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# Approximation by spline interpolating bases

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**Abstract.** Spline interpolating basis in $C^k(I)$ is constructed for $k = 2m+2$, with $m = -1, 0, \ldots$ The splines applied are of degree $k+1$ (without defect). The end conditions for the interpolating splines depend only on the values of the expanded function, not on its derivatives. It follows that the constructed sequence of splines of degree $k+1$ is simultaneously a basis in all of the spaces $C^l(I)$ with $l = 0, \ldots, k$.

Our construction is a slight but essential modification of the original one which is due to Shonefeld [18]. In our case the order of approximation by the partial sums is estimated by the moduli of smoothness of order $k+2$, for all $k$. The proof of these estimates is the same as in the periodic case is done by Subbotin [22].

**1. Introduction.** Our aim is to give a construction of a simultaneous interpolating basis for the space $C^k(I)$ of $k$-times continuously differentiable functions on $I = \langle 0, 1\rangle$ for $k = 2m+2$, $m \geqslant -1$, i.e., to construct such a sequence $\{\varphi_n^{(k)}\} \subset C^k(I)$ that

(1.1) $\{\varphi_n^{(k)}\}$ is a basis in $C^l(I)$ for $0 \leqslant l \leqslant k$ (cf. [18], [19]).

(1.2) For the countable dense sequence $\{t_0, t_1, \ldots\} \subset I$, i.e. for the dyadic points

$$t_i = \begin{cases} i & \text{for} \quad i = 0, 1, \\ (2\nu-1)/2N & \text{for} \quad i = 2, 3, \ldots \end{cases} \tag{1.3}$$

with $i = N+\nu$, $N = 2^\mu$, $\mu$ and $\nu$ being integers, $1 \leqslant \nu \leqslant N$, the following condition

$$\varphi_n^{(k)}(t_i) = \delta_{i,n} \quad \text{for} \quad 0 \leqslant i \leqslant n,\ n = 0, 1, \ldots \tag{1.4}$$

is satisfied, where $\delta_{i,n}$ denotes the Kronecker symbol [20].

$C^k(I)$ is treated here as a Banach space with the usual norm

$$\|f\|^{(k)} = \sum_{l=0}^{k} \|D^l f\| \quad \text{for} \quad f \in C^k(I),\ k \geqslant 0, \tag{1.5}$$

where $D^l f$ denotes the continuous derivative of $f$ of order $l$ and

$$\|g\| = \|g\|_I \quad \text{for} \quad g \in C(I), \tag{1.6}$$

$$\|g\|_S = \sup_{t \in S} |g(t)| \quad \text{for any bounded function } g \text{ on } S. \tag{1.7}$$

A Schauder basis of the Banach space $(X, \| \ \|)$ is any sequence $\{x_n\} \subset X$ such that each $x \in X$ has an expanonis

$$x = \sum_{n=0}^{\infty} a_n(x) x_n,$$

convergent in the norm $\| \ \|$, with uniquely determined functionals $a_n(x)$. According to the Banach–Steinhaus theorem the partial sums

$$S_n x = \sum_{i=0}^{n} a_i(x) x_i \tag{1.8}$$

are uniformly bounded as linear operators on $X$ and the functionals $a_i$ are continuous.

It should be noted that for the interpolating systems with nodes at $\{t_i\}$, the functionals $a_n(f)$, for $f \in C(I)$, are necessarily of the form

$$a_n(f) = \begin{cases} f(t_0) & \text{for} \quad n = 0, \\ f(t_n) - S_{n-1} f(t_n) & \text{for} \quad n \geqslant 1, \end{cases} \tag{1.9}$$

and therefore $a_n$ is a linear combination of the functionals $\delta_i f = f(t_i)$, $i = 0, \dots, n$, for $n \geqslant 0$.

In the case of periodic functions systems satisfying (1.1) and (1.2) were constructed by Schonefeld [19] (cf. also [18]), for even values of $k$, and independently by Subbotin [21], [22], for any $k$. In the last two papers the best estimates of the order of approximation by means of a higher order of moduli of smoothness are given.

In this paper we construct simultaneous interpolating bases for $C^k(I)$, $k = 0, 2, \dots$ modifying slightly the Schonefeld's construction proposed in [18]. Moreover, we complete here the proof of Schonefeld (see Section 6) and compare both of the constructions with respect to their approximation properties. In the proofs of our estimates we depend on the Subbotin's papers [21], [22].

**2. Spline functions.** The progressive difference of order $r$ with increment $h > 0$ of the function $f$ is denoted as usually, i.e.,

$$\Delta_h^r f(s) = \sum_{i=0}^{r} \binom{r}{i} (-1)^{r+i} f(s+ih) \quad \text{for} \quad r \geqslant 0, h > 0, \tag{2.1}$$

whenever $s, s+h, \dots, s+rh$ are in the domain of $f$. The symbol $[s_0, s_1, \dots, s_r; f(\cdot)]$ denotes the divided difference of order $r$ of $f$ at $\{s_0, s_1, \dots, s_r\}$, i.e., $[s_0; f(\cdot)] = f(s_0)$ and for $r \geqslant 1$

$$[s_0, \dots, s_r; f(\cdot)] = \big([s_1, \dots, s_r; f(\cdot)] - [s_0, \dots, s_{r-1}; f(\cdot)]\big)/(s_r - s_0)$$
$$= \sum_{i=0}^{r} f(s_i) \Big[\prod_{\substack{j=0 \\ j \neq 0}}^{r} (s_i - s_j)\Big]^{-1}.$$

These notions are related as follows

$$h^r r! [s, s+h, \dots, s+rh; f(\cdot)] = \Delta_h^r f(s). \tag{2.2}$$

For the properties of the divided differences the reader is referred e.g. to [11].

By $\mathscr{Z}$, $\mathscr{N}$ and $\mathscr{R}$ we denote the set of all integers, non-negative integers and reals, respectively. For a bounded function $f$ defined on an interval $S \subset \mathscr{R}$, the expression

$$\omega_r(f; \delta) = \sup \{|\Delta_h^r f(s)| : 0 < h \leqslant \delta, s, s+rh \in S\} \tag{2.3}$$

defines the moduli of smoothness of $f$ of order $r$, $r \in \mathscr{N}$. For $r = 0$ we shall use simply $\|f\|_S$ for $\omega_0(f; \delta)$, where $\delta \geqslant 0$.

In the sequel the following basic properties of the moduli of smoothness will be used and no references will be made to them: $\omega_r(f; \delta) \leqslant \omega_r(f; \delta')$ if $\delta' \geqslant \delta$; $\omega_{r+1}(f; \delta) \leqslant 2\omega_r(f; \delta)$; $\omega_r(f; n\delta) \leqslant n^r \omega_r(f; \delta)$; $\omega_{r+k}(f; \delta) \leqslant \delta^k \times \omega_r(D^k f; \delta)$.

In the last inequality and below in (2.5)–(2.6) we denote by $D^k f$ such a function $g$ of bounded variation, whenever it exists, for which,

$$f(s) = H_a^k g(s) + P_{k-1}^{(a)}(s) \tag{2.4}$$

holds for $s$ and $a$ in the domain of $f$ (an interval $S \subset \mathscr{R}$), with $H_a h(s) = \int_a^s h(t)\,dt$ and $P_{k-1}^{(a)}$ being a uniquely determined by $a$ and $f$ polynomial of degree not greater than $k-1$. In this case integration by parts leads to the identity

$$f(s) = \frac{1}{k!} \int_a^s (s-t)^k \, dD^k f(t) + P_{k-1}^{(a)}(s) + \frac{1}{k!} (s-a)^k D^k f(a) \tag{2.5}$$

for $s \geqslant a$, $k \geqslant 1$. Using $x_+$ for $\max\{0, x\}$ and applying the divided difference to both sides of (2.5) (with $k$ replaced by $k-1$) we get

$$k! [s_0, \dots, s_k; f(\cdot)] = \int_a^b B(s_0, \dots, s_k; t)\, dD^{k-1} f(t) \tag{2.6}$$

for $k \geqslant 1$ whenever $s_0, \ldots, s_k$ are in $(a, b) \subset S$ (the domain of $f$), $f$ being continuous at $s_1$ in the case of $k = 1$. Here

(2.7) $$B(s_0, \ldots, s_k; t) = k[s_0, \ldots, s_k; (\cdot - t)_+^{k-1}] \quad \text{for} \quad t \in \mathcal{R},\ k \geqslant 1,$$

is the *B-spline of order* $k-2$ (*degree* $k-1$) *with knots at* $\{s_0, \ldots, s_k\}$.

The $B$-splines may be defined equivalently as follows whenever $s_0 < s_1 < \ldots < s_k$ [7]:

(2.8) $$B(s_0, \ldots, s_k; \cdot) \in C^{k-2}(\mathcal{R}) \quad \text{for} \quad k \geqslant 2.$$

(2.9)

*In each* $I_j = \langle s_{j-1}, s_j)$ *it is polynomial of degree* $\leqslant k-1$ *for* $j = 1, 2, \ldots, k$.

(2.10) $$B(s_0, \ldots, s_k; t) = 0 \quad \textit{for} \quad t \notin \langle s_0, s_k),$$

(2.11) $$\int_{\mathcal{R}} B(s_0, \ldots, s_k; t)\, dt = 1.$$

In the above definition $k \geqslant 1$. It follows that

(2.12) $$\|B(s_0, \ldots, s_k; \cdot)\|_{\mathcal{R}} \leqslant k/|s_k - s_0| \quad \textit{for} \quad k \geqslant 1,$$

and

(2.13) $$B(s_0, \ldots, s_k; t) = 0 \quad \text{iff} \quad t \notin (s_0, s_k) \quad \textit{for} \quad k \geqslant 2.$$

In particular, if there is a piece-wise continuous version of $D^k f$ in $\langle s_0, s_k\rangle$, identity (2.6) implies the generalized Lagrange–Taylor formula: there are $\bar{s} \in (s_0, s_k)$ and $\alpha \in \langle 0, 1\rangle$ such that

(2.14) $$k![s_0, \ldots, s_k; f(\cdot)] = \alpha D^k f(\bar{s} - 0) + (1-\alpha) D^k f(\bar{s} + 0).$$

Let $\mathcal{S} = \{\ldots < s_{-1} < s_0 < s_1 < \ldots\}$ be a partition of the real line $\mathcal{R}$ without cluster points and let $I_j = \langle s_{j-1}, s_j)$, $j \in \mathcal{Z}$. By $C_{\mathcal{S}}^m(S)$ we denote the (linear) space of all spline functions of order $m$ defined on the interval $S$, with knots at $\mathcal{S} \cap S$, i.e., each $\varphi \in C_{\mathcal{S}}^m(S)$ is $m$-times continuously differentiable and $\varphi$ is polynomial of degree $\leqslant m+1$ in each $I_j \cap S \neq \emptyset$, with $m \geqslant -1$. For $m = -1$ no condition about the continuity of $\varphi$ is imposed. It is obvious that $D^k \varphi \in C_{\mathcal{S}}^{m-k}(S)$ whenever $\varphi \in C_{\mathcal{S}}^m(S)$ for $k = 0, \ldots, m+1$ (the continuous from the right version of $D^k \varphi$ is to be chosen).

LEMMA 1. *Let* $\varphi \in C_{\mathcal{S}}^m(\mathcal{R})$, $m \geqslant -1$, $\mathcal{S}$ *being a fixed set of knots. The jumps*

(2.15) $$J^{(m+2)}\varphi(s_i) = D^{m+1}\varphi(s_i + 0) - D^{m+1}\varphi(s_i - 0)$$

*of the* $(m+1)$*-th derivative of* $\varphi$ *satisfy then the equations*

(2.16) $$\sum_{j=0}^{m} B_i^{(m)}(s_{i+j}) J^{(m+2)}\varphi(s_{i+j}) = (m+2)!\,[s_{i-1}, \ldots, s_{i+m+1}; \varphi(\cdot)]$$



*or* $i \in \mathcal{Z}$, *whenever* $m \geqslant 0$, *and for* $m = -1$ *we have*

(2.16′) $$B_i^{(-1)}(s_{i-1}) J^{(+1)}\varphi(s_i) = [s_{i-1}, s_i; \varphi(\cdot)].$$

In this lemma we denoted by

(2.17) $$B_i^{(m)}(t) = (m+2)[s_{i-1}, \ldots, s_{i+m+1}; (\cdot - t)_+^{m+1}]$$

the *i-th B-spline* $B(s_{i-1}, \ldots, s_{i+m+1}; \cdot)$ *of order* $m$ corresponding to the given $\mathcal{S}$ (cf. (2.7)). This lemma follows immediatelly from the Peano-form (2.6) of the divided differences.

In the sequel we shall consider only the case of equi-distant knots, i.e., when

$$|I_j| = s_j - s_{j-1} = h = \text{const} \quad \text{for} \quad j \in \mathcal{Z}.$$

Moreover, we restrict our considerations to splines of even order (odd degree), although some theorems will concern their derivatives of any order.

It follows from the results of Schoenberg [17] that

(2.18) $$B_i^{(2m+2)}(s_{i+j}) = G_{j-m-1}^{(m)} \quad \text{for} \quad i, j \in \mathcal{Z},\ m \geqslant -1,$$

where

(2.18′) $$G_l^{(m)} = (M_0^{(m)}, M_l^{(m)}) \quad \text{for} \quad l \in \mathcal{Z},\ m \geqslant -1$$

denotes the scalar product of *standarized B-splines* (cf. (2.17))

$$M_l^{(m)}(t) = (m+2)[l-1, \ldots, l+m+1; (\cdot - t)_+^{m+1}].$$

The index $2m+2$ is chosen in order to be consistent with the notation of [8]. Using this notation we can conclude from Lemma 1

LEMMA 1′. *The jumps* $J^{(2m+4)}\varphi(ih)$ *of the* $(2m+3)$*-th derivative of* $\varphi \in C_{\mathcal{S}}^{2m+2}(\mathcal{R})$ *satisfy the equations*

$$\sum_{j=-m-1}^{m+1} G_j^{(m)} J^{(2m+4)}\varphi\big((i+j)h\big) = h^{-2m-3} \Delta_h^{2m+4}\big((i-m-2)h\big)$$

*whenever* $\mathcal{S} = \{ih\colon i \in \mathcal{Z}\}$, $h > 0$, *for* $i \in \mathcal{Z}$.

The following result was announced during the Colloquium on Constructive Theory of Functions, held in Cluj, September 1973.

THEOREM A [9]. *There are constants* $C_m$ *and* $q_m \in (0, 1)$ *depending on* $m$ *only, such that for the inverse matrices* $\mathbf{A}_n^{(m)} = (\mathbf{G}_n^{(m)})^{-1} = (A_{n;i,j}^{(m)}\colon i, j = 0, 1, \ldots, n)$ *we have*

$$|A_{n;i,j}^{(m)}| \leqslant C_m q_m^{|i-j|} \quad \textit{for} \quad i, j = 0, 1, \ldots, n,$$

*uniformly in* $n \in \mathcal{N} = \{0, 1, \ldots\}$, *where*

$$G_{n;i,j}^{(m)} = G_{j-i}^{(m)} \quad \textit{for} \quad i, j = 0, 1, \ldots, n,\ n \in \mathcal{N}.$$

The proof of this theorem is quite similar to that of Theorem 3, case $l = 1$, of [8], which is presented there in Sections 8–10. Therefore it is omitted here. The only difference is in the following

LEMMA 2. *The $(m+1)\times(m+1)$-submatrix $\mathbf{D}_{n(a;a)}$ with the elements*

$$D_{n;jsl} = \sum_{k=0}^{n} G_{k-j}^{(m)} (\gamma_{l+1}^{(m)})^k \quad \text{for} \quad j, l = 0, \ldots, m,$$

*is non-singular for $n \geqslant 2m-1$, $m \geqslant 0$, where $\gamma_l^{(m)}$, with $l = \pm 1, \pm 2, \ldots \ldots, \pm(m+1)$, denote the roots of the characteristic polynomial*

$$\chi(z) = \sum_{l=-m-1}^{m+1} G_l^{(m)} z^l, \tag{2.19}$$

*numbered in such a way that $|\gamma_l^{(m)}| > 1$, whenever $l \in \{1, \ldots, m+1\}$.*

Proof. All the roots $\gamma_l^{(m)}$, $l = \pm 1, \ldots, \pm(m+1)$ are simple and negative (cf. [17], Lemma 8). Thus this lemma is a trivial consequence of the identity

$$D_{n;j,l} = -\sum_{k=-m-1}^{-1} G_{k-j}^{(m)} (\gamma_{l+1}^{(m)})^k,$$

where $(G_{k-j}^{(m)}\colon k = -m-1, \ldots, -1, j = 0, \ldots, m)$ is a triangular matrix with positive elements on the main diagonal (cf. (2.10), (2.13) and (2.18)) and $\big((\gamma_{l+1}^{(m)})^k\colon k, l = 0, \ldots, m\big)$ is the Vandermonde matrix of (different and negative) roots of (2.19), for $m \geqslant 0$.

**3. The special splines.** For $n = 1, 2, \ldots$, and $m \geqslant -1$ let us denote by $C_n^m = C_{\mathscr{P}_n}^m(I) \subset C^m(I)$ the $(n+m+1)$-dimensional space of splines of order $m$ with $\mathscr{S}_n = \{i/n\colon i = 0, \ldots, n\}$ as a set of knots. For $n = 1$ we shall use equivalently $C_1^m = \mathscr{P}_{m+1}$ to denote the space of polynomials of degree $\leqslant m+1$, for $m \geqslant -1$. It is obvious that $C_n^m \subset C_{l\cdot n}^m$ for $l, n = 1, 2, \ldots$

Let us consider the subspace $C_{n;0}^{m'}$ of all $\varphi \in C_n^{m'}$, with $m' = 2m+2$ (cf. Section 2), which satisfy the condition (cf. (2.15))

$$J^{(m'+2)}\varphi(i/n) \equiv J_{n;i}^{(m'+2)} = 0 \quad \text{for} \quad i = 1, \ldots, m+1, \quad n-m-1, \ldots, n-1. \tag{3.1}$$

We assume $C_{n;0}^0 = C_n^0$. Because of the linearity of the jumps $J_{n;i}$ we have

$$\dim C_{n;0}^{m'} \geqslant n+1 \quad \text{for } n \geqslant m'+2 \equiv 2m+4, \ m \geqslant -1. \tag{3.2}$$

It follows from Lemma 1′ and (3.1) that each $\varphi \in C_{n;0}^{m'}$ satisfies the following equations

$$\sum_{j=m+2}^{n-m-2} G_{j-i}^{(m)} J_{n;j}^{(m'+2)} = n^{m'+1} \Delta_{1/n}^{m+2} \varphi\big((i-m-2)/n\big) \tag{3.3}$$



for $i = m+2, \ldots, n-m-2$. Using the matrices $A_n^{(m)} = (G_n^{(m)})^{-1}$ defined in Theorem A, we get

$$\sum_{j=m+2}^{n-m-2} A_{n-2m-4;i-m-2,j-m-2}^{(m)} \Delta_{1/n}^{2m+4} \varphi\big((j-m-2)/n\big) = n^{-2m-3} J_{n;i}^{2m+4} \tag{3.4}$$

$$\text{for} \quad i = m+2, \ldots, n-m-2,$$

with $n \geqslant m'+2 = 2m+4$.

LEMMA 3. *The mapping $V_n\colon C_{n;0}^{m'} \to R^{n+1}$, where $(V_n\varphi)_i = \varphi(i/n)$ for $\varphi \in C_{n;0}^{m'}$, $i = 0, 1, \ldots, n$, is one-to-one and onto.*

Proof. According to (3.2) it is sufficient to consider the kernel of $V_n$. For this let $\varphi(i/n) = 0$ for all $i$'s, it follows that $J_{n;i}^{(m'+2)} = 0$ for $i = 1, 2, \ldots, n-1$ (cf. (3.1) and (3.4)). Thus $\varphi \in C_1^{m'} = \mathscr{P}_{m'+1}$ and has $n+1 \geqslant m'+3$ zeros at $\{0, 1/n, \ldots, 1\}$. Hence $\varphi = 0$. ∎

COROLLARY 1. *To each vector $(f_i\colon i = 0, 1, \ldots, n) \in R^{n+1}$ there is exactly one $\varphi \in C_{n;0}^m$ such that*

$$\varphi(i/n) = f_i \quad \text{for} \quad i = 0, 1, \ldots, n. \tag{3.5}$$

COROLLARY 2. *Let $\varphi_j \in C_{n;0}^{m'}$ be defined by the conditions*

$$\varphi_j(i/n) = \delta_{i,j} \quad \text{for} \quad i, j = 0, 1, \ldots, n. \tag{3.6}$$

*Then the following estimate*

$$|J^{(m'+2)}\varphi_j(i/n)| \leqslant C_m n^{m'+1} q_m^{|i-j|} \quad \text{for} \quad i, j = 1, 2, \ldots, n-1 \tag{3.7}$$

*holds with some constants $C_m > 0$ and $q_m \in (0, 1)$ depending on $m$ only.*

COROLLARY 3. *The matrices $A_n^{(m)}$ defined in Theorem A are uniformly bounded in $n$ as operators in the $(n+1)$-dimensional real Banach space with the maximum norm. In particular, it follows* (cf. (3.4)) *that the estimates*

$$\max_i |J_{n;i}^{(m'+2)}| \leqslant C_m n^{m'+1} \max_i |\Delta_{1/n}^{m'+2} \varphi(i/n)| \tag{3.8}$$

*hold for $\varphi \in C_{n;0}^{m'}$ and $n \geqslant m'+2$, with $C_m > 0$ depending on $m$ only, where $m' = 2m+2$, $m \geqslant -1$.*

LEMMA 4. *There is a constant $C_m$ depending on $m$ only, $m \geqslant -1$, such that the estimate*

$$l!\,|[t_0, \ldots, t_l;\ D^{m'+2-l}\varphi(\cdot)]| \leqslant C_m n^{m'+2} \max_i |\Delta_{1/n}^{m'+2} \varphi(i/n)| / \beta(n|T|) \tag{3.9}$$

*holds for $\varphi \in C_{n;0}^{m'}$, $n \geqslant m'+2$, $l \in \{1, 2, \ldots, m'+2\}$ and for any set $\tau = \{t_0, \ldots, t_l\} \subset I = \langle 0, 1\rangle$, $t_0 < t_1 < \ldots < t_l$, $T = \langle t_0, t_l\rangle$, $|T| = t_l - t_0$, where $\beta(u) = \min\{u, 1\}$.*

Proof. According to Lemma 1 the L.H.S. of (3.9) may be estimated as follows (cf. (2.6) and (2.12))

$$L_{(3.9)} \leqslant \|B_\tau\|_T \cdot \sum_{i \in \mathscr{S}'} |J_{n;i}^{(m'+2)}| \leqslant (m'+2) \cdot \# \mathscr{S}' \cdot \max_i |J_{n;i}^{(m'+2)}| / |T|,$$

where $\mathscr{S}' = \{i\colon i/n \in T\}$, and therefore $\#\mathscr{S}' \leqslant |T|n+1$. The estimate (3.9) follows now from (3.8). ∎

The property expressed by the following Theorem 1 *for the periodic case* is due to Subbotin and is given in an implicite form in [21] and [22]. With a lower order of differences it is given also by Schonefeld in [19], Lemma 2.3. This problem in the case of the interval was brought to my attention by Z. Ciesielski.

THEOREM 1. *There exists a constant $C_m > 0$ depending on $m$ only, $m \geqslant -1$, such that the estimate*

$$\omega_l(D^k\varphi;\, 1/n) \leqslant C_m n^k \max_i |\Delta_{1/n}^{k+l}\varphi(i/n)| \tag{3.10}$$

*holds for $\varphi \in C_{n;0}^{m'}$, $n \geqslant m'+2 = 2m+4$, $0 \leqslant k \leqslant m'+1$, $l \geqslant 0$ and $l+k \leqslant 2m+4$.*

Proof. According to (2.2) and Lemma 4 we have for $m'+1 \geqslant k = m'+2-l \geqslant 0$

$$\begin{aligned}|\Delta_h^l D^{m'+2-l}\varphi(t)| &\leqslant h^l n^{m'+2} C_m \max_i |\Delta_{1/n}^{m'+2}\varphi(i/n)|/nh \\ &\leqslant n^{m'+2-l} C_m \max_i |\Delta_{1/n}^{m'+2}\varphi(i/n)|\end{aligned}$$

whenever $0 < h \leqslant 1/n$.

For $0 \leqslant l+k \leqslant m'+1$ it is sufficient to check the case of $l=0$. According to the generalized Lagrange–Taylor formula (2.14) we have then

$$\begin{aligned}|D^k\varphi(t)| &\leqslant |n^k\Delta_{1/n}^k\varphi(i/n)| + \Big|\int_{\bar{s}}^{t} dD^k\varphi(s)\Big| + \alpha|D^k\varphi(\bar{s}+0) - D^k\varphi(\bar{s}-0)| \\ &\leqslant n^k \max_i |\Delta_{1/n}^k\varphi(i/n)| + (k+3)A_k(\varphi)/n\end{aligned} \tag{3.11}$$

with $0 \leqslant \alpha \leqslant 1$, $|\bar{s}-t| \leqslant (k+1)/n$, where

$$A_k(\varphi) = \begin{cases} n\max\limits_i |J_{n;i}^{(m'+2)}| & \text{for } k = m'+1, \\ \|D^{k+1}\varphi\|_I & \text{for } k = 0, 1, \ldots, m', \end{cases}$$

whenever $t \in \langle i/n, (i+k)/n\rangle$. Without loss of generality we have supposed that $\bar{s} \leqslant t$. Inequality (3.10) with $l=0$ follows now from (3.8) for $k = m'+1$ and by induction with respect to decreasing $k = m', m'-1, \ldots, 0$.

As a simple corollary of Theorem 1 we obtain the following Bernstein-type inequality

$$\|D^k\varphi\|_I \leqslant C_m n^k \max_i |\varphi(i/n)| \tag{3.12}$$

for $\varphi \in C_{n;0}^{m'}$, $n \geqslant m'+2$, $0 \leqslant k \leqslant m'+1$, and also the following one



$$\omega_l(\varphi;\, 1/n) \leqslant C_m \max_i |\Delta_{1/n}^l\varphi(i/n)| \tag{3.13}$$

for $\varphi \in C_{n;0}^{m'}$, $n \geqslant m'+2$, $0 \leqslant l \leqslant m'+2$, with the constant $C_m$ depending on $m$ only, and $m' = 2m+2$, $m \geqslant -1$.

THEOREM 2. *Let $\varphi_j \in C_{n;0}^{m'}$ be defined as in (3.6). Then the following local estimate*

$$|\Delta_h^l D^k\varphi_j(t)| \leqslant n^k C_m q_m^{n|t-(j/n)|} \tag{3.14}$$

*holds for $j \in \{0, 1, \ldots, n\}$, $t \in I$, $0 \leqslant k \leqslant m'+1$, $l \geqslant 0$, $l+k \leqslant m'+2$, $0 < h \leqslant 1/n$, $n \geqslant m'+2$, $m' = 2m+2$, $m \geqslant -1$, with some $C_m > 0$ and $q_m \in (0,1)$ depending on $m$ only.*

Notice that the R.H.S.'s of (3.9) and (3.11) may be made more local. Then Theorem 2 follows from (3.7) like Theorem 1 follows from (3.8). Therefore the proof of Theorem 2 is omitted here.

**4. Interpolation and approximation by the special splines.** The projections $\pi_n^{(m')}\colon C(I) \to C_{n;0}^{m'}$ defined for $m' = 2m+2$, $m \geqslant -1$, according to Corollary 1 of Section 3 as follows

$$\pi_n^{(m')}f(i/n) = f(i/n) \quad \text{for} \quad i = 0, 1, \ldots, n \tag{4.1}$$

induce the following mappings $\pi_n^{(m',k)}\colon C(I) \to C_{n;0}^{(m',k)} = D^k(C_{n;0}^{m'})$, where

$$\pi_n^{(\,,k)}f \equiv \pi_n^{(m',k)}f = D^k\pi_n^{(m')}H^k f \quad \text{for} \quad 0 \leqslant k \leqslant m'+1, \tag{4.2}$$

whenever $f \in C(I)$ and $Hf(t) = \int_0^t f(s)\,ds$ for $t \in I$.

Obviously, $\pi_n^{(\,,k)}$ is a projection onto $C_{n;0}^{(m',k)} \equiv C_{n;0}^{(\,,k)})$, because the polynomials $\mathscr{P}_{k-1}$ (the kernel of $D^k$) are contained in $C_{n;0}^{m'}$, for $k = 0, 1, \ldots \ldots, m'+1$ and $\pi_n^{(m')}$ is a projection. It follows, moreover, that

$$D^k\pi_n^{(m',l)}f = \pi_n^{(m',l+k)}D^k f \tag{4.4}$$

for $f \in C^k(I)$, $k \geqslant 0$, $l \geqslant 0$, $k+l \leqslant m'+1$.

According to Theorem 1 the operators $\pi_n^{(\,,k)}$ are bounded in the sup-norm, uniformly in $n$. Indeed,

$$\begin{aligned}\|\pi_n^{(\,,k)}g\|_I &\equiv \|D^k\pi_n^{(m')}f\|_I \leqslant n^k C_m \max_i |\Delta_{1/n}^k\varphi(i/n)| \\ &\leqslant n^k C_m \max_i |\Delta_{1/n}^k f(i/n)| \leqslant C_m \|D^k f\|_I \equiv C_m \|g\|_I\end{aligned} \tag{4.5}$$

for $k = 0, 1, \ldots, m'+1$, where $f = H^k g$, $\varphi = \pi_n^{(m')}f$ (cf. (4.1)).

LEMMA 5. *The projections $\pi_n^{(\,,k)}$ are uniformly bounded as operators in $\big(C^l(I), \|\ \|^{(l)}\big)$, whenever $0 \leqslant l+k \leqslant m'$.*

Remark. In the case of $l+k = m'+1$ the statement of Lemma 5 fails because the derivatives $D^l\pi_n^{(\,,k)}f$ are not continuous.

For further considerations we need some basic notions of the approximation theory. Let

$$E_{X_0}(x) = \inf\{\|x-x'\|: x' \in X_0\}$$

denote the best approximation of $x \in (X, \|\ \|)$ by elements of $X_0 \subset X$, $X$ being a normed space. For finite-dimensional subspaces $X_0$ there exists $x' \in X_0$ such that $E_{X_0}(x) = \|x-x'\|$. In this case the standard estimate

$$E_{X_0}(x) \leqslant \|x-\pi_0(x)\| \leqslant (1+\|\pi_0\|)E_{X_0}(x) \tag{4.6}$$

may be obtained for any linear projection $\pi_0$ onto $X_0$.

The following result of Whitney (cf. [23], Theorem 1):

$$E_{P_k}(f) \leqslant C_k\omega_{k+1}(f; |T|) \quad \text{for} \quad f \in C(T),\ k \geqslant 0, \tag{4.7}$$

with $C_k$ depending on $k$ only, and also (4.6) give us

$$\|f-\pi_\tau f\|_T \leqslant [1+(k+1)(|T|/\underline{\tau})^{k+1}]C_k\omega_{k+1}(f; |T|), \tag{4.8}$$

where $\pi_\tau f$ denotes the unique polynomial of degree $\leqslant k+1$ which interpolates $f$ at $\tau = \{t_0, \ldots, t_k\} \subset T$, and $\underline{\tau} = \min\{|t_i-t_j|: i \neq j,\ i,j = 0, 1, \ldots, k\}$.

THEOREM 3. *Let $k \in \{0, 1, \ldots, m'+1\}$, $m' = 2m+2$, $m \geqslant -1$. The estimate*

$$\|D^kf - D^k\pi_n^{(m)}f\|_I \leqslant C_m\omega_{m'+2-k}(D^kf; 1/n) \tag{4.9}$$

*holds for $f \in C^k(I)$ and $n \geqslant m'+2$, with some $C_m > 0$ depending on $m$ only.*

The proof is the same as in the periodic case (cf. [21]):

Using the Rolle Theorem we can state that for large enough $n \geqslant n_m$ the difference $D^kf - D^k\pi_n^{(m')}f$ has at least $n+1-k$ zeros at $t_i^{(k)}$, which satisfy the conditions

$$0 \leqslant t_0^k < t_1^k < \ldots < t_{n-k}^{(k)} \leqslant 1, \quad t_0^{(k)} \leqslant k/n \quad \text{and} \quad 1 - t_{n-k}^{(k)} \leqslant k/n,$$
$$t_i^{(k)} - t_{i-1}^{(k)} \leqslant (2k+1)/n, \quad t_{i+2k}^{(k)} - t_i^{(k)} \geqslant 1/n$$

for all $i \in \{0, 1, \ldots, n-k\}$ for which the L.H.S.'s are sensefull. Thus we can cover the interval $I$ by subintervals $T_j \subset I$ of the length $|T_j| \leqslant C'_m/n$, i.e., $T_1 \cup T_2 \cup \ldots \cup T_{j_0} = I$ with

$$\tau_j = \{t_{j,0}, t_{j,1}, \ldots, t_{j,m'+1-k}\} \subset \{t_0^{(k)}, \ldots, t_{n-k}^{(k)}\}, \quad \tau_j \subset T_j,$$

for which $\underline{\tau_j} \geqslant 1/n$. Hence, according to (4.8), the estimate

$$\|D^kf - \pi_{\tau_j}D^kf\|_{T_j} \leqslant C_m\omega_{m'+2-k}(D^kf; 1/n) \tag{4.10}$$

holds for $f \in C^k(I)$, $n \geqslant n_m$ and $j \in \{1, \ldots, j_0\}$, with some $C_m > 0$ depending on $m$ only.

Moreover,

$$\pi_{\tau_j}D^kf = \pi_{\tau_j}D^k\pi_n^{(m')}f \quad \text{for} \quad j = 1, \ldots, j_0.$$



Thus, according to (4.10), Theorem 1, and (4.1)

$$\begin{aligned}\|D^k\varphi - \pi_{\tau_j}D^kf\|_{T_j} &\leqslant C_m\omega_{m'+2-k}(D^k\varphi; 1/n) \\ &\leqslant C_m\max_i|\Delta_{1/n}^{m'+2}\varphi(i/n)| \leqslant C_m\omega_{m'+2-k}(D^kf; 1/n),\end{aligned} \tag{4.11}$$

for all $j$, where $\varphi = \pi_n^{(m')}f$. Inequalities (4.10) and (4.11) imply (4.9). ∎

**5. Smooth interpolating bases.** As above, let the parameter $m' = 2m+2$, $m \geqslant -1$, be fixed. Let, moreover, $t_n$ denote the $n$th dyadic point of $I$ (cf. (1.3)). The interpolating sequence $(\varphi_n^{(m')}: n \geqslant 0) \subset C^{m'}(I)$ is now defined as follows (cf. (1.4)):

1° For $0 \leqslant n \leqslant m'+1$ $\varphi_n^{(m')}$ is the unique polynomial of degree not greater than $n$ with the property

$$\varphi_n^{(m')}(t_i) = \delta_{i,n} \quad \text{for} \quad 0 \leqslant i \leqslant n,\ n = 0, 1, \ldots, m'+1. \tag{5.1}$$

2° For $n \geqslant m'+2$ $\varphi_n^{(m')}$ is the unique element of $C_{2N;0}^{m'}$ which satisfies

$$\varphi_n^{(m')}(t_i) = \delta_{i,n} \quad \text{for} \quad 0 \leqslant i \leqslant 2N,\ n \geqslant m'+2. \tag{5.1'}$$

In this condition $N$ depends on $n$, cf. (1.3).

This definition is a generalization of the original Schauder basis in $C(I)$, which we obtain here for $m = -1$.

THEOREM 4. *For each $m \geqslant -1$ the sequence $(\varphi_n^{(m')}: n \geqslant 0)$ is a Schauder basis in each of the Banach spaces $(C^l(I), \|\ \|^l)$ for $l = 0, 1, \ldots, m'$, i.e., it is a simultaneous basis for $C^{m'}(I)$ in the sense of Schonefeld* (cf. [18], [19] and (1.1)).

*The coefficients $a_n^{(m')}$ of the expansion $f = \sum_{n=0}^{\infty} a_n^{(m')}(f)\varphi_n^{(m')}$ are determined as follows*

$$a_n^{(m')}(f) = \begin{cases} f(t_0) & \text{for} \quad n = 0, \\ f(t_n) - S_{n-1}^{(m')}f(t_n) & \text{for} \quad n \geqslant 1, \end{cases} \tag{5.2}$$

*where*

$$S_n^{(m')}f = \sum_{i=0}^{n} a_i^{(m')}(f)\varphi_n^{(m')} \quad \text{for} \quad n \geqslant 0,\ f \in C(I). \tag{5.3}$$

*Moreover, the estimate*

$$\|D^lf - D^lS_n^{(m')}f\|_I \leqslant C_m\omega_{m'+2-l}(D^lf, 1/n) \tag{5.4}$$

*holds for $f \in C^l(I)$, $l \in \{0, 1, \ldots, m'+1\}$ and $n \geqslant m'+2$ with $C_m > 0$ depending on $m$ only.*

Proof. It is sufficient to prove (5.4), because (5.2) was proved in Section 1, cf. (1.9). The proof of (5.4) is standard (cf. [19] and [21]):

Obviously,

$$(5.5)\qquad S_n^{(m')}f\in\begin{cases}\mathscr{P}_n & \text{for}\quad 0\leqslant n\leqslant m'+1,\\ C_{2N;0}^{m'} & \text{for}\quad n\geqslant m'+2,\end{cases}$$

for all $f\in C(I)$. Moreover,

$$S_n^{(m')}f(t_i)=f(t_i)\quad\text{for}\quad i=0,1,\ldots,n,\ n\geqslant 0.$$

Thus $S_{2N}^{(m')}f=\pi_{2N}^{(m')}f$ for $2N\geqslant m'+2$. Now, estimates (5.4) for $n=2^{n+1}=2N$ follow from Theorem 2. For other values of $n$ which satisfy the inequalities $2N>n>N'=\max\{m'+1,N\}$, let us write

$$g_n=S_{2N}^{(m')}f-S_n^{(m')}f.$$

Thus $g_n\in C_{2N;0}^{m'}$ and

$$g_n(t_i)=\begin{cases}0 & \text{for}\quad i=0,\ldots,n,\\ (S_{2N}^{(m')}f-S_N^{(m')}f)(t_i) & \text{for}\quad i=n+1,\ldots,2N.\end{cases}$$

The Bernstein inequality (3.12) allows us to estimate

$$\|D^lg_n\|_I\leqslant C_m(2N)^l\|S_{2N}^{(m')}f-S_N^{(m')}f\|_I$$

for all $n\geqslant m'+2$. Now, applying Theorem 3 to the R.H.S. of the last inequality we obtain (cf. also Theorem 1):

$$\begin{aligned}\|S_{2N}^{(m')}f-S_{N'}^{(m')}S_{2N}^{(m')}f\|&\leqslant C_m\omega_{m'+2}(S_{2N}^{(m')}f;1/N')\\&\leqslant C_m\max_i|\Delta_{1/2N}^{m'+2}S_{2N}^{(m')}f(i/2N)|\\&\leqslant C_m\max_i|\Delta_{1/2N}^{m+2}f(i/2N)|\\&\leqslant C_m\omega_{m'+2}(f;1/n),\end{aligned}$$

whence we infer easily (5.4). In the last estimates we have used the result of Whitney in the case of $N'=m'+1$ and the properties of moduli of smoothness.

COROLLARY 4. *For each $l\in\{0,\ldots,m'\}$ the sequence $(D^l\varphi_n^{(m')}\colon n\geqslant l)$ is a basis for $C(I)$, i.e., $(\varphi_n^{(m')};\ n\geqslant 0)$ is a simultaneous basis for $C^{m'}(I)$ in the sense of Ciesielski* (cf. [4]).

Indeed, let

$$(5.6)\qquad \|S_n^{(\cdot,l)}f-f\|_I\to 0\quad\text{as}\quad n\to\infty\quad\text{for}\quad f\in C(I),$$

where

$$(5.7)\qquad \sum_{i=l}^{n}a_i^{(\cdot,l)}(f)D^l\varphi_i^{(m')}=S_n^{(\cdot,l)}f\quad\text{for}\quad n\geqslant l,$$

with some continuous functionals $a_i^{(\cdot,l)}$ defined on $C(I)$. It follows from the uniform boundedness of $S_n^{(m')}$, that (cf. (4.3))

$$\|T_nH^lS_n^{(\cdot,l)}f-T_nH^lf\|\to 0\quad\text{as}\quad n\to\infty,$$



where $T_n=S_n^{(m')}-S_{l-1}^{(m')}$ for $n\geqslant l$. Thus, according to Theorem 1,

$$T_nH^lS_n^{(\cdot,l)}f\equiv\sum_{i=1}^{n}a_i^{(\cdot,l)}(f)\varphi_i^{(m')}\to H^lf-S_{l-1}^{(m')}H^lf,$$

and, by the uniqueness of the expansion of $H^lf$ with respect to $(\varphi_n^{(m')}\colon n\geqslant 0)$, we can state that

$$T_nH^lS_n^{(\cdot,l)}f=T_nH^lf\quad\text{for all } f\in C(I).$$

It follows that

$$S_n^{(\cdot,l)}f\equiv D^lT_nH^lS_n^{(\cdot,l)}f=D^lS_n^{(m')}H^lf$$

and this proves that

$$a_i^{(\cdot,l)}(f)\equiv a_i^{(m',l)}(f)=a_i^{(m')}H^lf\quad\text{for}\quad i\geqslant l,\ f\in C(I),$$

and, according to Theorem 4, the uniform convergence of (5.6). ■

**6. A comparison with the Schonefeld construction.** In the case of equi-distant knots it follows from Lemma 1 that

$$(6.1')\qquad \sum_{i=-m-1}^{m+1}G_j^{(m)}D^{m'+1}\varphi\big((i+j)h+0\big)=h^{-m'-1}\Delta_h^{m'+1}\varphi\big((i-m-1)h\big)$$

and also that

$$(6.1'')\qquad \sum_{j=-m-1}^{m+1}G_j^{(m)}D^{m'}\varphi\big((i+j)h\big)=h^{-m'}\Delta_h^{m'}\varphi\big((i-m-1)h\big)$$

holds for $\varphi\in C_{\mathscr{S}}^{m'}(\mathscr{R})$ with $\mathscr{S}=\{ih\colon i\in\mathscr{Z}\}$, for $i\in\mathscr{Z}$.

Schonefeld has used in [18] the $m'$-constant splines on $I$ with knots at $0,1/n,\ldots,1$, for $n\geqslant m'$, which are determined by conditions

$$D^{m'+1}\varphi(i/n)=0\quad\text{for}\quad i=0,\ldots,m,n-m-1,\ldots,n-1$$

(cf. Section 2, where the continuity from the right is supposed). Applying then (6.1″) he obtained equations for the vectors $\big(D^{m'}\varphi(i/n)\colon i=m+1,\ldots,n-m-2\big)$ with rather complicated matrices, corresponding to our matrices $G_n^{(m)}$(cf. Theorem A and Lemma 1′). The matrices are diagonally dominated only for $m=-1,0,1$ ($m'=0,2,4$) and therefore the proof presented in [18] was complete only for these values of $m'$. However, using (6.1′) we arrive at equations for $D^{m'+1}\varphi$ for which the matrices $G_n^{(m)}$ deal as coefficient matrices:

$$(6.2)\qquad \sum_{j=m+1}^{n-m-1}G_{j-i}^{(m)}D^{m'+1}\varphi(j/n+0)=n^{m'+1}\Delta_{1/n}^{m'+1}\varphi\big((i-m-1)/n\big)$$

for $\varphi$ being $m'$-constant spline on $I$ with knots at $0,1/n,\ldots,1$. Using now Theorem A we can obtain in the Schonefeld's case all of the above theorems if we change the order of differences replacing $m'+2$ by $m'+1$.

In particular, we have for the $m'$-constant splines

$$(6.3)\quad \omega_l(D^k\varphi; 1/n) \leqslant C_m n^k \max_i |\Delta_{1/n}^{k+l}\varphi(i/n)| \quad \text{for} \quad 0 \leqslant l+k \leqslant m'+1,$$

$$(6.4)\quad \|D^k f - D^k \pi_n^{(m')} f\|_I \leqslant C_m \omega_{m'+1-k}(D^k f, 1/n) \quad \text{for} \quad k = 0, \ldots, m'+1,$$

where $\pi_n^{(m')} f$ denotes the unique $m'$-constant spline with knots at $0, 1/n, \ldots$ $\ldots, 1$, which interpolates $f$ at these points, for $f \in C^k(I)$. The above estimates correspond to Theorems 1 and 3, respectively. These estimates allow to construct a simultaneous basis for $C^{m'}(I)$ which is interpolating with nodes at the dyadic points and a theorem similar to Theorem 4 may be proved. In fact, all the corresponding properties but estimate (5.4) are formulated by Schonefeld in [18].

**7. Final remarks.** The estimates obtained in Theorem 3 for the order of approximation by the moduli of smoothness are well known in the literature for approximation by splines being elements of the whole space $C_n^{m'} \supset C_{n;0}^{m'}$, for $n \geqslant m'+2$ (see e.g. [22], [2], [10] and for an extra moduli of smoothness in the case of free knots see [13]). For the dyadic partitions of the interval ($\{0, 1/n, \ldots, 1\}$ replaced by $\{t_0, t_1, \ldots, t_n\}$, cf. (3.1)) such estimates are given by Ciesielski [5] for the $L_p$-spaces. Similar results were obtained by Scherer [15] for a wider class of partitions of $I$.

The interpolation processes presented by Schonefeld and in this paper are especially interesting, they do not require any differentiability assumptions on the interpolated function. This property makes it similar to the local interpolation by splines of higher order degree (with defect) as is done by Riabenkiĭ and Filippov [14], cf. also [2].

It should be noted that according to theorem of Schonefeld [18] (cf. also [19] and [6]) the property of the basis constructed in Section 5, allows us to use the products $\varphi_{i_1(n)}^{(m')}(t_1)\cdot\varphi_{i_2(n)}^{(m')}(t_2)\cdot\ldots\cdot\varphi_{i_d(n)}^{(m')}(t_d)$, if suitably numbered, as a (simultaneous) interpolating basis for the space $C^{m'}(I^d)$ of $m'$-times continuously differentiable functions on the Euclidean cube, for $m' = 2m+2$, $m \geqslant -1$.

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