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Exhaustive measures in arbitrary topological vector spaces

by

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Abstract. In the paper bounded finitely additive measures with values in arbitrary topological vector spaces are investigated. This leads to a generalization of C-spaces of L. Schwartz and O-spaces investigated in different papers, for about thirty years now, by Orlicz. For every pair of infinite cardinal numbers $\mathfrak{m}, \mathfrak{n}$ we introduce topological vector spaces of type $C_\mathfrak{n}(\mathfrak{m})$ and $O_\mathfrak{n}(\mathfrak{m})$. In this classification C-space and O-space mentioned above are $C_{\aleph_0}(\aleph_0)$-space and $O_{\aleph_0}(\aleph_0)$-space, respectively. Necessary and sufficient conditions for a complete topological vector space to be a $C_\mathfrak{n}(\mathfrak{m})$- and an $O_\mathfrak{n}(\mathfrak{m})$-space are given. Examples of $C_\mathfrak{n}(\mathfrak{m})$-spaces which are not $O_\mathfrak{n}(\mathfrak{m})$-spaces are presented. In the last section exhaustivity properties of Radon measures are considered.

0. In this paper bounded finitely additive measures with values in arbitrary (Hausdorff) topological vector spaces are investigated. This leads to a generalization of C-spaces of L. Schwartz and O-spaces investigated in different papers, for about thirty years now, by Orlicz. For every pair of infinite cardinal numbers $\mathfrak{m}, \mathfrak{n}$ we introduce topological vector spaces of type $C_\mathfrak{n}(\mathfrak{m})$ and $O_\mathfrak{n}(\mathfrak{m})$. In this classification C-space and O-space mentioned above are $C_{\aleph_0}(\aleph_0)$-space and $O_{\aleph_0}(\aleph_0)$-space, respectively. Until now the only full characterization, in the setting of arbitrary topological vector spaces, was known (Kalton [3]) for C-spaces. Namely, a complete topological vector space X is a C-space iff X contains no subspace isomorphic to c_0. In the present paper, basing on a recent result of Drewnowski [1], necessary and sufficient conditions for a complete topological vector space to be a $C_\mathfrak{n}(\mathfrak{m})$-, and an $O_\mathfrak{n}(\mathfrak{m})$-space are given. Complete $C_\mathfrak{n}(\mathfrak{m})$-spaces, which are not $O_\mathfrak{n}(\mathfrak{m})$-spaces are constructed. In the last section exhaustivity properties of Radon measures are considered.

The results of this paper were anounced in [5], where the (probably most important) countable case was studied. The reader may find therein more references to the earlier work and the motivation for what follows.(1)

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(1) See, however, "added in proof".

1. Let Γ be a set and $\mathfrak{m}$, $\mathfrak{n}$ two infinite cardinal numbers. We denote by $\operatorname{card}\Gamma$ the cardinality of Γ, $\mathfrak{F}(\Gamma)$-the ring of finite subsets of Γ, $\mathfrak{P}(\Gamma)$-the power set of Γ. We say that a ring $\mathfrak{A}$ of sets is an $\mathfrak{m}$-*ring* if the union of any $\mathfrak{m}$ disjoint elements of $\mathfrak{A}$ belongs to $\mathfrak{A}$.

If the set Γ is fixed and $\operatorname{card}\Gamma = \mathfrak{m}$, then we denote $\mathfrak{D}_\mathfrak{n}(\mathfrak{m}) = \{E \subset \Gamma: \operatorname{card}E < \mathfrak{n}\}$. With this notation if $\mathfrak{n} > \mathfrak{m}$, then $\mathfrak{D}_\mathfrak{n}(\mathfrak{m}) = \mathfrak{P}(\Gamma)$.

Let X be a Hausdorff topological vector space (tvs), $\mathfrak{A}$ a ring of sets. A finitely additive set function is called *measure*; a measure $x(:)$: $\mathfrak{A} \to X$ (denoted also $\mathfrak{A} \ni E \mapsto x_E \in X$) is said to be $\mathfrak{m}$-*exhaustive* if for any family $(E_\iota)_{\iota\in I} \subset \mathfrak{A}$ of pairwise disjoint sets $\operatorname{card}\{x(E_\iota): x(E_\iota) \notin V\} < \mathfrak{m}$ for each neighbourhood V of zero in X. $x(:)$ is said to be *bounded* if its range, $x(\mathfrak{A})$, is a bounded set in X; it is said to be *convexly bounded* if the convex hull of its range, $\operatorname{conv}(x(\mathfrak{A}))$, is a bounded set in X.

We will say that X *contains a discrete copy of a ring* $\mathfrak{A}$ if there exist a neighbourhood W of zero in X, and a measure $x(:)$: $\mathfrak{A} \to X$ such that for any $E_1, E_2 \in \mathfrak{A}$

$$E_1 \neq E_2 \Rightarrow x(E_1) - x(E_2) \notin W.$$

In the sequel $l^\infty(\Gamma) \equiv l^\infty(\mathfrak{m})$ denotes the Banach space of bounded real-valued functions on Γ (with $\operatorname{card}\Gamma = \mathfrak{m}$) endowed with the supremum norm. $c_\mathfrak{n}(\mathfrak{m})$ is the Banach subspace of $l^\infty(\mathfrak{m})$ consisting of those $c(\cdot) \in l^\infty(\mathfrak{m})$ that $\operatorname{card}\{\gamma: |c(\gamma)| > \varepsilon\} < \mathfrak{n}$ for any $\varepsilon > 0$. With this notation, for $\mathfrak{n} > \mathfrak{m}$, $c_\mathfrak{n}(\mathfrak{m}) = l^\infty(\mathfrak{m})$; the space $c_{\aleph_0}(\mathfrak{m})$ is denoted as usually by $c_0(\mathfrak{m}) \equiv c_0(\Gamma)$. $s_\mathfrak{n}(\mathfrak{m})$ is the subspace of $c_\mathfrak{n}(\mathfrak{m})$ consisting of finitely many valued functions. It may be treated as the space of simple functions over $\mathfrak{D}_\mathfrak{n}(\mathfrak{m})$.

A measure $x(:)$: $\mathfrak{D}_\mathfrak{n}(\mathfrak{m}) \to X$ is convexly bounded iff, taking simple functions

$$\sum_{i=1}^n a_i \chi_{E_i} = s \in s_\mathfrak{n}(\mathfrak{m})$$

with $\|s\| \leqslant 1$, the corresponding set of values $\{\sum_{i=1}^n a_i x_{E_i}\}$ is bounded in X; χ_E denotes the indicator function of the set $E \subset \Gamma$. A tvs X is said to be a $C_\mathfrak{n}(\mathfrak{m})$-*space* if every convexly bounded X-valued measure on a ring $\mathfrak{D}_\mathfrak{n}(\mathfrak{m})$ is $\mathfrak{m}$-exhaustive. A tvs X is said to be an $O_\mathfrak{n}(\mathfrak{m})$-*space* if every bounded X-valued measure on a ring $\mathfrak{D}_\mathfrak{n}(\mathfrak{m})$ is $\mathfrak{m}$-exhaustive.

Every $O_\mathfrak{n}(\mathfrak{m})$-space is a $C_\mathfrak{n}(\mathfrak{m})$-space.

A complete metrizable tvs is called an *F-space*. Everywhere below Γ denotes a set with $\operatorname{card}\Gamma = \mathfrak{m}$.

2. The measure

$$\mathfrak{D}_\mathfrak{n}(\mathfrak{m}) \ni E \mapsto \chi_E \in c_\mathfrak{n}(\mathfrak{m})$$

will be called the *canonical indicator measure* of $c_\mathfrak{n}(\mathfrak{m})$. It provides example of a bounded non-$\mathfrak{m}$-exhaustive Banach space valued measure on rings and $\mathfrak{n}$-rings of sets.

A tvs X is said to have *bounded multiplier property* (bmp) iff it fulfils the following condition:

For each neighbourhood U of zero in X there is a neighbourhood $V \subset U$ of zero in X such that for any finite sequence $(x_1, \ldots, x_n) \subset X$:

$$A = \Big\{\sum_{i\in E} x_i:\ E \subset \{1, 2, \ldots, n\}\Big\} \subset V \text{ implies}$$

(BMP)

$$\Big\{\sum_{i\in E} a_i x_i:\ E \subset \{1, 2, \ldots, n\},\ 0 \leqslant a_i \leqslant 1\Big\} = \operatorname{conv}(A) \subset U.$$

Lemma 1. *A tvs X has bmp iff it can be embedded in a product $\prod_{i\in I} X_i$ of metrizable tvs's having bmp.*

Proof. Let $\mathfrak{U}$ be a base of neighbourhoods of zero in X. For every U in $\mathfrak{U}$ we construct a sequence (V_m) of balanced neighbourhoods of 0 with the following properties:

(1) $V_1 = U$,

(2) V_{m-1} being defined, V_m is chosen such that $V_m + V_m \subset V_{m-1}$ and for V_m condition (BMP) is satisfied with V_{m-1}.

Denoting by p_U the F-semi-norm generated by (V_m), the space (X, p_U) has bmp. Moreover, the the family $(p_U: U \in \mathfrak{U})$ of F-semi-norms defines the original topology of X which implies the "only if" part of the lemma. The "if" part is easy and the proof will be omitted.

Lemma 2. *A metrizable tvs X has bmp iff every unconditionally Cauchy series in X is bounded multiplier Cauchy (i.e., if $\sum_{n=1}^\infty x_n$ is unconditionally Cauchy, then for each $(a_n) \in l^\infty$, $\sum_{n=1}^\infty a_n x_n$ is Cauchy).*

Proof. Let $\|\ \|$ be an F-norm defining the topology of X. The "only if" part being contained in Lemma 3 below, we proceed to the proof of the "if" part.

Take $\varepsilon_k > 0$ such that $\sum_{k=1}^\infty \varepsilon_k < \infty$. If (BMP) is not satisfied, then we find $\delta > 0$ and ε_n, $n \geqslant n_0$, such that for each ε_n there exists (x_i^n), $i \in K_n \in \mathfrak{F}(\boldsymbol{N}$ with the property

(1) $\|\sum_{i\in E} x_i^n\| \leqslant \varepsilon_n$ for each $E \subset K_n$,

(2) $\|\sum_{i\in E} a_i x_i^n\| > \delta > 0$ for some $E \subset K_n$ and $0 \leqslant a_i \leqslant 1$.

Choosing (x_i^n) for which (2) holds and arranging them one after another, we obtain an infinite sequence, (x_m) say, which is unconditionally Cauchy. In the same time for some sequence $(a_m) \in l^\infty$, the series $\sum_{m=1}^{\infty} a_m x_m$ is not Cauchy, which proves the lemma.

In view of the above lemmas and recent results of Kashin, Maurey, Pisier and Turpin (see e.g. [4], [12]) the class of spaces having bmp contains, except semi-convex spaces, quite a large number of other spaces.

LEMMA 3. *Let $\mathfrak{A}$ be a ring of sets, X a tvs having bmp and $x(:)\colon \mathfrak{A}\to X$ a bounded measure. Then $x(:)$ is convexly bounded.*

Proof. Given a neighbourhood U of zero in X, let V be such a neighbourhood of zero in X that (BMP) is satisfied. Since $x(:)$ is bounded, we can find $a > 0$ such that $ax(\mathfrak{A}) \subset V$. Then clearly, $a\operatorname{conv}x\big((\mathfrak{A})\big) \subset U$.

THEOREM 1. *Let X be a sequentially complete tvs having bmp, $\mathfrak{n}$ an infinite cardinal number. The following are equivalent.*

(i) *X is an $O_{\mathfrak{n}}(\mathfrak{m})$-space.*

(ii) *$x \not\supset c_{\mathfrak{n}}(\mathfrak{m})$, i.e., no subspace of X is linearly homeomorphic to $c_{\mathfrak{n}}(\mathfrak{m})$.*

With the same notation

COROLLARY. A. *The following are equivalent.*

(i) *Every X-valued bounded measure on an $\mathfrak{m}$-ring of sets is $\mathfrak{m}$-exhaustive.*

(ii) *$X \not\supset l^\infty(\mathfrak{m})$.*

B. *The following are equivalent.*

(i) *Every X-valued bounded measure on a ring of sets is $\mathfrak{m}$-exhaustive.*

(ii) *$X \not\supset c_0(\mathfrak{m})$.*

Proof of the theorem. If X contains $c_{\mathfrak{n}}(\mathfrak{m})$, then the canonical indicator measure of $c_{\mathfrak{n}}(\mathfrak{m})$, treated as X-valued, is a bounded non-$\mathfrak{m}$-exhaustive measure. Hence (i) $\Rightarrow$ (ii). (ii) $\Rightarrow$ (i). Assume $x(:)\colon \mathfrak{O}_{\mathfrak{n}}(\mathfrak{m})\to X$ is bounded, hence convexly bounded by Lemma 3. If $x(:)$ is not $\mathfrak{m}$-exhaustive, we can find $\mathfrak{m}$ disjoint sets in $\mathfrak{O}_{\mathfrak{n}}(\mathfrak{m})$, $(E_\iota)_{\iota\in I}$ say, such that $x(E_\iota)\notin U$ for some neighbourhood U of zero in X and $\iota\in I$. We may identify I with Γ and the ring $\mathfrak{O}_{\mathfrak{n}}(I)$ generated on I with $\mathfrak{O}_{\mathfrak{n}}(\mathfrak{m})$. Transporting $x(:)$ on $\mathfrak{O}_{\mathfrak{n}}(I)$ in a usual way, we get a measure on $\mathfrak{O}_{\mathfrak{n}}(\mathfrak{m})$, which we denote still by $x(:)$, with the property $x(\{\gamma\})\notin U$ for any $\gamma\in\Gamma$. As $x(:)$ is convexly bounded, its integral extension $\tilde{x}\colon s_{\mathfrak{n}}(\mathfrak{m})\to X$ is continuous, and consequently extends by continuity to a bounded linear map (denoted still by $\tilde{x}$) $\tilde{x}\colon c_{\mathfrak{n}}(\mathfrak{m})\to X$. The result now follows by Drewnowski's Theorem [1].

An examination of the proof shows that we have proved also the following

THEOREM 2. *Let X be a sequentially complete tvs, $\mathfrak{n}$ an infinite cardinal number. The following are equivalent.*

(i) *X is a $C_{\mathfrak{n}}(\mathfrak{m})$-space.*

(ii) *$X \not\supset c_{\mathfrak{n}}(\mathfrak{m})$.*

In connection with Theorem 1 it may be worth to notice that among F-spaces arising in a natural manner in analysis, one encounters both — those which do not "contain" c_0, and those which "contain" l^∞. Probably the most important class of such F-spaces is that of Orlicz spaces L^φ. They are "proper" F-spaces (i.e., not locally bounded in general) and, as proved in [4], [12], they have bmp. Also it is known, mainly thank to the work of Orlicz himself, that L^φ with absolutely continuous norm are $O_{\aleph_0}(\aleph_0)$-spaces, hence do not "contain" c_0. On the other hand, if L^φ has the norm which is not absolutely continuous, then L^φ contains a subspace isomorphic to l^∞.

When no convexity properties of a given tvs X are known, a result similar to Theorem 1 could hardly be expected. In fact, we will give later examples of complete $C_{\mathfrak{n}}(\mathfrak{m})$-spaces which are not $O_{\mathfrak{n}}(\mathfrak{m})$-spaces. First, necessary and sufficient conditions for a tvs X to be an $O_{\mathfrak{n}}(\mathfrak{m})$-space will be given. Our argument relies on the already mentioned paper of Drewnowski [1].

3. Let $\mathfrak{A}(\Gamma)$ be a ring of subsets of Γ such that

1° $\forall_{\gamma\in\Gamma}\ \{\gamma\}\in\mathfrak{A}(\Gamma)$,

2° $\forall_{E\in\mathfrak{A}(\Gamma)}\ \forall_{\Delta\in\Gamma}\ E\cap\Delta\in\mathfrak{A}(\Gamma)$,

i.e., an ideal containing finite subsets of Γ. In particular $\mathfrak{A}(\Gamma)$ may be the ring $\mathfrak{O}_{\mathfrak{n}}(\mathfrak{m})$.

If $\Delta\subset\Gamma$, then $\mathfrak{A}(\Delta)$ denotes the subring $\{E\cap\Delta\colon E\in\mathfrak{A}(\Gamma)\}$; the same convention will be applied to the symbol $\mathfrak{B}(\Gamma)$ which will appear below.

THEOREM 3. *Let X be a tvs, $x(:)\colon \mathfrak{A}(\Gamma)\to X$ a bounded measure. Assume that for some neighbourhood U of zero in X the set Γ'' of all such $\gamma\in\Gamma$ that $x(\{\gamma\})\notin U$ is infinite. Then there exists a subset Γ' of Γ'' with $\operatorname{card}\Gamma' = \operatorname{card}\Gamma''$ such that the restriction $x(:)\colon \mathfrak{A}(\Gamma')\to X$ has the following property:*

There is a neighbourhood W of zero in X such that for each $E_1, E_2\in\mathfrak{A}(\Gamma')$ with $E_1\neq E_2$, $x(E_1)-x(E_2)\notin W$.

In other terms, X contains a bounded discrete copy of $\mathfrak{A}(\Gamma')$.

Proof. In the proof we will represent Γ as a set of ordinal numbers. If α is an ordinal number, then P_α will denote the set of all ordinals less than α. Let $\mathfrak{m} = \operatorname{card}\Gamma$, and let μ be the least ordinal number with $\operatorname{card}P_\alpha = \mathfrak{m}$. We may and shall assume $\Gamma = P_\mu$. For each $\alpha<\mu$ we denote F_α

$=\{\beta\colon \alpha\leqslant\beta<\mu\}$. Further on, we may assume that $\Gamma''=\Gamma$. Also, we may and shall identify each element of $\mathfrak{A}(\Gamma)$ with its indicator function. Denote by $\mathfrak{B}(\Gamma)$ the set of all simple functions over $\mathfrak{A}(\Gamma)$ which take values $0, 1, -1$, i.e., if $A\in\mathfrak{B}(\Gamma)$, then $A=E_1-E_2$, where $E_1, E_2\in\mathfrak{A}(\Gamma)$ and are disjoint. It is clear that $x(:)\colon \mathfrak{A}(\Gamma)\to X$ can be uniquely extended to $\mathfrak{B}(\Gamma)$. Indeed, if $A=E_1-E_2\in\mathfrak{B}(\Gamma)$, then we put $\tilde{x}(A)=x(E_1)-x(E_2)$. Obviously $\tilde{x}\colon \mathfrak{B}(\Gamma)\to X$ is still bounded.

Arguing exactly as in the proof of Theorem in [1] we will get:

(*) There is a subset Δ of Γ with $\operatorname{card}\Gamma=\operatorname{card}\Delta$ such that for every $\sigma\in\Delta$ $\tilde{x}(\{\sigma\}+A)\notin W$ if $A\in\mathfrak{B}(\Delta\cap F_{\sigma+1})$, where W is a balanced neighbourhood of zero(²) (and $\{\sigma\}=\chi_{\{\sigma\}}$ by our identification).

By the symmetry $\tilde{x}(-\{\sigma\}+A)=-\tilde{x}(\{\sigma\}+(-A))\notin W$, since W is balanced. This means precisely that $\tilde{x}(A)\notin W$ if $A\neq\emptyset$, $A\in\mathfrak{B}(\Delta)$. Indeed, if $\emptyset\neq A=E_1-E_2$, $E_1, E_2\in\mathfrak{A}(\Delta)$, let σ be the least ordinal in $E_1\cup E_2$. We have $A=\{\sigma\}+B$ or $A=-\{\sigma\}+B$, where $B\in\mathfrak{B}(\Delta\cap F_{\sigma+1})$ and (*) applies.

Take any two different sets $E_1, E_2\in\mathfrak{A}(\Delta)$ and consider $x(E_1)-x(E_2)$. We have $x(E_1)-x(E_2)=x(E_1-E_2)-x(E_2-E_1)=\tilde{x}(A)$, where $\emptyset\neq A\in\mathfrak{B}(\Delta)$. So $x(E_1)-x(E_2)\notin W$. This proves the theorem.

With the same notation

COROLLARY 1. *Exactly one of the two following possibilities holds.*

(i) $x(:)\colon \mathfrak{D}_{\mathfrak{n}}(\mathfrak{m})\to X$ *is* $\mathfrak{m}$*-exhaustive.*

(ii) $X\supset\mathfrak{D}_{\mathfrak{n}}(\mathfrak{m})$, *i.e., X contains a bounded discrete copy of* $\mathfrak{D}_{\mathfrak{n}}(\mathfrak{m})$.

COROLLARY 2. *The following are equivalent.*

(i) *X is an $O_{\mathfrak{n}}(\mathfrak{m})$-space.*

(ii) $X\not\supset\mathfrak{D}_{\mathfrak{n}}(\mathfrak{m})$, *i.e., X contains no bounded discrete copy of* $\mathfrak{D}_{\mathfrak{n}}(\mathfrak{m})$.

In particular, since $\operatorname{card}\mathfrak{P}(\Gamma)>\operatorname{card}\Gamma$:

COROLLARY 3. *Let X be a tvs and assume that the character of density of X is at most $\mathfrak{m}$. Then every bounded measure on an $\mathfrak{m}$-ring of sets, with values in X, is $\mathfrak{m}$-exhaustive.*

As the character of density of $c_{\mathfrak{n}}(\mathfrak{m})$, for $\mathfrak{n}\leqslant\mathfrak{m}$, is precisely $\mathfrak{m}$, the following generalization of Phillips–Sobczyk theorem holds (compare Pełczyński and Sudakov [6]):

COROLLARY 4. *There is no continuous projection from $l^\infty(\mathfrak{m})$ onto $c_{\mathfrak{n}}(\mathfrak{m})$*, $\mathfrak{n}\leqslant\mathfrak{m}$.

(²) W is chosen as in [1] but for $\tilde{x}$ instead of $x(:)$ (which corresponds to T in [1]).

In fact, the composition of such a projection with the canonical indicator measure of $l^\infty(\mathfrak{m})$ would be $\mathfrak{m}$-exhaustive. And it is not as $\chi_{\{\gamma\}}\not\to 0$ in $c_{\mathfrak{n}}(\mathfrak{m})$.

4. Consider the space $s_{\mathfrak{n}}(\mathfrak{m})$ of simple functions, where $\mathfrak{n}$ is an infinite cardinal number. It contains the range of the canonical indicator measure of $c_{\mathfrak{n}}(\mathfrak{m})$ which is isomorphic to a bounded discrete copy of $\mathfrak{D}_{\mathfrak{n}}(\mathfrak{m})$.

We will endow $s_{\mathfrak{n}}(\mathfrak{m})$ with the suitable for our purposes complete linear topology $\tau_{\mathfrak{n}}(\mathfrak{m})$. Let for each $N\in\boldsymbol{N}$, X_N denote the subset of $s_{\mathfrak{n}}(\mathfrak{m})$ of simple functions of the form $\sum_{i=1}^{N}a_i\chi_{E_i}$, where $|a_i|\leqslant N$, $E_i\in\mathfrak{D}_{\mathfrak{n}}(\mathfrak{m})$ and are pairwise disjoint. For instance, which will be of importance for us, X_1 is a balanced hull of the range of the canonical indicator measure of $c_{\mathfrak{n}}(\mathfrak{m})$. Sets X_N, $N\in\boldsymbol{N}$, have the following properties:

(1) X_N's are balanced,

(2) $X_N+X_N\subset X_{(1+N)^2}$,

(3) $\bigcup_{N=1}^{\infty}X_N=s_{\mathfrak{n}}(\mathfrak{m})$,

(4) X_N's are complete (in the original topology of uniform convergence),

(5) X_1 contains a discrete bounded copy of $\mathfrak{D}_{\mathfrak{n}}(\mathfrak{m})$.

Now, by a theorem of Turpin [10] (see also [11]) the strongest linear topology $\tau_{\mathfrak{n}}(\mathfrak{m})$ which induces the original topology on each X_N is (Hausdorff) complete. Since X_N's are bounded in the topology of uniform convergence, they are bounded in the topology $\tau_{\mathfrak{n}}(\mathfrak{m})$ as well. We claim that

PROPOSITION. $(s_{\mathfrak{n}}(\mathfrak{m}), \tau_{\mathfrak{n}}(\mathfrak{m}))$, *a complete tvs, contains a discrete bounded copy of $\mathfrak{D}_{\mathfrak{n}}(\mathfrak{m})$ and does not contain any subspace isomorphic to $c_{\mathfrak{n}}(\mathfrak{m})$. Thus it is a $C_{\mathfrak{n}}(\mathfrak{m})$-space which is not an $O_{\mathfrak{n}}(\mathfrak{m})$*-space.

Proof. Let us assume the contrary. Then the sets $X'_N=X_N\cap c_{\mathfrak{n}}(\mathfrak{m})$ cover $c_{\mathfrak{n}}(\mathfrak{m})$. By Baire's Theorem one of them, X'_M say, contains a ball B. Every element of X'_M is of the form $\sum_{i=1}^{M}c_i\chi_{E_i}$, $|c_i|\leqslant M$, and $c_{\mathfrak{n}}(\mathfrak{m})=\bigcup_{n=1}^{\infty}nB$. Consequently, every $x\in c_{\mathfrak{n}}(\mathfrak{m})$ would be of the form $\sum_{i=1}^{M}c_i\chi_{E_i}$, where c_i are now arbitrary. This is impossible since the collection of functions of such form is not a vector space unless it is at most M-dimensional.

The above examples show that the necessary and sufficient conditions given in Corollary 2 cannot be much improved. Notice, however, that $X=(s_{\mathfrak{n}}(\mathfrak{m}), \tau_{\mathfrak{n}}(\mathfrak{m}))$ is not an F-space. For if it were so, for some $N\in\boldsymbol{N}$, X_N would be a neighbourhood, hence absorbent. As above, this is impossible unless X is finite dimensional.

Finally in the same line of ideas, Rolewicz and Ryll-Nardzewski gave in [8] (see also [7]) an example of an F-space without bmp. This proves automatically the existence in an F-space of an O-series [5] (even subseries convergent) which is not a C-series. Indeed, general results of Thomas [9] (see also Kalton [2]) on integral extensions of Radon maps imply in particular that every converging C-series is bounded multiplier convergent (of course, one can also give a relatively simple direct proof of this fact). In view of the same example, it is as well possible that the convex hull of a set of finite partial sums (which is a subset of a compact metrizable set of all partial sums of a subseries convergent series) is not bounded.

5. The notion of exhaustivity may be investigated when dealing with Radon measures (see [2], [3], [9]), or even considering homomorphisms on Riesz groups ([2]), as well. In this section we are interested mainly in exhaustivity properties of Radon measures. However, in order to obtain more symmetry we will place us first in the slightly more general setting of M-spaces.

We recall that an M-space F is a Banach lattice with the property:

$$\text{(M)} \qquad f, g \in F \quad \text{and} \quad f \wedge g = 0 \Rightarrow \|f \vee g\| = \|f\| \vee \|g\|.$$

Let X be a tvs, F an M-space, $\mathfrak{m}$ an infinite cardinal number. A linear operator $\Phi\colon F \to X$ is said to be $\mathfrak{m}$-*exhaustive* if:

For any family $(f_\gamma)_{\gamma\in\Gamma} \subset F$ such that $f_\gamma > 0$, $f_\gamma \wedge f_\delta = 0$ for $\gamma \neq \delta$, $\|f_\gamma\| \leqslant 1$, $\operatorname{card}\{\Phi(f_\gamma)\colon \Phi(f_\gamma) \notin V\} < \mathfrak{m}$ for any neighbourhood V of zero in X.

A Banach lattice F is said to be $\mathfrak{m}$-complete if any family $(f_\gamma)_{\gamma\in\Gamma} \subset F$ with $\operatorname{card}\Gamma < \mathfrak{m}$ which has an upper bound, has a supremum in F.

Let K be a locally compact Hausdorff space. We denote by $C(K)$ the space of bounded, continuous, real-valued functions on K with the supremum norm, by $C_0(K)$ its subspace consisting of functions vanishing at infinity.

THEOREM 4 (cf. Theorem 2). *Let $\mathfrak{m}, \mathfrak{n}$ be infinite cardinal numbers, X a tvs. The following are equivalent.*

(i) *Given any $\mathfrak{n}$-complete M-space F, and any bounded linear operator $\Phi\colon F \to X$, then Φ is $\mathfrak{m}$-exhaustive.*

(ii) $X \not\supset c_{\mathfrak{n}}(\mathfrak{m})$.

Proof. (i) $\Rightarrow$ (ii). If Φ is not $\mathfrak{m}$-exhaustive, we find positive $(f_\gamma)_{\gamma\in\Gamma}$ in F such that: $\operatorname{card}\Gamma = \mathfrak{m}$, $f_\gamma \wedge f_\delta = 0$ for $\gamma \neq \delta$, $\|f_\gamma\| \leqslant 1$ and $\Phi(f_\gamma) \notin V$ for some neighbourhood V of zero in X and $\gamma \in \Gamma$. Let $c = (c_\gamma) \in c_{\mathfrak{n}}(\mathfrak{m})$ $(\equiv c_{\mathfrak{n}}(\Gamma))$, $c > 0$, $\|c\| \leqslant 1$. Define $\Psi(c) = \sup_{\gamma\in\Gamma} c_\gamma f_\gamma$ $(= \sup_{E\in\mathfrak{F}(\Gamma)} \sum_{\gamma\in E} c_\gamma f_\gamma =$ the order-sum $0-\sum_{\gamma\in\Gamma} c_\gamma f_\gamma$, where $\mathfrak{F}(\Gamma)$ is ordered by inclusion)(³). As $c \in c_{\mathfrak{n}}(\mathfrak{m})$ and F is $\mathfrak{n}$-complete, $\Psi(c) \in F$. Extending Ψ on the whole $c_{\mathfrak{n}}(\mathfrak{m})$ we get the linear bounded operator $\Psi\colon c_{\mathfrak{n}}(\mathfrak{m}) \to F$. Consider $\Phi\Psi\colon c_{\mathfrak{n}}(\mathfrak{m}) \to X$ It is bounded linear and

$$\Phi\Psi(\chi_{\{\gamma\}}) = \Phi(f_\gamma) \notin V.$$

Applying Drewnowski's Theorem [1] $X \supset c_{\mathfrak{n}}(\mathfrak{m})$.

(ii) $\Rightarrow$ (i). If $X \supset c_{\mathfrak{n}}(\mathfrak{m})$ it is sufficient to consider the identity on $c_{\mathfrak{n}}(\mathfrak{m})$ as X-valued map (which is bounded non-$\mathfrak{m}$-exhaustive).

COROLLARY (cf. Corollary after Theorem 1). A. *Let K be an $\mathfrak{m}$-Stonian compact Hausdorff space. The following are equivalent.*

(i) *Every bounded linear operator* $(=$ *Radon measure*$)$ $\Phi\colon C(K) \to X$ *is $\mathfrak{m}$-exhaustive.*

(ii) $X \not\supset l^\infty(\mathfrak{m})$.

B. *Let K be a locally compact Hausdorff space. The following are equivalent.*

(i) *Every bounded linear operator* $(=$ *Radon measure*$)$ $\Phi\colon C_0(K) \to X$ *is $\mathfrak{m}$-exhaustive.*

(ii) $X \not\supset c_0(\mathfrak{m})$.

Added in proof. 1) The *Théorème* as stated in [5] is false. The assumption "If $x(:)$ is not exhaustive" must be replaced therein by "If $x\ (\{n\}) \not\to 0$".

2) In the meantime Drewnowski found an extremely simple, avoiding the transfinite induction, proof of his theorem. This method may be applied in our situation as well, see [13].

(³) This definition of Ψ must be changed if (and only if) $\mathfrak{n} = \aleph_0$. Noting that if $c_{\mathfrak{n}}(\mathfrak{m}) = c_0(\mathfrak{m})$ the series $\sum_{\gamma\in\Gamma} c_\gamma f_\gamma$ is summable in F, in that case we define $\Psi(c) = \sum_{\gamma\in\Gamma} c_\gamma f_\gamma$.

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Anosov endomorphisms*

by

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Abstract. We generalize notions of Anosov diffeomorphisms and expanding maps by introducing Anosov endomorphisms. For such an endomorphism f we assume the existence of an invariant hyperbolic splitting of $T_{x_n}(M)$ along every f-trajectory (x_n). The main result of this paper is a construction of an uncountable family of pairwise nonconjugated Anosov endomorphisms contained in a small open subset of $C^1(M, M)$. We construct also an Anosov endomorphism which has an arc of unstable manifolds at some point. We prove some technical lemmas in more general situation of hyperbolic sets or Axiom A.

§ 0. Introduction. Let M be a compact, connected, boundaryless finite-dimensional C^∞ manifold.

DEFINITION. A diffeomorphism $f\colon M\to M$ is called an *Anosov diffeomorphism* if there is a continuous splitting of the tangent bundle $TM = E^s+E^u$ which is preserved by the derivative Df and if there are constants $C>0$, $0<\mu<1$ and a Riemannian metric $\langle\cdot,\cdot\rangle$ on TM such that for $n=0,1,\ldots$ we have

(1) $$\|Df^n(v)\|\leqslant C\mu^n\|v\| \quad \text{for} \quad v\in E^s,$$

(2) $$\|Df^n(v)\|\geqslant C^{-1}\mu^{-n}\|v\| \quad \text{for} \quad v\in E^u.$$

For the main properties of Anosov diffeomorphisms see [4], [8].

DEFINITION. A map $f\in C^1(M, M)$ is called *expanding* if there are constants $C>0$, $0<\mu<1$ and a Riemannian metric $\langle\cdot,\cdot\rangle$ on TM such that for $n=0,1,\ldots$ we have

$$\|Df^n(v)\|\geqslant C\mu^{-n}\|v\|$$

(see, for example, [5], [9]).

In this paper we generalize the above notions as follows:

DEFINITION. We call a regular map $f\in C^1(M, M)$ an *Anosov endomorphism* if there exist constants $C>0$, $0<\mu<1$ and a Riemannian metric $\langle\cdot,\cdot\rangle$ on TM such that for every f-trajectory (x_n) (a sequence of points in M satisfying $f(x_n)=x_{n+1}$ for every integer n) there is a splitting of $\bigcup_{n=-\infty}^{+\infty} T_{x_n}M = E^s\oplus E^u = \bigcup_{n=-\infty}^{+\infty} E^s_{x_n}\oplus E^u_{x_n}$ which is preserved by the derivative Df and conditions (1), (2) are satisfied.

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