\chi) - \prod_{n=1}^{p} \hat{\nu}_n(\chi)\Big| \leqslant \Big|\prod_{n=p+1}^{r} \hat{\nu}_n(\chi) - 1\Big| \\ &= \Big|\int (\bar{\chi} - 1)\, d(\overset{r}{\underset{n=p+1}{*}} \nu_n)\Big| \leqslant \sup\{|\chi(x) - 1| : x \in U_p - U_p\}, \end{aligned}$$

because $U_{n+1} + U_{n+1} \subset U_n$ for all $n \geqslant 1$. Since $\{U_n\}_{n=1}^{\infty}$ is a local base at 0, it follows that the sequence $(\overset{p}{\underset{n=1}{*}} \nu_n)\hat{\ }(\chi)$ converges uniformly in $(x, \chi) \in U \times K$ for each compact subset K of Γ. Therefore the product $\overset{\infty}{\underset{n=1}{*}} \nu_{x'n}$ converges to some $\nu_x \in M(G)$ in the weak* topology of $M(G)$ for each $x \in U$ (notice that all the measures under consideration are carried by the compact set $2U_1 - 2U_1$). The second assertion in our lemma is obvious by the above arguments.

Now let H be the subgroup of G generated by U_1. Clearly, if Theorem 1 holds for H, then so does it for G as well. Therefore we may assume $G = H$. Then G is σ-compact and metrizable (by hypothesis), and so Γ is separable. We choose and fix an arbitrary countable dense subset $\{\psi_k\}_{k=1}^{\infty}$ of Γ.

LEMMA 2. *Let $\alpha \in D$ and $\nu \in M(G)$. Suppose that to each $N \geqslant 1$ there corresponds a $\chi_N \in \Gamma$ such that $|\alpha\hat{\nu}(\psi_k) - \hat{\nu}(\chi_N\psi_k)| < 1/N$ for all $1 \leqslant k \leqslant N$. Then the constant α belongs to the weak* closure of Γ in $L^\infty(\nu)$.*

Proof. Let $\{\chi_N\}_{N=1}^{\infty}$ be as above. Then we have

$$\lim_{N\to\infty} \int \bar{\chi}_N \psi\, d\nu = \int \alpha\psi\, d\nu \tag{1}$$

for every $\psi \in \{\psi_k\}_{k=1}^{\infty}$. Since the last set is dense in Γ, (1) holds for all $\psi \in \Gamma$, and hence for all $\psi \in L^1(\nu)$ (cf. [7], 31.4). In other words, the sequence $\{\bar{\chi}_N\}_{N=1}^{\infty}$ converges to α in the weak* topology of $L^\infty(\nu)$.

Let $(a, b, c) \in L$, $|z| = 1$, and $\alpha = a + bz + c\bar{z}$ be given. Set

$$E(\alpha, N) = \bigcap_{\chi \in \Gamma} \bigcup_{k=1}^{N} \{x \in U : |\alpha\hat{\nu}_x(\psi_k) - \hat{\nu}_x(\chi\psi_k)| \geqslant 1/N\}$$

for $N = 1, 2, \ldots$

LEMMA 3. *The set $E(\alpha, N)$ is closed in U. If G is an I-group, then $E(\alpha, N)$ has no interior point.*

Proof. For each $\chi \in \Gamma$, $\hat{\nu}_x(\chi)$ is a continuous function of $x \in U$ by Lemma 1. Therefore $E(\alpha, N)$ is closed in U.

Now suppose G is an I-group. To force a contradiction, assume that $E(\alpha, N)$ has non-empty interior. Then there exist finitely many non-empty sets $V_n \subset U_n$, $1 \leqslant n < M$, such that

$$V_1 \times \ldots \times V_{M-1} \times \prod_{M}^{\infty} U_n \subset E(\alpha, N).$$

We may assume M satisfies

$$\max\{|a - a_M|, |b - b_M|, |c - c_M|\} < 1/(8N), \tag{1}$$

$$\|\psi_k - 1\|_{U_M} < 1/(8N) \quad (1 \leqslant k \leqslant N). \tag{2}$$

Choose any points $x_n \in V_n$, $1 \leqslant n < M$. Since G is an I-group, we can find an $x_M \in U_M$ such that $px_M \notin Gp(\{x_n : 1 \leqslant n < M\})$ for all nonzero integers p (for the proof, see [8], 5.2.3). Then there exists a $\chi \in \Gamma$ such that

$$|\chi(x_n) - 1| < 1/(8MN) \quad (1 \leqslant n < M), \tag{3}$$

$$|\chi(x_M) - \bar{z}| < 1/(8N). \tag{4}$$

Setting $x = (x_1, x_2, \ldots, x_M, 0, 0, \ldots) \in E(\alpha, N)$ and $\nu_x = \overset{M}{\underset{n=1}{*}} \nu_n$, we have for $1 \leqslant k \leqslant N$

$$\begin{aligned} |\alpha\hat{\nu}_M(\psi_k) - \hat{\nu}_M(\chi\psi_k)| &\leqslant |\alpha\{\hat{\nu}_M(\psi_k) - 1\}| + \\ &+ |a + bz + c\bar{z} - a_M - b_M\bar{\chi}(x_M) - c_M\bar{\chi}(-x_M)| + |\hat{\nu}_M(\chi) - \hat{\nu}_M(\chi\psi_k)| \\ &< 1/(8N) + 5/(8N) + 1/(8N) = 7/(8N) \end{aligned}$$

by (2), (1) and (4), since $\alpha = a + bz + c\bar{z}$. It follows from (3) that $1 \leqslant k \leqslant N$ imply

$$\begin{aligned} |\alpha\hat{\nu}_x(\psi_k) - \hat{\nu}_x(\chi\psi_k)| &= \Big|\alpha \prod_{n=1}^{M} \hat{\nu}_n(\psi_k) - \prod_{n=1}^{M} \hat{\nu}_n(\chi\psi_k)\Big| \\ &\leqslant \sum_{n=1}^{M-1} |\hat{\nu}_n(\psi_k) - \hat{\nu}_n(\chi\psi_k)| + |\alpha\hat{\nu}_M(\psi_k) - \hat{\nu}_M(\chi\psi_k)| \\ &< (M-1)/(8MN) + 7/(8N) < 1/N. \end{aligned}$$

Hence $x \notin E(\alpha, N)$, which contradicts our choice of x.

Proofs of Theorems 1 and 2. Suppose G is an I-group, and take any countable dense subset A of the set

$$(*)\qquad \{a+bz+c\bar{z}:\ (a,b,c)\in L \text{ and } |z|=1\}.$$

If $x\in U$ is not in $\bigcup\{E(a,N):\ a\in A \text{ and } N\geqslant 1\}$, then we have $A\subset S_x$ by Lemma 2. On the other hand, it is easy to see that S_x is a compact semigroup in D for every $x\in U$. Therefore, for each x as above, S_x contains the compact semigroup generated by the set $(*)$. Thus the first assertion of Theorem 1 follows from Lemma 3.

Now assume that G is not an I-group. Then G contains an open subgroup of the form $R^n\times H$, where $n\geqslant 0$ is an integer and H is a compact abelian group. Since G is not an I-group, $n=0$ and H must be torsion ([6], 25.10). So H is a compact open torsion subgroup of G. Let n_0 be a positive integer such that $n_0x=0$ for all $x\in H$. Define $a_n=0$ and $b_n=c_n=1/2$ for all $n\geqslant 1$; hence $L=(0,1/2,1/2)$. If $\chi\in\Gamma$ and $x\in H$, then we have either $\chi(x)=1$ or $|\mathrm{Re}\,\chi(x)|\leqslant|\cos(2\pi/n_0)|$, since $\{\chi(x)\}^{n_0}=\chi(n_0x)=1$. Let $\{U_n\}_{n=1}^{\infty}$ be any admissible local base at $0\in H$. Then, for every $x\in U$ and $\chi\in\Gamma$, we have either $\hat{\nu}_x(\chi)=\prod_1^\infty \mathrm{Re}\,\chi(x_n)=1$, $\hat{\nu}_x(\chi)=-1$ (if $n_0=2$), or $|\hat{\nu}_x(\chi)|\leqslant|\cos(2\pi/n_0)|$ (if $n_0\geqslant 3$). Therefore every S_x is disjoint from the open interval $(-1,1)$ if $n_0=2$ and from $\big(|\cos(2\pi/n_0)|,1\big)$ if $n_0\geqslant 3$. This establishes Theorem 1. The proof of Theorem 2 is almost the same as that of Theorem 1, and so we omit the details.

To prove Theorem 3, we need the following fact.

LEMMA 4. *Let d be a real positive number, and p a natural number larger than $\max\{32\pi^2d^{-2},1\}$. Then, for each natural number s, there exist four non-negative integers p_k $(1\leqslant k\leqslant 4)$ such that*

$$2\pi p_k/(sp+1)<d\ \ (1\leqslant k\leqslant 4)\quad \textit{and}\quad p_1^2+p_2^2+p_3^2+p_4^2=s(sp+2).$$

Proof. By hypothesis, we have $2p\leqslant 4^{-1}p^2(2\pi)^{-2}d^2\leqslant[p(2\pi)^{-1}d]^2$, where [ ] denotes integer part. Let s be a given natural number. By the four-square theorem of Lagrange ([4], p. 302), there exist non-negative integers p_k $(1\leqslant k\leqslant 4)$ such that $p_1^2+\ldots+p_4^2=s(sp+2)$. Then we have

$$p_k^2\leqslant s(sp+2)\leqslant 2s^2p\leqslant s^2[p(2\pi)^{-1}d]^2<\{(sp+1)(2\pi)^{-1}d\}^2,$$

as was required.

Proof of Theorem 3. Let $\{(a_n,b_n,c_n)\}_{n=1}^{\infty}$, $(d_n)_{n=1}^{\infty}$, B, and $x_0\in T=[0,2\pi)$ be as in Theorem 3. We first note that if a, b, c and t are real numbers with $a+b+c=1$, then

$$\begin{aligned}|1-a-be^{-it}-ce^{it}| &= |b(1-e^{-it})+c(1-e^{it})|\\ &\leqslant |b+c|(1-\cos t)+|(b-c)\sin t|\leqslant 2^{-1}|b+c|t^2+|(b-c)t|.\end{aligned}$$

It follows that $x\in B$ and $k\in\mathbf{Z}$ imply

$$\begin{aligned}\sum_1^\infty &|1-a_n-b_n\exp(-ikx_n)-c_n\exp(ikx_n)|\\ &\leqslant 2^{-1}\sum_1^\infty(kx_n)^2+\sum_1^\infty|(b_n-c_n)kx_n|\\ &\leqslant 2^{-1}k^2\sum_1^\infty x_n^2+|k|\Big(\sum_1^\infty(b_n-c_n)^2\Big)^{1/2}\Big(\sum_1^\infty x_n^2\Big)^{1/2}<\infty\end{aligned}$$

by Schwarz' inequality. This assures that the convolution product defined in part (i) of Theorem 3 converges in the weak* topology of $M(\mathbf{T})$ and that

$$\hat{\nu}_x(k)=\prod_1^\infty\{a_n+b_n\exp(-ikx_n)+c_n\exp(ikx_n)\}\qquad(k\in\mathbf{Z})$$

for all $x\in B$. A similar argument shows that if $\sum_1^\infty d_n^2<\infty$, then $x\to\hat{\nu}_x(k)$ is a continuous function of $x\in B$ for every $k\in\mathbf{Z}$. In this case, the proof of Theorem 3 proceeds on the same lines as that of Theorem 1. Consequently, we shall hereafter assume that $\sum_1^\infty d_n^2=\infty$.

Next notice that if $b=\limsup_{n\to\infty}b_n$, then $(1-2b,b,b)$ is a limit point of $\{(a_n,b_n,c_n)\}_{n=1}^{\infty}$, since $\lim_{n\to\infty}(b_n-c_n)=0$. Moreover, the compact semigroup in D generated by the set $\{1-2b+bz+b\bar{z}:\ |z|=1)=[1-4b,1]$ is $[1-4b,1]$ if $b\geqslant 1/4$, $[0,1]$ if $0<b<1/4$, and $\{1\}$ if $b=0$.

Now we want to prove part (ii). Let $|z|=1$, $(a,b,c)\in L$, N a natural number, and $j\in\mathbf{Z}$ be given. Put $\alpha=a+bz+c\bar{z}$, and define $F_j=F(\alpha,N,j)$ to be the set

$$F_j=\bigcup_{k=-N}^{N}\{x\in B:\ |\alpha\hat{\nu}_x(k)-\hat{\nu}_x(k+j)|\geqslant 1/N\}.$$

By Lemma 2 and the last remark, we need only confirm that $E(\alpha,N)=\bigcap_1^\infty\bar{F}_j$ has no interior point. (Notice that F_j may not be closed in B, since, in general, the correspondence $x\to\hat{\nu}_x(k)$ is not continuous for any fixed $k\neq 0$; see [2], Lemma 2.)

Suppose this is false for some α and N. Then there exist finitely many open intervals $I_n\subset[0,d_n]$ $(1\leqslant n\leqslant M-2,\ M>2)$ such that

$$(1)\qquad \emptyset\neq B\cap\Big(I_1\times\ldots\times I_{M-2}\times\prod_{n=M-1}^{\infty}[0,d_n]\Big)\subset E(\alpha,N).$$

Setting $I_n = (0, d_n)$ for all $n \geqslant M-1$ and replacing M by a larger number, we may assume the following:

(2) $$\max\{|a_M - a|, |b_M - b|, |c_M - c|\} < 1/(24N);$$

(3) $$\sum_M^{\infty} (b_n - c_n)^2 < 1;$$

there exist $y_n \in I_n$ $(1 \leqslant n \leqslant M-2)$ such that

$$0 < C - (y_1^2 + \ldots + y_{M-2}^2) < \min\{(8N)^{-4}, d_M^2\}.$$

(Notice that we have assumed $\sum_1^{\infty} d_n^2 = \infty$.) We can demand that $x_0, \pi, y_1, \ldots, y_{M-2}$ are rationally independent. (The reason x_0 is treated here is to prove part (iii).) Let p be any natural number satisfying $(2\pi)^2/p < C - (y_1^2 + \ldots + y_{M-2}^2)$ and $p > 32\pi^2 d^{-2}$, where $d = \min\{d_{M+k}\colon 1 \leqslant k \leqslant 4\}$. Then there is a number $y_{M-1} \in I_{M-1}$ such that

(4) $$(2\pi)^2/p < C - (y_1^2 + \ldots + y_{M-1}^2) < \min\{(8N)^{-4}, d_M^2\},$$

and such that $x_0, \pi, y_1, \ldots, y_M$ are rationally independent, where

(5) $$y_M = \{C - (y_1^2 + \ldots + y_{M-1}^2) - (2\pi)^2/p\}^{1/2}.$$

Hence y_M is in I_M. By the well-known Kronecker theorem ([8], 5.1.3, we can find a natural number $s > N$ so that

(6) $$|\exp i(sp+1)y_n - 1| < 1/(8MN) \quad (1 \leqslant n < M),$$

(7) $$|\exp i(sp+1)y_M - \bar{z}| < 1/(8N)^2,$$

$(7)_0$ $$|\exp i(sp+1)x_0 - \bar{w}| < 1/(8N),$$

where w is an arbitrary, but preassigned, complex number of absolute value one. (The requirement $(7)_0$ is only needed in the proof of part (iii).) By Lemma 4 and our choice of p, there are four non-negative integers p_k $(1 \leqslant k \leqslant 4)$ such that $2\pi p_k/(sp+1) < d_{M+k}$ for $1 \leqslant k \leqslant 4$ and $\sum_1^4 p_k^2 = s(sp+2)$. Set $y_{M+k} = 2\pi p_k/(sp+1)$ for $1 \leqslant k \leqslant 4$ and $y_n = 0$ for $n > M+4$. Then we have $y = (y_1, y_2, \ldots) \in B$, and

(8) $$C - \sum_{n=1}^{M+4} y_n^2 = (2\pi)^2/p - (2\pi)^2 s(sp+2)/(sp+1)^2 = (sp+1)^{-2}(2\pi)^2/p$$

by (5).

Now we define V to be the set of all $x \in B$ satisfying these conditions:

(5)′ $$x_M < (8N)^{-2},$$

(6)′ $$|\exp i(sp+1)x_n - 1| < 1/(8MN) \quad (1 \leqslant n < M),$$

(6)″ $$|\exp i(sp+1)x_n - 1| < 1/(32N) \quad (M < n \leqslant M+4),$$

(7)′ $$|\exp i(sp+1)x_M - \bar{z}| < 1/(8N)^2,$$

(8)′ $$C - \sum_{n=1}^{M+4} x_n^2 < 2(sp+1)^{-2}(2\pi)^2/p.$$

Then V is open in B and contains the element y. Hence

(9) $$\emptyset \neq W = V \cap \Big(I_1 \times \ldots \times I_{M-1} \times \prod_{n=M}^{\infty} U_n\Big) \subset E(\alpha, N)$$

by (1). We claim this contradicts the definition of $E(\alpha, N)$.

Let $x \in W$, and k any integer with $|k| \leqslant N$. Upon setting

$$\nu_n = a_n \delta(0) + b_n \delta(x_n) + c_n \delta(-x_n),$$

we have

(10) $$\Big|\prod_{n=1}^{M-1} \hat{\nu}_n(k) - \prod_{n=1}^{M-1} \hat{\nu}_n(k+sp+1)\Big| \leqslant \sum_{n=1}^{M-1} |1 - \exp i(sp+1)x_n| < 1/(8N)$$

by (6)′. Similarly, we have

(11) $$\Big|\prod_{n=M+1}^{M+4} \hat{\nu}_n(k) - \prod_{n=M+1}^{M+4} \hat{\nu}_n(k+sp+1)\Big| < 1/(8N)$$

by (6)″. On the other hand,

(12) $$\begin{aligned} &|\alpha\hat{\nu}_M(k) - \hat{\nu}_M(k+sp+1)| \\ &\leqslant |\alpha(\hat{\nu}_M(k) - 1)| + |\alpha - \hat{\nu}_M(sp+1)| + |\hat{\nu}_M(sp+1) - \hat{\nu}_M(k+sp+1)| \\ &\leqslant |\exp(ikx_M) - 1| + |1 - \exp(ikx_M)| + \\ &\quad + |a + bz + c\bar{z} - a_M - b_M \exp(-i(sp+1)x_M) - c_M \exp(i(sp+1)x_M)| \\ &\leqslant N/(8N)^2 + N/(8N)^2 + 3/(24N) + 2/(8N)^2 < 2/(8N) \end{aligned}$$

by (5)′, (2) and (7)′. Since $|k| \leqslant N < s$, we also have

$$(13)\qquad \Big|1-\prod_{n=M+5}^{\infty}\hat{\nu}_n(k+sp+1)\Big|$$

$$\leqslant 2^{-1}(k+sp+1)^2\sum_{n=M+5}^{\infty}x_n^2+(k+sp+1)\Big\{\sum_{n=M+5}^{\infty}(b_n-c_n)^2\Big\}^{1/2}\Big(\sum_{n=M+5}^{\infty}x_n^2\Big)^{1/2}$$

$$\leqslant 2(sp+1)^2\sum_{n=M+5}^{\infty}x_n^2+2(sp+1)\Big(\sum_{n=M+5}^{\infty}x_n^2\Big)^{1/2}$$

$$\leqslant 2(sp+1)^2\Big(C-\sum_{n=1}^{M+4}x_n^2\Big)+2(sp+1)\Big(C-\sum_{n=1}^{M+4}x_n^2\Big)^{1/2}$$

$$\leqslant 4(2\pi)^2/p+4(2\pi)/p^{1/2} < 4/(8N)^4+4/(8N)^2 < 1/(8N)$$

by (3), (8)′ and (4). Similarly, we have

$$(14)\qquad \Big|1-\prod_{n=M+5}^{\infty}\hat{\nu}_n(k)\Big| < 1/(8N).$$

Combining (10)–(14), we conclude that

$$|\alpha\hat{\nu}_x(k)-\hat{\nu}_x(k+sp+1)| = \Big|\alpha\prod_{n=1}^{\infty}\hat{\nu}_n(k)-\prod_{n=1}^{\infty}\hat{\nu}_n(k+sp+1)\Big|$$
$$< 1/(8N)+1/(8N)+2/(8N)+1/(8N)+1/(8N)$$
$$< 6/(8N)$$

for all $x \in W$ and all $k \in \boldsymbol{Z}$ with $|k| \leqslant N$. But this implies $W \cap F_j = \emptyset$ and so $W \cap \overline{F}_j = \emptyset$ for $j = sp+1$, because W is open in B. Hence $W \cap E(\alpha, N) = \emptyset$, which contradicts (9) and therefore completes the proof of part (ii).

The proof of part (iii) is almost the same as that of part (ii), and so we only give a sketch of the proof. Let α be as before, and choose an arbitrary complex number w of absolute value one. Recalling $\mu_x = \delta(x_0)*\nu_x$, we redefine F_j to be

$$F_j = \bigcup_{k=-N}^{N}\{x \in B\colon\ |w\alpha\hat{\mu}_x(k)-\hat{\mu}_x(k+j)| \geqslant 1/N\},$$

and set $E(\alpha, w, N) = \bigcap_{j=1}^{\infty}\overline{F}_j$. Define V and W as before; then $x \in W$ and $|k| \leqslant N$ imply

$$|w\alpha\hat{\mu}_x(k)-\hat{\mu}_x(k+sp+1)| < 7/(8N),$$

which gives us the desired contradiction.

Proof of Corollaries 1 and 2. Suppose that the hypotheses of Corollary 1 hold. Then, for quasi-all $x \in U$, S_x contains a complex number α with $0 < |\alpha| < 1$ by Theorems 1 and 2. For such x and α, there exists an $f \in \Delta[M(G)]$ such that $f_{\nu(x)} = \alpha$ ($\nu(x)$-a.e.). Then we have

$$f_{\delta(y)*\nu(x)^m} = f(\delta(y))\,\alpha^m \qquad (\delta(y)*\nu(x)^m\text{-a.e.})$$

and

$$f_{\nu(x)^n} = \alpha^n \qquad (\nu(x)\text{-a.e.}).$$

Since $|f(\delta(y))| = 1$ for all $y \in G$ and since $|\alpha|^m \neq |\alpha|^n$ unless $m = n$, this establishes Corollary 1. The proof of Corollary 2 is similar.

§ 3. Further results. Under the hypotheses of Corollary 1, ν_x has independent powers for quasi-all $x \in U$. For such ν_x, it is known that $\{f_{\nu(x)}\colon f \in \Delta[M(G)]\}$ contains all constant functions of absolute value one (see [3]). More is true; $S_x = D$ for quasi-all $x \in U$ under certain conditions. The following lemma is strong enough for our purpose.

LEMMA 5 (cf. Lemma (4.1) of [5]). *Let a_{nk} be non-negative real numbers for $1 \leqslant n \leqslant N$ and $|k| \leqslant K$, and*

$$a(z) = \prod_{n=1}^{N}\Big(\sum_{k=-K}^{K}a_{nk}z^k\Big) \qquad (|z| = 1).$$

Suppose (i) $\sum_k a_{nk} = 1$ *for all* n, (ii) $\max\{a_{nk}\colon |k| \leqslant K\} < 1$ *for some* n, *and* (iii) $\sum_n\sum_k k a_{nk} \neq 0$. *Then the semigroup generated by all* $a(z)$, $|z| = 1$, *is dense in the closed unit disk* D.

Proof. We have $|a(z)| \leqslant 1$ for all z by (i) and $|a(z)| < 1$ for some z by (ii). Therefore the compact semigroup S generated by all $a(z)$ is contained in D and contains $[0, 1]$, because $|a(z)|^2 = a(z)a(\bar{z})$ is in S whenever $|z| = 1$. Since $e^{it} = 1+it+O(t^2)$ as $t \to 0$, we also have

$$a(e^{it}) = \prod_{n=1}^{N}\Big\{1+it\sum_{k=-K}^{K}k a_{nk}+O(t^2)\Big\} \quad \text{as} \quad t \to 0.$$

It follows that $\lim_m a(e^{it/m})^m = \exp(it\sum_n\sum_k k a_{nk})$ for all real t. Therefore S contains the circle $\{|z| = 1\}$ by (iii); hence $D = [0, 1]\cdot\{|z| = 1\} \subset S$, as was required.

Finally, we state two results without proofs. The former of them follows from Theorem 1 and Lemma 5 while the latter can be proved along the same lines as Theorem 3 was proved.

COROLLARY 3. *Suppose that G is a metrizable* LCA *I-group, and that $\{U_n\}_{n=1}$ and $\{(a_n, b_n, c_n)\}_{n=1}$ are as in* § 1. *If there exists a point (a, b, c) in L such that $\max\{a, b, c\} < 1$ and $b \neq c$, then quasi-all $x \in U$ have the following property: to each $|z| \leqslant 1$ there corresponds an $f \in \overline{\Gamma}$ such that $f_{\nu(x)} = z$ (ν_x-a.e.). Here $\overline{\Gamma}$ denotes the closure of Γ in $\Delta[M(G)]$.*

THEOREM 3′ (cf. [2], Remark 3). *Let $\{(a_n, b_n, c_n)\}_{n=1}^{\infty}$ and $(d_n)_{n=1}^{\infty}$ be an (arbitrary) admissible sequence and a sequence of positive real numbers, respectively, and let $C > 0$ be such that C/π is irrational. Setting*

$$B' = \Big\{x \in \prod_{n=1}^{\infty} [0, d_n]\colon \sum_{n=1}^{\infty} x_n \leqslant C\Big\},$$

we then have (i) *for every $x \in B'$, the convolution product ν_x defined as in* § 1 *converges in the weak* topology of $M(\boldsymbol{T})$, and* (ii) *for quasi-all $x \in B'$, the weak* closure of $\hat{\boldsymbol{T}}$ in $L^\infty(\nu_x)$ contains all the constants in the compact semigroup defined as in Theorem* 1.

Remarks. (a) Suppose $d_n > 0$ for all n and $\sum_1^\infty d_n < \infty$ (resp. $\sum_1^\infty d_n < \infty$). Then the set B in Theorem 3 (resp. B' in Theorem 3′) may be replaced by $\{x \in \prod_1^\infty [0, d_n]\colon \sum_1^\infty f(x_n) \leqslant 1\}$, where f is an arbitrary non-decreasing continuous function of $t \geqslant 0$ such that $f(0) = 0$.

(b) Under the circumstances of Theorem 3, put $E = [1-4b, 1]$ if $b \geqslant 1/4$, $E = [0, 1]$ if $0 < b < 1/4$, and $E = \{1\}$ if $b = 0$. Let Y be a countable subset of $\boldsymbol{T}$. Then quasi-all $x \in \boldsymbol{B}$ have the following property: given $m \in \boldsymbol{Z}$, $\{z_1, z_2, \ldots, z_N\} \subset \{|z| = 1\}$, and $\alpha \in E$, there exists a sequence $(r_j)_1^\infty$ of natural numbers such that (i) $\lim_j \exp(ir_j y) = \exp(imy)$ for $y \in Y$, (ii) $\lim_j \exp(ir_j x_n) = z_n$ for $1 \leqslant n \leqslant N$, and (iii) $\lim_j \exp(ir_j t) = \alpha$ in the weak* topology of $L^\infty(\nu(x, N))$, where $\nu(x, N) = \mathop{*}\limits_{N+1}^{\infty} \nu_n$. In particular, etting $\hat{\boldsymbol{Z}}_x$ denote the weak* closure of $\hat{\boldsymbol{T}}$ in $L^\infty(\nu_x)$, we conclude (for quasi-all $x \in B$) that $\hat{\boldsymbol{Z}}_x$ contains many functions which are not of the form $\beta \exp(int)$, where $\beta \in \boldsymbol{C}$ and $n \in \boldsymbol{Z}$, and that the measures $\delta(x) * \nu(x, N)$, $x \in Gp(\{x_1, \ldots, x_N\})$, are mutually singular for $N = 1, 2, \ldots$ Similar assertions hold under the circumstances of Theorems 1, 2 and 3′.

(c) Replacing the set U (B or B') by the countable cartesian product of sets of the same type, we have some obvious generalizations of the results established in this note. Furthermore, as Lemma 5 suggests, our methods used here apply equally well for convolution products of measures of the form $\sum_{|k| \leqslant K} a_k \delta(kx)$, where K is a fixed natural number, $a_k \geqslant 0$ for all k and $\sum_k a_k = 1$.

(d) Suppose $\mu \in M(G)$, $g \in \Delta[M(G)]$, and $g_\mu = \alpha$ (μ-a.e.) for some α with $0 < |\alpha| < 1$. Then, to each $z \in D$ there corresponds an $f \in \Delta[M(G)]$ such that $f = z$ (μ-a.e.); for the proof, see [9]. Therefore we have the following result under the hypotheses of Corollary 1: for quasi-all $x \in U$ and all $z \in D$ there exists an $f \in \Delta[M(G)]$ such that $f_{\nu(x)} = z$ (ν_x-a.e.).

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UNIVERSITY OF OREGON, EUGENE, OREGON
and
TOKYO METROPOLITAN UNIVERSITY, SETAGAYA, TOKYO

Chung Lin's current address:
University of British Columbia,
Vancouver, B.C.

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