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Received February 21, 1975 (960)

STUDIA MATHEMATICA, T. LVIII. 3. (1976)

Anosov endomorphisms*

by

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Abstract. We generalize notions of Anosov diffeomorphisms and expanding maps by introducing Anosov endomorphisms. For such an endomorphism f we assume the existence of an invariant hyperbolic splitting of $T_{x_n}(M)$ along every f-trajectory (x_n). The main result of this paper is a construction of an uncountable family of pairwise nonconjugated Anosov endomorphisms contained in a small open subset of $C^1(M, M)$. We construct also an Anosov endomorphism which has an arc of unstable manifolds at some point. We prove some technical lemmas in more general situation of hyperbolic sets or Axiom A.

§ 0. Introduction. Let M be a compact, connected, boundaryless finite-dimensional C^∞ manifold.

DEFINITION. A diffeomorphism $f\colon M\to M$ is called an *Anosov diffeomorphism* if there is a continuous splitting of the tangent bundle $TM = E^s + E^u$ which is preserved by the derivative Df and if there are constants $C > 0$, $0 < \mu < 1$ and a Riemannian metric $\langle\cdot,\cdot\rangle$ on TM such that for $n = 0, 1, \ldots$ we have

(1) $$\|Df^n(v)\| \leqslant C\mu^n\|v\| \quad \text{for} \quad v\in E^s,$$

(2) $$\|Df^n(v)\| \geqslant C^{-1}\mu^{-n}\|v\| \quad \text{for} \quad v\in E^u.$$

For the main properties of Anosov diffeomorphisms see [4], [8].

DEFINITION. A map $f\in C^1(M, M)$ is called *expanding* if there are constants $C > 0$, $0 < \mu < 1$ and a Riemannian metric $\langle\cdot,\cdot\rangle$ on TM such that for $n = 0, 1, \ldots$ we have

$$\|Df^n(v)\| \geqslant C\mu^{-n}\|v\|$$

(see, for example, [5], [9]).

In this paper we generalize the above notions as follows:

DEFINITION. We call a regular map $f\in C^1(M, M)$ an *Anosov endomorphism* if there exist constants $C > 0$, $0 < \mu < 1$ and a Riemannian metric $\langle\cdot,\cdot\rangle$ on TM such that for every f-trajectory (x_n) (a sequence of points in M satisfying $f(x_n) = x_{n+1}$ for every integer n) there is a splitting of $\bigcup_{n=-\infty}^{+\infty} T_{x_n}M = E^s\oplus E^u = \bigcup_{n=-\infty}^{+\infty} E^s_{x_n}\oplus E^u_{x_n}$ which is preserved by the derivative Df and conditions (1), (2) are satisfied.

* This is a part of the author's PhD. thesis written under supervision of Dr. K. Krzyżewski and Dr. W. Szlenk.

In § 1 of this paper we prove (see Theorem 1.16) that the set of all Anosov endomorphisms is an open subset of $C^1(M, M)$.

From the definition of an Anosov endomorphism it does not follow that there exists a splitting of the whole tangent bundle $TM = E^s \oplus E^u$. Notice that $E^u_{x_0}$ depends on the whole trajectory (x_n), therefore it may happen that $E^u_{x_0} \neq E^u_{y_0}$ though $x_0 = y_0$ (but $(x_n) \neq (y_n)$). Such a phenomenon is impossible for $E^s_{x_0}$, it depends only on x_0. Indeed, if $v \in E^s_{x_0}$ then $\|Df^n(v)\| \xrightarrow[n\to\infty]{} 0$, hence $v \in E^s_{y_0}$.

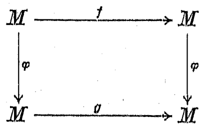
In this paper we consider also special Anosov endomorphisms for which E^u_x does not depend on the trajectory containing x. A classical example of such an endomorphism is the algebraic endomorphism of the torus: $\begin{bmatrix} n & 1 \\ 1 & 1 \end{bmatrix}$ for $n \geq 2$. Special Anosov endomorphisms were first introduced by Shub in [9] and called there *Anosov endomorphisms*; however, the theorems formulated there and concerning such endomorphisms are false (see Theorem 2.18 and Corollary 2.19).

Definition. We say that two maps $f, g \in C^1(M, M)$ are *topologically conjugate* if there exists a homeomorphism $h\colon M \to M$ such that $f \circ h = h \circ g$.

A map $f \in C^1(M, M)$ is said to be *structurally stable* if there is a neighbourhood U of f in the C^1-topology on $C^1(M, M)$ such that $g \in U$ implies that f and g are topologically conjugate.

It is well known that Anosov diffeomorphisms and expandings are structurally stable. In this paper we claim that these are no other Anosov endomorphisms satisfying the property of structural stability. This is an immediate consequence of our main result (see § 4):

Theorem 4.11. *Every non-empty open in the C^r-topology subset of the set of all Anosov endomorphisms of class C^r, which are not diffeomorphisms or expandings, contains an uncountable subset such that, if $f \neq g$ are any elements of its, then there exists no surjective map $\varphi \in C^0(M, M)$ which makes the diagram*



commute.

Now we shall try to explain this strange behaviour of Anosov endomorphisms. Consider f-trajectories (x_n), (y_n) such that $x_0 = y_0$ and there exists a neighbourhood U of y_{-1} for which

$$\operatorname{dist}\big(U, \{x_n\}_{n=-\infty}^{\infty} \cup \{y_n\}_{n=-\infty}^{-2}\big) > 0.$$

Let us perturb f inside U to an f' such that there exists $y'_{-1} \in W^s_{y_{-1},\mathrm{loc}}$, $y'_{-1} \neq y_{-1}$, $f'(y'_{-1}) = y_0$. ($W^s_{y_{-1},\mathrm{loc}}$ is the local stable manifold of f at y_{-1} — see Theorem 2.1.)

Just as in the case of diffeomorphisms (see Corollary 1.14) there exists exactly one f'-trajectory in the small uniform neighbourhood of (x_n). This f'-trajectory is of course (x_n). Thus if there exists a conjugating homeomorphism h close to the identity, then $h(x_0) = x_0$. Hence $h(y_{-1}) = y'_{-1}$, and so $h(y_{-n}) = y'_{-n}$ for some f'-trajectory (y'_n). This contradicts the fact that as long as $\varrho(y_{-n}, y'_{-n})$ is small, it grows at least exponentially.

We may give also another explanation of the nonstability of Anosov endomorphisms which are not diffeomorphisms or expandings. An operator $f_*^{-1} - \mathrm{Id}$ in the space of continuous vector fields on M, where $f_*^{-1}(V(\cdot)) = Df^{-1}(\cdot) \circ V \circ f(\cdot)$, is not invertible. Indeed, let (x_n), (y_n) be f-trajectories as above. Then it is easy to check that a vector field V which is zero outside U and $V(y_{-1}) \in E^s_{y_{-1}} - \{0\}$ does not belong to the image of $f_*^{-1} - \mathrm{Id}$.

The above ideas in fact explain only the non-ε-stability (ε-stability means that the conjugating map can be chosen arbitrary close to the identity if the perturbation is sufficiently C^1-small) and will not be used in this paper (1).

In § 2 (see Theorems 2.15 and 2.18), we construct an endomorphism which is close to an algebraic Anosov endomorphism and at a point has many different local unstable manifolds. This construction can be reckogned as another explanation of the nonstability of Anosov endomorphisms.

In our proof of Theorem 4.11 we use the following topological invariant: if x and z are periodic points of f, then there exists a non-periodic f-trajectory (y_n) such that $y_0 = x$ and $\varrho(y_{-tn}, z) \to 0$ for $n \to \infty$, where t is the period of z.

Some technical statements will be proved in the situation of a hyperbolic set Λ (similarly to the well-known case of diffeomorphisms) It is useful to consider the inverse limit $\tilde{\Lambda}$ of a system $\ldots \xleftarrow{f|\Lambda} \Lambda \xleftarrow{f|\Lambda} \Lambda \xleftarrow{f|\Lambda} \ldots$ We claim (see Theorem 2.5) the continuity of the transformation $\tilde{\Lambda} \ni (x_n) \mapsto W^u_{x_0,\mathrm{loc}}$ (with C^1-topology in the set of local unstable manifolds regarded as embeddings of the disk).

In § 3, we introduce Axiom A endomorphisms which are generalizations of Smale's Axiom A diffeomorphisms (see [8]). We claim that the set of nonwandering points in the inverse limit of Axiom A endomorphism with a shift satisfies conditions of Axiom A* (see [1], [2]). This permits

(1) After sending this paper for publication, the paper of R. Mañé and C. Pugh "*Stability of endomorphisms*" appeared in Warwick Dynamical Systems. R. Mañé and C. Pugh proved that Anosov endomorphism (called by them "weak Anosov endomorphism") is ε-structurally stable iff it is either expanding or a diffeomorphism.

us to use results of Bowen on the existence of the unique invariant measure with maximal entropy.

The paper of Katok [4] has been very helpful in the preparation of this paper. I wish to express my thanks to M. Krych and M. Misiurewicz who read a preliminary version of the paper and made many valuable suggestions.

Symbols:

$B(x, t)$	the open ball with centre x and radius t;
$\mathrm{dist}_\varrho(A, B)$	$\inf\{\varrho(x, y)\colon x \in A, y \in B\}$, where A, B are subsets of a metric space with a metric ϱ;
$\mathrm{Diff\,An}^r(M)$	the set of all C^r-Anosov diffeomorphisms on a manifold M, $1 \leqslant r \leqslant \infty$;
$\mathrm{End\,An}^r(M)$	the set of all C^r-Anosov endomorphisms on a manifold M, $1 \leqslant r \leqslant \infty$;
$\mathrm{End\,An}'^r(M)$	$\mathrm{End\,An}^r(M)$ — {Expandings};
$E^{u(s)}_{f,x_i}$	the unstable (stable) subspace for an f-trajectory (x_n) in a point x_i (often we shall omit the subscript f);
$E^{u(s)}_{x_i,t}$	$E^{u(s)}_{x_i} \cap B(O_{T_{x_i}M}, t)$;
$E^u_{f,x}$	the unstable subspace at $x \in \mathrm{Per}(f)$ for the periodic trajectory of x;
$E^{u(s)\perp}_{x_i}$	the orthogonal subspace to $E^{u(s)}_{x_i}$ in $T_{x_i}M$ with an inner product;
$\mathscr{E}_{x,y}$	the parallel translation from T_yM into T_xM along the unique shortest gheodesic joining x and y (under the assumption that $\varrho(x, y)$ is sufficiently small);
$f\|A$	a mapping f restricted to a subset A;
$h_{f'f}$	the conjugating map between $\tilde{A}$ and $\tilde{A}'$ (see Notation 1.18, and Theorem 1.20);
k_{f,x_i}	see Theorem 2.8;
Λ	a hyperbolic set (see Definition 1.1);
$\tilde{\Lambda}$	$\underleftarrow{\lim}(\Lambda, f)$ — the inverse limit of a system $\ldots \xrightarrow{f} \Lambda \xrightarrow{f} \Lambda \xrightarrow{f} \ldots$, where Λ is a hyperbolic set of an endomorphism f;
$N_n(f)$	the number of all fixed points for f^n;
$\mathrm{Orb}_f(x)$	$\{f^n(x)\}_{n\geqslant 0}$ for $x \in \mathrm{Per}(f)$;
$\mathrm{ord}_f(x)$	the minimal period of $x \in \mathrm{Per}(f)$;
$\Omega(f)$	the set of all nonwandering points of f (see Definition 3.1);
$\mathrm{Per}(f)$	the set of all periodic points of f;
$\mathrm{Per}_n(f)$	$\{x \in \mathrm{Per}(f)\colon \mathrm{ord}_f(x) \leqslant n\}$;
$\mathrm{Per}^j(f)$	jth class of $\mathrm{Per}(f)$ under Spectral Decomposition (see Theorem 3.11, Definition 3.12, Proposition 3.13);
$\mathrm{Per}^j_n(f)$	$\mathrm{Per}^j(f) \cap \mathrm{Per}_n(f)$;

π_i	the standard projection $\pi_i\colon \tilde{\Lambda} \to \Lambda$, $\pi_i((x_n)) = x_i$;
$\varrho(x, A)$	$\mathrm{dist}_\varrho(\{x\}, A)$, for x — a point, A — a set;
$\varrho(f, g)$	$\sup_{x\in M} \varrho(f(x), g(x))$, for $f, g \in C^0(M, N)$, M, N — Riemannian manifolds;
$\varrho_{C^1}(f, g)$	the standard C^1-distance betwen maps $f, g \in C^1(M, N)$ for which $\varrho(f, g)$ is sufficiently small;
$r(x_0, y_0)$	the unique point of the intersection $W^s_{x_0,\Delta} \cap W^u_{y_0,\Delta}$ for $(x_n), (y_n) \in \tilde{\Lambda}$, $\varrho(x_0, y_0) < \nu(\Delta)$ (see Proposition 2.3 and Notation 2.4);
$r((x_n), (y_n))$	the unique point of the intersection $\tilde{W}^s_{(x_n),\delta^{**}} \cap \tilde{W}^u_{(y_n),\delta^{**}}$ for $(x_n), (y_n) \in \tilde{\Lambda}$, $\tilde{\varrho}((x_n), (y_n))$ sufficiently small (see Proposition 3.7);
R	the constant described in Theorem 2.1;
S^1	the circle in R^2 with centre 0 and radius 1;
$S\,\mathrm{End\,An}^r(M)$	the set of all C^r-special Anosov endomorphisms (see Definition 2.12);
$\mathrm{spec}(L)$	the spectrum of a linear operator L;
$\theta_{f'f}$	the conjugating map between $\mathrm{Per}(f) \cap \Lambda$ and $\mathrm{Per}(f') \cap \Lambda'$ (see Remark 1.23);
$W^{u(s)}_{x_i,\eta}$	local unstable (stable) manifold (see Theorem 2.1);
$W^s_{x,\mathrm{loc}} \pitchfork W^u_{y,\mathrm{loc}}$	submanifolds $W^s_{x,\mathrm{loc}}$ and $W^u_{y,\mathrm{loc}}$ intersect each other transversally;
W^u_{f,x_i}	global unstable manifold for f-trajectory (x_n) at a point x_i;
$W^u_{f,x_i,a}$	$W^u_{f,x_i} \cap B(x_i, a)$;
$W^u_{f,x_i,a,\varrho u}$	see Theorem 2.8, Definition 2.9, and Notation 2.10;
$W^u_{f,x}$	global unstable manifold at $x \in \mathrm{Per}(f)$ for the periodic trajectory of x.

§ 1. Local hyperbolic structure of endomorphisms.

1.1. Definition. Let U be some open subset of M, let $f\colon U \to M$ be a regular C^1-map (such a map will be called an *endomorphism*). Let Λ be a closed subset of M such that $f(\Lambda) = \Lambda \subset U$. Then Λ is called a *hyperbolic set* for this endomorphism iff there exist real constants: $C > 0$, μ: $0 < \mu < 1$ and a continuous Riemannian metric $\langle\cdot,\cdot\rangle$ on TM such that for every f-trajectory $(x_n)_{n=-\infty}^{+\infty}$ of points in Λ and for every integer i we have:

$$T_{x_i}M = E^s_{x_i} \oplus E^u_{x_i},$$

$$Df(E^s_{x_i}) = E^s_{x_{i+1}}, \quad \|Df^n_{x_i}(v)\| \leqslant C\mu^n\|v\| \quad \text{for} \quad v \in E^s_{x_i},$$

$$Df(E^u_{x_i}) = E^u_{x_{i+1}}, \quad \|Df^n_{x_i}(v)\| \geqslant (1/(C\mu^n))\|v\| \quad \text{for} \quad v \in E^u_{x_i}$$

$$\text{for} \quad n = 0, 1, \ldots$$

This family of splittings will be called hyperbolic.

1.2. Remark. Let $\Lambda = M$ be a hyperbolic set for f. Then f is an Anosov endomorphism. Of course, in this case f is a covering map, hence there exists a positive integer N called the *degree* of f such that every point has exactly N counterimages. The diameter of f is defined as

$$\sup\{\operatorname{diam} B\colon B \text{ is a ball in } M, f^{-1}(B) = \bigcup_1^N B_i,\ B_i \text{ are open, disjoint sets, } f|B_i\colon B_i \to B \text{ is a homeomorphism for } i = 1, \ldots, N\}.$$

1.3. EXAMPLE. An example of an Anosov endomorhpism of degree n is the algebraic endomorphism of the torus T^2: $\begin{bmatrix} n+1 & 1 \\ 1 & 1 \end{bmatrix}$.

Remark. The Cartesian product of two Anosov endomorphisms is also an Anosov endomorphism.

1.4. PROPOSITION. *For any hyperbolic set Λ for f there exists a smooth Riemannian metric $\langle\cdot,\cdot\rangle_\Lambda$ adapted to Λ on TM such that for some λ: $0 < \lambda < 1$ and for every f-trajectory $\{x_n\} \subset \Lambda$*

$$\|Df_{x_i}(v)\|_\Lambda \leqslant \lambda \|v\|_\Lambda \quad \text{for} \quad v \in E^s_{x_i},$$
$$\|Df_{x_i}(v)\|_\Lambda \geqslant (1/\lambda)\|v\|_\Lambda \quad \text{for} \quad v \in E^u_{x_i}.$$

$\|\cdot\|_\Lambda$ is equivalent to $\|\cdot\|$.

(From now on, a hyperbolic set Λ being fixed, we shall use in general the metric $\langle\cdot,\cdot\rangle_\Lambda$ and we shall omit the subscript Λ.)

Proof. We define a new metric $\langle\cdot,\cdot\rangle'$ on $T\Lambda$ as in [5]:

$$\langle v_1, v_2\rangle' = \sum_{j=0}^N \langle Df^j_x(v_1), Df^j_x(v_2)\rangle,$$

where N is such that $C\mu^N \leqslant K < 1$ for some K.

In what follows we extend $\langle\cdot,\cdot\rangle'$ to a continuous Riemannian metric on TM and approximate it by a smooth metric.

We first prove some technical facts concerning hyperbolic sets.

1.5. DEFINITION. Let E', E'' be subspaces of R^n with an inner product $\langle\cdot,\cdot\rangle$. Define

$$\tan \sphericalangle (E', E'') = \sup\{\|w_1\|/\|w_2\|\colon w_1 \in E''^{\perp}, w_2 \in E'', w_1 + w_2 \in E' - \{0\}\}.$$

1.6. Remark. If $\dim E' = \dim E''$, then $\tan \sphericalangle (E', E'') = \tan \sphericalangle (E'', E')$.

If $\dim E'' < \dim E'$, then $\tan \sphericalangle (E', E'') = +\infty$.

1.7. PROPOSITION. *Let f be an endomorphism with a hyperbolic set Λ. There exist real numbers $\xi > 0$, $\alpha > 0$ such that for all f-trajectories $\{x_n\}, \{y_n\} \subset \Lambda$, $\varrho(x_0, y_0) < \xi$ implies*

$$\tan \sphericalangle \big(\mathscr{E}_{x_0y_0}(E^u_{y_0}), E^{s\perp}_{x_0}\big) < \alpha \quad \text{and} \quad \tan \sphericalangle \big(\mathscr{E}_{x_0y_0}(E^s_{y_0}), E^{u\perp}_{x_0}\big) < \alpha.$$

Proof. We shall prove the first assertion. Let $\zeta > 0$ be a number such that $(1/\lambda) - \lambda - \zeta > 0$. Choose ξ such that

$$\|Df_x - \mathscr{E}_{f(x)f(y)} \circ Df_y \circ \mathscr{E}_{yx}\| < \zeta \quad \text{for} \quad x, y \in \Lambda,\ \varrho(x, y) < \xi.$$

There exists a constant $C > 0$ such that $\|Df(v)\| \leqslant C\|v\|$ for every v tangent to M at any point of Λ.

Let $(x_n), (y_n)$ be f-trajectories satisfying our assumptions, let $w \in \mathscr{E}_{x_0y_0}(E^u_{y_0})$, $w = w_1 + w_2$, $0 \neq w_1 \in E^s_{x_0}$, $w_2 \in E^s_{x_0}$. Then

$$\begin{aligned}\|Df_{x_0}(w_2)\| &\geqslant \|Df_{x_0}(w)\| - \|Df_{x_0}(w_1)\| \\ &> \|\mathscr{E}_{f(x_0)f(y_0)} \circ Df_{y_0} \circ \mathscr{E}_{y_0x_0}(w)\| - \zeta\|w\| - \lambda\|w_1\| \\ &\geqslant \big((1/\lambda) - \zeta\big)\|w\| - \lambda\|w_1\| \geqslant \big((1/\lambda) - \lambda - \zeta\big)\|w_1\|.\end{aligned}$$

So

$$\frac{\|w_1\|}{\|w_2\|} < \frac{\|w_1\|}{(1/C)\big((1/\lambda) - \lambda - \zeta\big)\|w_1\|} = C/\big((1/\lambda) - \lambda - \zeta\big).$$

Therefore it is sufficient to define $\alpha = C/\big((1/\lambda) - \lambda - \zeta\big)$.

In particular, it follows from Proposition 1.7 that if $\varrho(x_0, y_0) < \xi$, then $O_{T_{x_0}M}$ is the unique common point of $\mathscr{E}_{x_0y_0}(E^u_{y_0})$ and $E^s_{x_0}$. So we have:

1.8. COROLLARY. $\dim E^u_{x_0}$ *depends only on* x_0 (i.e., it does not depend on the whole sequence (x_n)). *The maps $x \mapsto \dim E^{u(s)}_x$ are locally constant.*

We now investigate the continuity of the splitting $T_{x_i}M = E^s_{x_i} \oplus E^u_x$ with respect to (x_n). We shall need the following

1.9. LEMMA. *Let us take a sequence of finite-dimensional linear spaces $B_0, B_1, \ldots$ with inner products and a sequence of linear isomorphisms L_i: $B_i \to B_{i+1}$. Furthermore, put $B_i = E^s_i \oplus E^u_i$, $L_i(E^{s(u)}_i) = E^{s(u)}_{i+1}$ and suppose that*

$$\|L_i(v)\| \leqslant \lambda\|v\| \quad \text{for } v \in E^s_i,$$
$$\|L_i(v)\| \geqslant (1/\lambda)\|v\| \text{ for } v \in E^u_i \text{ and for some } \lambda \text{ with } 0 < \lambda < 1.$$

Let $\alpha, \beta > 0$ and suppose $\tan \sphericalangle (E^s_i, E^{u\perp}_i) < \alpha$ for $i = 0, 1, \ldots$ *Then for every $w \in B_0$ for which $\tan \sphericalangle (w, E^{s\perp}_0) < \beta$ there exists a constant $K > 0$ and a positive integer N such that*

$$\tan \sphericalangle \big(L_{n-1} \circ \ldots \circ L_0(w), E^u_n\big) \leqslant K\lambda^{2n} \quad \text{for all } n \geqslant N.$$

In fact, K and N depend only on the triple (α, β, λ).

Proof. For a vector $v \in B_i$, let $v = v^1 + v^2 = v^s + v^u$ where $v^1 \in E^{u\perp}_i$, $v^2 \in E^u_i$, $v^s \in E^s_i$, $v^u \in E^u_i$. Of course, $(v^s)^1 = v^1$. Let a vector w satisfy the conditions of the lemma. Then for sufficiently large n we have

$$\tan\sphericalangle\left(L_{n-1}\circ\ldots\circ L_0(w),\,E_n^u\right)=\frac{\|(L_{n-1}\circ\ldots\circ L_0w)^1\|}{\|(L_{n-1}\circ\ldots\circ L_0w)^2\|}$$
$$\leqslant\frac{\|(L_{n-1}\circ\ldots\circ L_0w^s)^1\|}{\left|\|(L_{n-1}\circ\ldots\circ L_0w^u)^2\|-\|(L_{n-1}\circ\ldots\circ L_0w^s)^2\|\right|}$$
$$\leqslant\left|\frac{\|L_{n-1}\circ\ldots\circ L_0w^u\|}{\|L_{n-1}\circ\ldots\circ L_0w^s\|}-\frac{\|(L_{n-1}\circ\ldots\circ L_0w^s)^2\|}{\|(L_{n-1}\circ\ldots\circ L_0w^s)^1\|}\right|^{-1}$$
$$\leqslant\left(\frac{(1/\lambda^n)w^u}{\lambda^n\|w^s\|}-\alpha\right)^{-1}\leqslant\Big((1/\lambda^{2n})\big(1/(1+\alpha+\beta)\big)-\alpha\Big)^{-1}$$
$$\leqslant\lambda^{2n}\Big(\big(1/(1+\alpha+\beta)\big)-\lambda^{2n}\alpha\Big)^{-1}\leqslant K\lambda^{2n}.$$

We shall apply this Lemma to the case of a sequence of tangent spaces and derivatives $Df_{x_n}\colon T_{x_n}M\to T_{x_{n+1}}M$.

1.10. THEOREM. *Let all objects be as in Definition 1.1. Then the map:*

$$f\text{-trajectory }(x_n)\mapsto\text{ the splitting }E^s_{x_i}\oplus E^u_{x_i}$$

is continuous, i.e.,

$(\forall\varepsilon>0)(\exists\delta>0)$ $(\exists N$ positive integer$)$ $(\forall f$-trajectories $\{x_n\},\{y_n\}\subset\Lambda)$

$$\big(\varrho(x_i,y_i)\leqslant\delta\ \text{ for }\ -N\leqslant i\leqslant N\big)\ \Rightarrow\ \tan\sphericalangle\big(E^{s(u)}_{x_0},\,\mathscr{E}_{x_0y_0}(E^{s(u)}_{y_0})\big)\leqslant\varepsilon.$$

Proof. We shall prove the last inequality for unstable subspaces. Let $\varepsilon>0$. Let $\alpha=\beta$ be equal to the number α described in Proposition 1.7. For a triple (α,β,λ) take numbers K and N as in Lemma 1.9 such that $K\lambda^{2N}\leqslant\varepsilon/2$.

There exists a δ such that for any two f-trajectories $\{x_n\},\{y_n\}\subset\Lambda$ if $\varrho(x_i,y_i)\leqslant\delta$, $i=-N,\ldots,0$, then

(1) $\quad\tan\sphericalangle\big(\mathscr{E}_{x_iy_i}(E^u_{y_i}),\,\tilde{E}^u_i\big)\leqslant\varepsilon/4$

$$\text{where}\quad\tilde{E}^u_i=Df_{x_{i-1}}\circ\ldots\circ Df_{x_{-N}}\circ\mathscr{E}_{x_{-N}y_{-N}}(E^u_{y_{-N}}),$$

(2) $\quad\tan\sphericalangle(E^u_{-N},E^{s\perp}_{-N})<\alpha.$

Now it suffices to apply the result of Lemma 1.9 to the sequence of isomorphisms $Df_{x_i}\colon E^s_{x_i}\oplus E^u_{x_i}\to E^s_{x_{i+1}}\oplus E^u_{x_{i+1}}$ and to any vector $w\in\tilde{E}^u_{-N}$. Hence $\tan\sphericalangle(E^u_{x_0},\tilde{E}^u_0)\leqslant\varepsilon/2$. Thus, using (1), we obtain

$$\tan\sphericalangle\big(\mathscr{E}_{x_0y_0}(E^u_{y_0}),\,E^u_{x_0}\big)\leqslant\varepsilon.$$

Now we formulate important Theorems 1.11, 1.12, 1.13 which are very similar to the analogous results of [4], [6], [10], [11].

1.11. THEOREM (*Anosov's theorem on uniform perturbations*). *Let $(B_i)^{+\infty}_{-\infty}$ be a sequence of finite-dimensional linear spaces with inner products, and let $L_i\colon B_i\to B_{i+1}$, $i=\ldots-1,0,1,\ldots$, be a sequence of linear isomorphisms such that $\sup_i\|L_i\|\leqslant A$ and $\sup_i\|L_i^{-1}\|\leqslant A$ for some real number A.*

Write $B_i=E^s_i\oplus E^u_i$, $L_i(E^{s(u)}_i)=E^{s(u)}_{i+1}$ and suppose that

$$\|L_i(v)\|\leqslant\lambda\|v\|\quad\text{for}\quad v\in E^s_i,$$
$$\|L_i^{-1}(v)\|\leqslant\lambda\|v\|\quad\text{for}\quad v\in E^u_{i+1}\ \text{and for some }\lambda\text{ with }0<\lambda<1.$$

Let r be a positive number and let, for any integer i, V_i denote the open ball with centre at the origin in B_i and radius r. Then for every ε there exists a δ such that for every sequence of diffeomorphisms $F_i\colon V_i\to F_i(V_i)\subset B_{i+1}$ if $\varrho_{C^1}(F_i,L_i|V_i)<\delta$ for all i, then:

(a) *The set W^s_i of all points of V_i, whose images under F_i, $F_{i+1}\circ F_i$, $F_{i+2}\circ F_{i+1}\circ F_i,\ldots$ are in $V_{i+1},V_{i+2},\ldots$, respectively, is a C^1-submanifold of B_i,*

(b) *The set W^u_i of all points of V_i whose images under F^{-1}_{i-1}, $F^{-1}_{i-2}\circ F^{-1}_{i-1},\ldots$ are in $V_{i-1},V_{i-2},\ldots$, respectively, is a C^1-submanifold of B_i.*

(c) *Submanifolds W^s_i and W^u_i intersect each other at exactly one point b_i. Furthermore, $W^s_i\pitchfork W^u_i$.*

(d) $\varrho_{C^1}(W^s_i,E^s_i|V_i)\leqslant\varepsilon$, $\varrho_{C^1}(W^u_i,E^u_i|V_i)\leqslant\varepsilon$.

(e) $\|F_i(v_1)-F_i(v_2)\|\leqslant\big((1+\lambda)/2\big)\|v_1-v_2\|$ *for* $v_1,v_2\in W^s_i$,

$\quad\|F_i^{-1}(v_1)-F_i^{-1}(v_2)\|<\big((1+\lambda)/2\big)\|v_1-v_2\|$ *for* $v_1,v_2\in W^u_{i+1}$.

Under the condition that $F_i(0)=0$ for every i, δ depends in fact only on numbers λ, ε and A.

1.12. THEOREM. *Let λ and A be real numbers, $0<\lambda<1$. Then for δ_0 sufficiently small and for any ε there exist a $\delta>0$ and a positive integer N such that for*

(i) *an arbitrary number $r>0$,*

(ii) *arbitrary sequences of linear spaces $(B^{(1)}_i)^{+\infty}_{i=-\infty}$, $(B^{(2)}_i)^{+\infty}_{i=-\infty}$ and sequences of linear isomorphisms $L^{(j)}_i\colon B^{(j)}_i\to B^{(j)}_{i+1}$, $j=1,2$, with $\|L^{(j)}_i\|\leqslant A$, $\|(L^{(j)}_i)^{-1}\|\leqslant A$, for which there exists the hyperbolic splitting $B^{(j)}_i=E^{(j)s}_i\oplus E^{(j)u}_i$ with coefficient λ,*

(iii) *arbitrary sequences of C^1-mappings $F^{(j)}_i\colon V^{(j)}_i\to B^{(j)}_{i+1}$ ($V^{(j)}_i$ as in Theorem 1.11) such that*

$$\varrho_{C^1}(F^{(j)}_i,L^{(j)}_i|V^{(j)}_i)<\delta_0\quad\text{and}\quad F^{(j)}_i(0)=0,$$

the following assertion holds:

if $B^{(1)}_i=B^{(2)}_i$ and $\varrho_{C^1}(F^{(1)}_i,F^{(2)}_i)<\delta$ for $i=0,1,\ldots,N$, then

$$\varrho_{C^1}(W^{(1)s}_0,W^{(2)s}_0)<\varepsilon\quad\text{and}\quad\varrho_{C^1}(W^{(1)u}_N,W^{(2)u}_N)<\varepsilon.$$

1.13. THEOREM (*Anosov's theorem on families of ε-trajectories*). *Let Λ be a hyperbolic set of $f\colon U\to M$. Then there exist a neighbourhood $U(\Lambda)$ of the set Λ, and numbers $\varepsilon_0,\eta_0>0$ with the following properties:*

For any η $(\eta_0>\eta>0)$ there exists $\varepsilon>0$ such that for any endomor-

phism $f'\colon U\to M$ *with* $\varrho_{C^1}(f,f')<\varepsilon_0$, *for any topological space* X, *any homeomorphism* $g\colon X\to X$ *and any continuous map* $\varphi\colon X\to U(\Lambda)$ *such that* $\varrho(\varphi g, f'\varphi)<\varepsilon$, *there exists a continuous map* $\psi\colon X\to U$ *such that*:

(1) $\psi g = f'\psi$,

(2) $\varrho(\psi,\varphi)<\eta$,

(3) *if for some* $\psi'\colon X\to U$ *we have* $\psi' g = f'\psi'$ *and* $\varrho(\varphi,\psi')<\eta_0$ *then* $\psi'=\psi$

It is essential in Theorem 1.13 that g is a homeomorphism and that is why a proof of the structural stability of Anosov diffeomorphisms in which the above theorem is applied does not work in the case of Anosov endomorphisms.

1.14. COROLLARY. *If* $f, U, \Lambda, U(\Lambda), \varepsilon_0, \eta, \varepsilon, f'$ *satisfy the assumptions of Theorem* 1.13, *then for any* ε-f'-*trajectory* $(y_n)\subset U(\Lambda)$ *there exists exactly one* f'-*trajectory* (x_n) *such that* $\varrho(x_i,y_i)<\eta$ *for every* i.

Proof. In Theorem 1.13 one can take as X the set of all integers with the shift to the right.

1.15. COROLLARY. *Under the assumptions of Theorem* 1.13, *if* (y_n) *is an* ε-f'-*trajectory such that* $y_n=y_{n+p}$ *for some* p *and for every* n, *then the* f'-*trajectory* (x_n) *from Corollary* 1.14 *has period* p.

Proof. One can take $X=\{0,1,\ldots,p-1\}$, $g(i)=i+1 \pmod p$.

Now we can easily prove the following

1.16. THEOREM. *The set of all Anosov endomorphisms is open in* $C^1(M,M)$.

Proof. *The first method.* Let f be an Anosov endomorphism with a hyperbolic coefficient λ. Let $\varepsilon_1>0$. Let ε_2 be equal to δ from Theorem 1.11 chosen for ε_1, λ and

$$A=\max\big(\sup_{x\in M}\|Df_x\|,\ \sup_{x\in M}\|(Df_x)^{-1}\|\big).$$

Let $\varepsilon_3>0$ be such that for every $f'\in C^1(M,M)$ and $x,y\in M$ if $\varrho_{C^1}(f',f)<\varepsilon_3$ and $\varrho(x,y)<\varepsilon_3$ then

$$\|Df'_x-\mathscr{E}_{f'(x)f(y)}\circ Df_y\circ\mathscr{E}_{yx}\|<\varepsilon_2.$$

Let ε_4 be equal to ε from Theorem 1.13 chosen for $\varepsilon_3=\eta$. Take $\varepsilon_5=\min(\varepsilon_4,\varepsilon_3)$. Let $f'\in C^1(M,M)$ satisfy the condition $\varrho_{C^1}(f,f')<\varepsilon_5$. Then any f'-trajectory is an ε_5-f-trajectory. From Corollary 1.14 it follows that for a given f'-trajectory (x_n) there exists an f-trajectory (y_n) with $(x_n,y_n)<\varepsilon_3$ for all n. We define

$$L_n=\mathscr{E}_{x_{n+1}y_{n+1}}\circ Df_{y_n}\circ\mathscr{E}_{y_nx_n}\colon T_{x_n}M\to T_{x_{n+1}}M.$$

It is clear that the splitting $T_{x_n}M=\mathscr{E}_{x_ny_n}(E^u_{y_n})\oplus\mathscr{E}_{x_ny_n}(E^s_{y_n})$ is invariant under L_n. All the assumptions of Theorem 1.11 are satisfied (the role of

maps F_i is played by the derivatives Df'_{x_i}). The resulting manifolds E^s_{f',x_i} and E^u_{f',x_i} are linear subspaces of $T_{x_i}M$ because Df'_{x_i} are linear. Moreover, the splitting $T_{x_i}M=E^s_{f',x_i}\oplus E^u_{f',x_i}$ is invariant under Df'_{x_i} and hyperbolic (with the constant $(1+\lambda)/2$).

We shall give another proof of this theorem. For this purpose we shall use the following theorem.

1.17. THEOREM (A globalization of J. Mather's theorem, see [12], [4], Theorem 1.1). *Let* X *be a normal topological space and let* $\pi\colon TX\to X$ *be a finite-dimensional vector bundle with a continuous Riemannian metric* $\langle\cdot,\cdot\rangle$. *Let* $Df\colon TX\to TX$ *be a linear bundle automorphism covering* $f\colon X\to X$ *and satisfying* $\sup_{x\in X}\|Df_x\|<\infty$. *Let* $\mathscr{V}$ *denote the Banach space of all bounded continuous sections of* TX *with the topology of uniform convergence. We define* $f\colon\mathscr{V}\to\mathscr{V}$ *by* $f_*(V)=Df\circ V\circ f^{-1}$. *Then*:

$$\operatorname{spec}(f_*)\cap S^1=\emptyset$$

iff there exists a hyperbolic continuous splitting of TX *which is preserved by* Df.

Proof is the same as in [4].

Proof of Theorem 1.16. *The second method.* Let f be an Anosov endomorphism with a hyperbolic coefficient λ. Let $f'\in C^1(M,M)$. For a non-periodic f'-trajectory (x_n), denote by φ the natural transformation $n\mapsto x_n$. The transformation φ induces in the usual way a vector bundle $\varphi^*(TM)$ over Z (the set of all integers) such that the following diagram commutes:

$$\begin{array}{ccc}\varphi^*(TM) & \xrightarrow{\ \pi^*(\varphi)\ } & TM\\ \Big\downarrow{\scriptstyle\varphi^*(\pi)} & & \Big\downarrow{\scriptstyle\pi}\\ Z & \xrightarrow{\ \ \varphi\ \ } & M\end{array}$$

φ is injective. Therefore there exists $(\pi^*(\varphi))^{-1}$ (defined on the image of $\pi^*(\varphi)$). For brevity we shall denote this inverse map by a. (In the case of a periodic trajectory with the minimal period p one considers a vector bundle over Z_p instead of a bundle over Z.)

Denote by

$\mathscr{V}$ the space of all bounded sections of $\varphi^*(TM)$,

$\mathscr{V}^s$ the space of all bounded sections of $\varphi^*(E^s)$,

$\mathscr{V}^u$ the space of all bounded sections of $\bigcup_n E^u_n$,

where for each j, E^u_j is defined in the following way: x_j being fixed, we first take an f-trajectory (y^j_n) by putting

(1) $y^j_j=x_j$;

(2) to define y_i^j for $i<j$ we consider the counterimage $P=(f^{j-i})^{-1}(x_j)$; y_i^j is defined as the point of P lying closest to x_j (if there are several points in the same distance to x_j we choose any one of them).

Now we define E_j^u by $E_j^u = a(E^u_{y_j^i})$.

Of course, $\mathscr{V} = \mathscr{V}^s \oplus \mathscr{V}^u$. To simplify our notation we shall denote the shift on Z also by f'. Define

$$Df'_n\colon a(T_{x_n}M)\to a(T_{x_{n+1}}M)$$

as the map induced by $Df'_{x_n}\colon T_{x_n}M\to T_{x_{n+1}}M$. Proceeding in the same way as in Theorem 1.17 we define f'_*.

Put

$$f'_* = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix}\colon \mathscr{V}^s\oplus\mathscr{V}^u\to\mathscr{V}^s\oplus\mathscr{V}^u.$$

Using Theorem 1.10 and Proposition 1.7, one can check that for any $\eta>0$ there exists $\delta(\eta)>0$ such that if $\varrho_{C^1}(f,f')<\delta$, then

$$\text{(3)}\qquad \begin{aligned} &\|A_{11}\| < (1+\lambda)/2, && \|A_{21}\| < \eta,\\ &\|A_{22}^{-1}\| < (1+\lambda)/2, && \|A_{12}\| < \eta. \end{aligned}$$

We shall show that if η is sufficiently small, then inequalities (3) imply $(\operatorname{spec}(f'_*))\cap S^1 = \emptyset$. This and Theorem 1.17 will yield the existence of the structure of an Anosov endomorphism for f.

Write

$$A = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22}\end{bmatrix}.$$

Let us estimate the norm of the resolvent of A for $\xi\in S^1$

$$\|R(A,\xi)\| = \left\|\begin{bmatrix} (A_{11}-\xi\mathbf{1})^{-1} & 0 \\ 0 & (A_{22}-\xi\mathbf{1})^{-1}\end{bmatrix}\right\|.$$

For $V\in\mathscr{V}^s$ we have

$$\|(A_{11}-\xi\mathbf{1})V\| \geqslant \|\xi\mathbf{1}(V)\| - \|A_{11}(V)\| \geqslant \big(1-((1+\lambda)/2)\big)\|V\| = \big((1-\lambda)/2\big)\|V\|.$$

For $V\in\mathscr{V}^u$ we have

$$\|(A_{22}-\xi\mathbf{1})V\| \geqslant \|A_{22}(V)\| - \|V\| \geqslant \big(((1+\lambda)/2)^{-1}-1\big)\|V\| \geqslant \big((1-\lambda)/2\big)\|V\|.$$

For any $v\in T_nZ$ we consider the decompositions

$$v = v^1+v^2,\qquad v^1\in E_n^s,\ v^2\in E_n^{s\perp}$$

and

$$v = v^s+v^u,\qquad v^s\in E_n^s,\ v^u\in E_n^u.$$

Let a be the number defined in Proposition 1.7. We have

$$\|v^u\| \leqslant \|(v^u)^1\| + \|(v^u)^2\| \leqslant (a+1)\|(v^u)^2\| = (a+1)\|v^2\| \leqslant (a+1)\|v\|.$$

In the same way we can get $\|v^s\|\leqslant (a+1)\|v\|$.

Thus, for any $V\in\mathscr{V}$, we obtain

$$\|(A-\xi\mathbf{1})^{-1}V\| \leqslant \big(2/(1-\lambda)\big)\sup_{n\in Z}\big(\|V^s_{(n)}\| + \|V^u_{(n)}\|\big) \leqslant \big(4(1+a)/(1-\lambda)\big)\|V\|.$$

Therefore

$$\inf_{\xi\in S^1}\|R(A,\xi)\|^{-1} \geqslant (1-\lambda)/\big(4(1+a)\big).$$

It follows from the above inequality and from Remark IV, 3.2 of [3], p. 208, that if

$$\text{(4)}\qquad \|f'_*-A\| = \left\|\begin{bmatrix} 0 & A_{12} \\ A_{21} & 0\end{bmatrix}\right\| < (1-\lambda)/\big(4(1+a)\big),$$

then

$$(\operatorname{spec} f'_*)\cap S^1 = \emptyset.$$

For any $V\in\mathscr{V}$:

$$\left\|\begin{bmatrix} 0 & A_{12} \\ A_{21} & 0\end{bmatrix}(V)\right\| \leqslant \|A_{12}(V^s)\| + \|A_{21}(V^u)\| \leqslant \eta\big(\|V^s\|+\|V^u\|\big) \leqslant \eta\cdot 2(1+a)\|V\|.$$

Thus, in order that inequality (4) be satisfied, it suffices to have $\eta\cdot 2(1+a) < (1-\lambda)/\big(4(1+a)\big)$.

Remark. The above method is more economic than the first method of the proof of Theorem 1.16. Our considerations concerning the operator $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix}$ are in fact hidden in the first method in proofs of Theorems 1.11 and 1.13.

1.18. Notation. Let Λ be a hyperbolic set for the endomorphism $f\colon U\to M$. Denote $\tilde\Lambda = \underleftarrow{\lim}(\Lambda, f) \subset \prod_{-\infty}^{+\infty} M$. Define $\tilde f\colon \tilde\Lambda\to\tilde\Lambda$ by $\pi_k\tilde f((x_n)) = x_{k+1}$.

It will be useful to prepare the following essentially topological

1.19. Lemma. *Let $g\colon U\to M$. Then, for any compact set $P\subset \underleftarrow{\lim}\big(\bigcap_{-\infty}^{+\infty} g^n(U), g\big)$ invariant under $\tilde g$*

$$P = \underleftarrow{\lim}\big(\pi_0(P), g\big).$$

Proof. It is obvious that $\underleftarrow{\lim}(\pi_0(P), g)\supset P$. Now let (x_n) be a g-trajectory in $\underleftarrow{\lim}(\pi_0(P), g)$. For any i there exists $a_i\in P$ such that $\pi_j(a_i) = x_j$ for $j\geqslant i$. Thus $a_i\to(x_n)$ and by compactness of P we have $(x_n)\in P$.

Using Theorem 1.13 we can obtain an easy proof of the following

1.20. THEOREM. $(\exists \eta_0)(\forall \eta < \eta_0)(\exists \varepsilon > 0)(\forall f'\colon U \to M)$ *if* $\varrho_{C^1}(f, f')$, *then*:

(a) *there exist* $\Lambda' \subset U$ *such that* $f'(\Lambda') = \Lambda'$ *and a homeomorphism* $h\colon \tilde{\Lambda} \to \tilde{\Lambda}' = \overleftarrow{\lim}(\Lambda', f')$ *such that*

(i) $h \circ \tilde{f} | \tilde{\Lambda} = (\tilde{f}' | \tilde{\Lambda}') \circ h$, *and*

(ii) $\varrho(h, \mathrm{id}) < \eta$,

(b) *if* $P \subset \overleftarrow{\lim} \bigcap_{n=-\infty}^{+\infty} (f'^n(U), f') \subset \mathop{\mathrm{P}}_{-\infty}^{+\infty} M$ *and if a homeomorphism* $h_1\colon \tilde{\Lambda} \to P$ *satisfies conditions*: $h_1 \circ \tilde{f}|_{\tilde{\Lambda}} = (\tilde{f}'|_P) \circ h_1$ *and* $\varrho(h_1, \mathrm{id}) < \eta_0$, *then* $P = \tilde{\Lambda}'$ *and* $h_1 = h$.

Moreover, for each f'*-trajectory* $(x_n) \in \tilde{\Lambda}'$ *there exists a hyperbolic splitting.*

Proof. In Theorem 1.13 one can take $X = \tilde{\Lambda}$, $g = \tilde{f}$ and $\varphi = \pi_0$. Put

$$\Lambda' = \psi(\tilde{\Lambda}) \quad \text{and} \quad h\big((x_n)\big) = \Big(\psi\big(\tilde{f}^k\big((x_n)\big)\big)\Big)_{k=-\infty}^{+\infty}$$

($\psi\colon \tilde{\Lambda} \to U$ is constructed in Theorem 1.13 and fulfils $\psi \circ \tilde{f} = f' \circ \psi$). By Lemma 1.19, $h(\tilde{\Lambda}) = \overleftarrow{\lim}(\Lambda', f')$. In the same way one can construct $\psi'\colon \tilde{\Lambda}' \to U$ such that $\psi' \circ \overleftarrow{f'} = f \circ \psi$. Define

$$h'\big((x_n)\big) = \Big(\psi'\big(\tilde{f}^k\big((x_n)\big)\big)\Big)_{k=-\infty}^{+\infty}.$$

Since $\varrho(\psi \circ h', \pi_0) < \eta_0$ and $\varrho(\psi' \circ h, \pi_0) < \eta_0$, we have $\psi \circ h' = \pi_0$, $\psi' \circ h = \pi_0$. Hence $h \circ h' = \mathrm{id}_{\tilde{\Lambda}'}$, and $h' \circ h = \mathrm{id}_{\tilde{\Lambda}}$, so h is a homeomorphism.

Let $h_1\colon \tilde{\Lambda} \to P$ satisfy the conditions of (b). Then $\varrho(\pi_0 \circ h_1, \pi_0) < \eta_0$, so $\pi_0 \circ h_1 = \psi$, which implies the assertion of (b).

Sometimes we shall use the notation $h_{f'f}$ for h.

Since Anosov endomorphisms are not structurally stable (cf. §§ 0, 4), therefore h need not be a lift of any map. In fact, if h were a lift of $h_1\colon M \to M$, then $f' \circ h_1 = h_1 \circ f$.

1.21. Remark. If $f\colon M \to M$ is an Anosov endomorphism, then $h_{f'f}\big(\overleftarrow{\lim}(M, f)\big) = \overleftarrow{\lim}(M, f')$.

Hence, taking into account Proposition 3.4, one can obtain

1.22. COROLLARY. *The property* $\Omega(f) = M$ *is stable. More exactly*:

$$(\forall f \in \mathrm{End\,An}^1(M))(\exists \varepsilon > 0)(\forall f')\ \big(\varrho_{C^1}(f, f') < \varepsilon \ \&\ \Omega(f) = M\big) \Rightarrow \Omega(f') = M$$

(see § 3 for the definition of nonwandering points).

1.23. Remark. $\pi_0|\mathrm{Per}(\tilde{f})$ is a one-one map, so $h_{f'f}$ defined in Theorem 1.20 induces a one-one map $\theta_{f'f}\colon \mathrm{Per}(f) \cap \Lambda \to \mathrm{Per}(f') \cap \Lambda'$ given by the formula $\theta_{f'f} = \pi_0 \circ h_{f'f} \circ \big(\pi_0|\big(\mathrm{Per}(f) \cap \Lambda\big)\big)^{-1}$.

If $f\colon M \to M$ is an Anosov endomorphism, then $\theta_{f'f}$ maps $\mathrm{Per}(f)$ onto $\mathrm{Per}(f')$.

§ 2. Stable and unstable manifolds.

2.1. THEOREM. *Let* Λ *be a hyperbolic set for an endomorphism* $f\colon U \to M$ *with the hyperbolic coefficient smaller than a* $\lambda < 1$. *Then there exist* $R > 0$ *and* $\mu > 0$ *such that for any* $f'\colon U \to M$ $\varrho_{C^1}(f, f') < \mu$ *and for any* f'*-trajectory* $(x_n) \in h_{f'f}(\tilde{\Lambda})$ (see definitions in Theorem 1.19) *the following conditions are satisfied*:

(a) *for each integer* i *the set*

$$W^s_{f', x_i, R} = \{y \in M\colon (\forall t \geqslant 0)\ \varrho\big(f'^t(y), f'^t(x_i)\big) < R\}$$

is a manifold (called as in the case of a diffeomorphism, the *local stable manifold*);

(b) *for each integer* i *the set*

$$W^u_{f', x_i, R} = \{y \in M\colon (\exists (y_n)_{n=-\infty}^{0})(\forall n < 0)\ [f'(y_n) = y_{n+1} \,\&\, y_n \in U \ \&\ \big((\forall t \geqslant 0)\ \varrho(y_{-t}, x_{i-t}) < R\big)]\}$$

is a manifold (called the *local unstable manifold*);

(c) $(\forall z, y \in W^s_{f', x_i, R})\ \varrho\big(f'^{t+1}(y), f'^{t+1}(z)\big) \leqslant (2+\lambda)/3 \cdot \varrho\big(f'^t(y), f'^t(z)\big)$,

$(\forall z, y \in W^u_{f', x_i, R})\ \varrho(y_{-t-1}, z_{-t-1}) \leqslant (2+\lambda)/3 \cdot \varrho(y_{-t}, z_{-t})$

for $t = 0, 1, \ldots$

(d) $T_{x_i} W^{s(u)}_{f, x_i, R} = E^{s(u)}_{f, x_i}$ *for* $(x_i) \in \tilde{\Lambda}$.

Furthermore,

(e) $(\forall \varepsilon > 0)(\exists 0 < \eta \leqslant R)\big(\forall (x_n) \in \tilde{\Lambda}\big)$ $W^{s(u)}_{f, x_i, \eta}$, $E^{s(u)}_{f, x_i, \eta}$ *are* ε*-*C^1*-close*

(*for the definition of* ε*-*C^1*-close see* [13]).

Proof. Let μ_0 be such a number that if $\varrho_{C^1}(f', f) < \mu_0$, then the hyperbolic coefficient of f' on $\pi_0(\Lambda')$ is smaller than λ and $\max\big(\sup_x \|Df'_x\|, \sup_x \|(Df'_x)^{-1}\|\big) \leqslant \max\big(\sup_x \|Df_x\|, \sup_x \|(Df_x^{-1})\|\big) + 1 = A$.

Choose a number $\delta > 0$ to λ, ε and A as in Theorem 1.11 on uniform perturbations. Then there exist numbers $0 < \mu < \mu_0$, $0 < \eta < \min\big(1, \inf\{\mathrm{dist}(\Lambda', M - U)\colon \varrho_{C^1}(f', f) < \mu\}\big)$ such that

$$(\forall f', \varrho_{C^1}(f, f') < \mu)\ \big(\forall y \in \pi_0(\tilde{\Lambda}')\big)(\forall z \in M)$$

$$\text{(1)} \qquad \varrho(y, z) < \eta \ \Rightarrow\ \big\|D\big(\exp^{-1}_{f'(y)}(y) \circ f' \circ \exp_y\big)_{\exp_y^{-1}(z)} - Df'_y\big\| \leqslant \delta$$

and

$$(\forall v \in T_z M)\ \left(\sqrt{\frac{1+\lambda}{2}} \Big/ \sqrt{\frac{2+\lambda}{3}}\right)\|v\| \leqslant \|(D\exp_y^{-1})_z(v)\| \leqslant \left(\sqrt{\frac{2+\lambda}{3}} \Big/ \sqrt{\frac{1+\lambda}{2}}\right)\|v\|.$$

For any f' with $\varrho_{C^1}(f, f') < \mu$ and for any $(x_n) \in \tilde{\Lambda}'$ consider the sequence

$$\ldots \to T_{x_i}M \xrightarrow{Df'_{x_i}} T_{x_{i+1}}M \to \ldots, \qquad (x_n) \in \tilde{\Lambda}'.$$

Define $F_{i,\eta}\colon B(0, \eta) \to T_{x_{i+1}}M$ by

$$F_{i,\eta}(v) = \exp^{-1}_{x_{i+1}} \circ f' \circ \exp_{x_i}(v).$$

Then inequality (1) implies

$$\|F_{i,\eta}(v) - Df_{x_i}(v)\| \leqslant \|v\| \sup_{w:\ \|w\| \leqslant \|v\|} \|(DF_{i,\eta})_w - Df'_{x_i}\| \leqslant \|v\|\,\delta \leqslant \delta \quad \text{if} \quad \|v\| \leqslant \eta;$$

inequality (1) implies also

$$\|(DF_{i,\eta})_v - D(Df'_{x_i})_v\| \leqslant \delta.$$

So it follows from Theorem 1.11 that there exist manifolds $W^s_{f',i,\eta}$ and $W^u_{f',i,\eta}$ contained in the balls $B(O_{T_{x_i}M}, \eta)$ such that $\varrho_{C^1}(W^{u(s)}_{i,\eta}, E^{u(s)}_{i,\eta}) \leqslant \varepsilon$.

Write $W^{u(s)}_{f',x_i,\eta} = \exp_{x_i}(W^{u(s)}_{i,\eta})$. The number R may now be defined as an arbitrary η satisfying the above conditions. (d) now easily follows from the fact that for $\eta_1 \leqslant \eta_2$

$$W^{u(s)}_{i,\eta_1} \subset W^{u(s)}_{i,\eta_2}.$$

Using Theorem 1.12 one can prove the following

2.2. Proposition. *Let f and R be as in Theorem 2.1. Then*

$$(\forall \varepsilon)(\exists \mu)(\forall f'\colon U \to M)\big(\forall (x_n) \in \tilde{\Lambda}\big)(\forall i)$$
$$\varrho_{C^1}(f, f') < \mu \Rightarrow \varrho_{C^1}(W^{u(s)}_{f',\pi_i h_{f'f}((x_n)),R}, W^{u(s)}_{f,x_i,R}) < \varepsilon.$$

2.3. Proposition (see [4], p. 161). *Let Λ be the hyperbolic set for $f\colon U \to M$. For sufficiently small $\Delta < R$ and sufficiently small μ we have*

$$\big(\forall f'\colon U \to M,\ \varrho_{C^1}(f', f) < \mu\big)(\forall 0 < \delta \leqslant \Delta)(\exists \nu)\big(\forall (x_n), (y_n) \in h_{f'f}(\tilde{\Lambda})\big)(\forall i)$$
$$\big(\varrho(x_i, y_i) < \nu\big) \Rightarrow W^s_{x_i,\delta} \pitchfork W^u_{y_i,\delta}$$

and the intersection consists of exactly one point.

Proof. The proposition easily follows from Theorem 2.1 and from the following generalization of Proposition 1.7:

If f is an endomorphism with Λ the hyperbolic set, then there exist $\mu > 0$, $\xi > 0$, $\alpha > 0$ such that

$$(\forall f'\colon U \to M)\big(\forall (x_n) \in h_{f'f}(\tilde{\Lambda})\big)\big(\forall (y_n) \in h_{f'f}(\tilde{\Lambda})\big)$$
$$\big(\varrho_{C^1}(f, f') < \mu \,\&\, \varrho(x_0, y_0) < \xi\big) \Rightarrow \tan \sphericalangle\big(\mathscr{E}_{x_0y_0}(E^{u(s)}_{f',y_0}), (E^{s(u)\perp}_{f',x_0})\big) < \alpha.$$

2.4. Notation. If $\varrho(x_i, y_i) < \nu = \nu(\Delta)$, we shall denote (as in [4]) by $r(x_i, y_i)$ the unique common point of the manifolds $W^s_{f,x_i,\Delta}$ and $W^u_{f,y_i,\Delta}$.

2.5. Theorem. *Let $(x^k_n)^{+\infty}_{n=-\infty}$, $k = 0, 1, \ldots$, $(x^k_n) \in \tilde{\Lambda}$, be a family of f-trajectories. If $(x^k_n) \xrightarrow[k\to\infty]{} (x^0_n)$* (the convergence in the topology of $\tilde{\Lambda}$), *then*

$$(\forall n) \quad W^{u(s)}_{f,x^k_n,R} \xrightarrow{C^1} W^{u(s)}_{f,x^0_n,R}.$$

Proof. The theorem is an easy corollary of Theorem 1.12.

2.6. Corollary. *If families $(x^k_n), (y^k_n) \in \tilde{\Lambda}$, $k = 0, 1, \ldots$, satisfy $(x^k_n) \xrightarrow[k\to\infty]{} (x^0_n)$ and $(y^k_n) \xrightarrow[k\to\infty]{} (y^0_n)$, then*

$$r(x^k_i, y^k_i) \xrightarrow[k\to\infty]{} r(x^0_i, y^0_i)$$

for every i (of course, provided that $r(x^k_i, y^k_i)$ is defined).

Now we define global unstable manifolds.

2.7. Definition. Let Λ be the hyperbolic set for $f\colon U \to M$. Let $(x_n) \in \tilde{\Lambda}$. The *global unstable manifold* is the set

$$W^u_{x_i} = \bigcup_{n=0}^{+\infty} f^n(W^u_{x_{i-n},R}).$$

More exactly:

$$W^u_{x_i} = \bigcup_{n=0}^{+\infty} \underbrace{f\big(\ldots f\big(f}_{n}(W^u_{x_{i-n},R}) \cap U\big) \cap U\big) \ldots \cap U.$$

Remark. If $U \neq M$, then $W^u_{x_i}$ may be disconnected.

One can in a standard way prove the following

2.8. Theorem (cf. the analogous theorem for diffeomorphisms in [4]). *Let a C^r-map $f\colon M \to M$ be an endomorphism on a neighbourhood of its hyperbolic set Λ. Then for $(x_n) \in \tilde{\Lambda}$ there exist diffeomorphisms $g_{x_ix_{i+1}}\colon R^m \to R^m$ ($m = \dim E^u_{x_i}$) and C^r-maps $k_{x_i}\colon R^m \to M$, $k_{x_i}(R^m) = W^u_{x_i}$, $k_{x_i}(0) = x_i$ such that the diagram*

$$\begin{array}{ccccc}
\ldots \to & R^m & \xrightarrow{\ g_{x_ix_{i+1}}\ } & R^m & \to \ldots \\
 & \downarrow{\scriptstyle k_{x_i}} & & \downarrow{\scriptstyle k_{x_{i+1}}} & \\
\ldots \to & W^u_{x_i} & \xrightarrow{\ \ f\ \ } & W^u_{x_{i+1}} & \to \ldots
\end{array}$$

commutes.

If f is regular on the whole manifold M, then k_{x_i} are immersions. Note that k_{x_i} need not be one-one maps.

2.9. Definition. Let us define a *metric* $\varrho^u_{x_i}$ as the metric on R^m determined by the Riemannian metric on TR^m induced from the Riemannian metric on TM by the immersion

$$k_{x_i}\colon R^m \to W^u_{x_i} \subset M$$

(the subscript x_i in the symbol $\varrho^u_{x_i}$ will be often omitted).

2.10. Notation. Write $W^n_{x_i,\tau,\varrho^u} = \{x \in R^m\colon \varrho^u_{x_i}(x, 0) \leqslant \tau\}$.

Note that this set is compact.

2.11. Notation. Let $f: M \to M$ be an endomorphism. Then we write

$$G_x(f) = \{(x_n) \in \varprojlim(M, f): x_0 = x\}.$$

In the sequel we obtain some results concerning the structure of the set $(W^u_{x_0,R})(x_n) \in G_x(f)$.

To this purpose we distinguish a certain class of Anosov endomorphisms $S\mathrm{End\,An}^r(M)$; its elements will be called *special endomorphisms.*

2.12. Definition. $f \in S\mathrm{End\,An}^r(M)$ if f is an Anosov endomorphism of class C^r and, for every x, $(x_n) \in G_x$, $(y_n) \in G_x$, we have $E^u_{x_0} = E^u_{y_0}$.

2.13. Proposition. *If $f \in S\mathrm{End\,An}^1(M)$, then for every x, $(x_n) \in G_x$, $(y_n) \in G_x$, we have $W^u_{x_0} = W^u_{y_0}$.*

2.14. Remark. Each algebraic hyperbolic endomorphism on the torus is special.

2.15. Theorem. *Let A be an algebraic hyperbolic endomorphism on the torus T^m with a symmetric matrix. Let $\lambda_1 \in \operatorname{spec} A$, $|\lambda_1| = \max_{\lambda \in \operatorname{spec} A} |\lambda|$. Let $\mu_1 \in \operatorname{spec} A$, $|\mu_1| = \max_{\lambda \in \operatorname{spec} A, |\lambda|<1} \lambda$.*

Denote by σ the degree of the endomorphism A $(= |\det A|)$.

If $\sigma - 1 > \lambda_1/\mu_1$, then

$(\forall \varepsilon > 0)(\forall x \in T^m)\big(\exists f \in C^1(T^m, T^m),\ \varrho_{C^1}(f, A) < \varepsilon\big)(\exists \mathscr{R} > 0)$ the family of sets $\{W^u_{x_0,\mathscr{R}}: (x_n) \in G_x(f)\}$ with the Hausdorff metric contains a subset homeomorphic to the interval $\langle 0, 1\rangle$.

The same is true for the family $\{E^u_{x_0} \cap B(0,1): (x_n) \in G_x(f)\}$ with the Hausdorff metric.

(We call the last property "*the property $I_{\mathscr{R}}$ of f at the point x*".)

Proof. It suffices to prove the theorem for non-periodic points because each periodic point is an image of a non-periodic point. In fact, since $\deg A = \sigma > 1$, we have that the inverse-image of any point contains at least two points and at most one of them can be periodic. Let $\nu_1, \bar{\nu}_1, \nu_2, \bar{\nu}_2, \ldots, \nu_p, \bar{\nu}_p, \nu_{p+1}, \ldots, \nu_q$ be all the eigenvalues of A (only the last $q-p$ numbers are real).

Here $p+q = n$. Denote the real parts of the eigenspaces by

$$E_1, E_2, \ldots, E_p, E_{p+1}, \ldots, E_q, \quad \text{respectively}. \tag{1}$$

For every i, $1 \leq i \leq p$, we choose a basis $\{v_i, w_i\}$ in E_i such that $A|E_i = \begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix}$. Choose in R^m an inner product satisfying, for every $1 \leq i \leq p$, the conditions $v_i \perp w_i$, $\|v_i\| = \|w_i\| = 1$ and $E_i \perp E_j$ for $i \neq j$, $1 \leq i, j \leq q$.

Finally, we equip $T(T^m) = T^m \times R^m$ with the Riemannian metric induced by the above inner product and denote the metric on T^m induced by this Riemannian metric by ϱ.

A is a covering map, so there exists a number $\tau > 0$ such that for every $z_1, z_2 \in T^m$ if $A(z_1) = A(z_2)$ and $z_1 \neq z_2$, then $\varrho(z_1, z_2) > 4\tau$.

Let x be a non-periodic point. Let us denote the points of $\bigcup_{n=1}^{+\infty} A^{-n}(\{x\})$ by n-tuples $(\sigma_1, \ldots, \sigma_n)$, $0 \leq \sigma_i < \sigma$, in the following way (defined by induction):

If $(\sigma_1, \ldots, \sigma_n) \in A^{-n}(\{x\})$, then $\sigma - 1$ of counterimages of $(\sigma_1, \ldots, \sigma_n)$ lay outside the 2τ-ball with centre x. We denote those points by $(\sigma_1, \ldots, \sigma_n, 0)$, $(\sigma_1, \ldots, \sigma_n, 1)$, $\ldots$, $(\sigma_1, \ldots, \sigma_n, \sigma-2)$ and the last counterimage by $(\sigma_1, \ldots, \sigma_n, \sigma-1)$.

Denote by $(\sigma_1, \sigma_2, \ldots)$ the trajectories which are elements of $G_x(A)$.

Further on we shall define certain disjoint neighbourhoods of certain points contained in $\bigcup_{n=1}^{+\infty} A^{-n}(\{x\})$, in which we shall perturb the function A.

Now we introduce further notation. Let an inner product be defined in R^n and consider a splitting into orthogonal subspaces $R^n = P_1 \oplus P_2 \oplus \ldots \oplus P_k$, $k \geq 2$. This gives for any $y \in R^n$ the representation $y = y_1 + \ldots + y_k$. Write

$$K_{\alpha_1,\ldots,\alpha_k} = \{x \in P_1: \|x\| \leq \alpha_1\} + \ldots + \{x \in P_k: \|x\| \leq \alpha_k\}. \tag{2}$$

Let $\dim P_1 = \dim P_k = 1$. Define $\varphi_c: K_{\alpha_1,\ldots,\alpha_k} \to K_{\alpha_1,\ldots,\alpha_k}$ as follows:

$$\varphi_c(y_1, \ldots, y_k) = \big(y_1, \ldots, y_{k-1}, y_k + c\cdot\alpha_1\psi_1(y_1/\alpha_1)\cdot\psi_2(\|y_2\|/\alpha_2)\cdot\ldots\cdot\psi_2(y_k/\alpha_k)\big),$$

where $\psi_1, \psi_2: R \to R$ are functions of class C^∞ such that $\operatorname{supp}\psi_1 \subset \langle -1, 1\rangle$, $\psi_1|\langle -1/2, 1/2\rangle$ is linear, $\psi_1(1/2) = -\psi_1(-1/2) = 1$, $|\psi_1(x)| \leq 2$ for every $x \in R$, $\operatorname{supp}\psi_2 \subset \langle -1, 1\rangle$, $\psi_2|\langle -1/2, 1/2\rangle \equiv 1$ and for every $x \in R$ we have $0 \leq \psi_2(x) \leq 1$.

$$\text{If } \Big|2c(\alpha_1/\alpha_k)\sup_{t \in R}|d\psi_2/dt|\Big| < 1, \text{ then } \varphi_c \text{ is a diffeomorphism.} \tag{3}$$

Let us return to the torus T^m. Let $E_{i_1}, \ldots, E_{i_r}$ (respectively, $E_{j_1}, \ldots, E_{j_t}$) be linear subspaces of R^m such that $\{E_{i_1}, \ldots, E_{i_r}\} \subset \{E_1, \ldots, E_q\}$ ($\{E_{j_1}, \ldots, E_{j_t}\} \subset \{E_1, \ldots, E_q\}$) (see (1)) and $E_{i_1} + E_{i_2} + \ldots + E_{i_r} = E^u$ ($E_{j_1} + \ldots + E_{j_t} = E^s$). Let E' (E'') be one-dimensional linear subspace of R^m contained in the A-invariant subspace corresponding to λ_1 (μ_1) (if λ_1 (μ_1) is not real, then E' (E'') is not A-invariant).

Denote by $\lambda_{i_1}, \ldots, \lambda_{i_r}, \mu_{j_1}, \ldots, \mu_{j_t}$ the eigenvalues of A corresponding to $E_{i_1}, \ldots, E_{i_r}$, $E_{j_1}, \ldots, E_{j_t}$.

Let us fix the splitting:

$$T_x(T^m) = \bigoplus_{i=1}^{k} P_i$$

$$= E' \oplus (E_{i_1} \cap E'^{\perp}) \oplus \ldots \oplus (E_{i_r} \cap E'^{\perp}) \oplus (E_{j_1} \cap E''^{\perp}) \oplus \ldots \oplus (E_{j_t} \cap E''^{\perp}) \oplus E''.$$

Consider the splitting

$$T_{(\sigma_1,\ldots,\sigma_n)}(T^m) = \bigoplus_{i=1}^{k} (DA^n_{(\sigma_1,\ldots,\sigma_n)})^{-1}(P_i).$$

Denote the set $\exp_z(K)$ by $K^{(z)}$ and $\exp_z \varphi_c \exp_z^{-1}$ by $\varphi_c^{(z)}$, where K is the set defined in (2).

Write $(\lambda_1/\mu_1)/(\sigma-1) = \vartheta$. For any point $(\sigma_1, \ldots, \sigma_n)$ we define

$$U_{(\sigma_1,\ldots,\sigma_n)} = K^{(\sigma_1,\ldots,\sigma_n)}_{a_1,\ldots,a_k}$$

for the following values of a_i: $a_1 = \tau/k \cdot (1/\lambda_1)^n$, $a_q = \tau/k \cdot (1/\lambda_{i_{q-1}})$ for $q = 2, \ldots, r+1$ and $a_q = \tau/k$ for $q > r+1$.

It is easy to see that $A(U_{(\sigma_1,\ldots,\sigma_n,\sigma_{n+1})}) = U_{(\sigma_1,\ldots,\sigma_n)}$, hence if $0 \leqslant \sigma_1, \ldots, \sigma_{n_1}, \sigma'_1, \ldots, \sigma'_{n_2} \leqslant \sigma - 2$, then:

$$(\sigma_1, \ldots, \sigma_{n_1}) \neq (\sigma'_1, \ldots, \sigma'_{n_2}) \Rightarrow U_{(\sigma_1,\ldots,\sigma_{n_1})} \cap U_{(\sigma'_1,\ldots,\sigma'_{n_2})} = \emptyset.$$

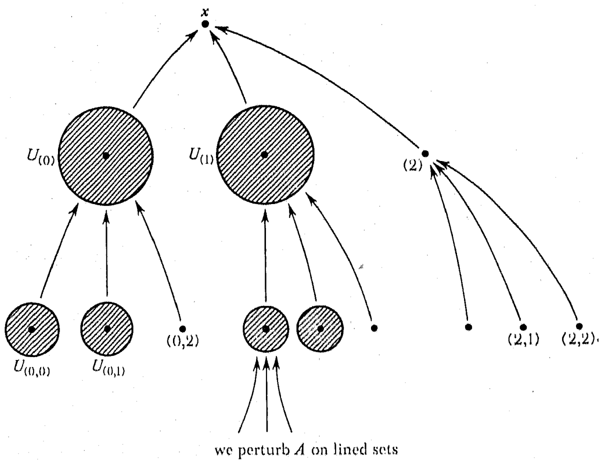


Fig. 1

We now define a transformation $f\colon T^m \to T^m$ as follows:

$$f(y) = \begin{cases} A(y) & \text{for } y \notin \bigcup_n \bigcup_{\substack{(\sigma_1,\ldots,\sigma_n)\\ 0 \leqslant \sigma_i \leqslant \sigma-2}} U_{(\sigma_1,\ldots,\sigma_n)}, \\ A \circ \varphi^{(\sigma_1,\ldots,\sigma_n)}_{c_1 \cdot \sigma_n \cdot \vartheta^n}(y) & \text{for } y \in U_{(\sigma_1,\ldots,\sigma_n)}. \end{cases}$$

It is easy to see that

$$\|D\varphi^{(\sigma_1,\ldots,\sigma_n)}_{c_1 \sigma_n \vartheta^n} - \mathrm{id}\| < c_2 \vartheta^n \quad \text{for some constant } c_2.$$

Hence and by (3), if $c_1 > 0$ is small enough, then f is a C^1-diffeomorphism. c_2 can be chosen arbitrarily small, provided c_1 is small enough. Therefore, $\varrho_{C^1}(A, f) < \varepsilon$.

It is obvious that if $\sigma_1, \ldots, \sigma_n \leqslant \sigma - 2$, then

$$W^u_{f,\pi_0((\sigma_1,\sigma_2,\ldots,\sigma_n,0,0,\ldots)),\tau/k} = f^n(W^u_{A,(\sigma_1,\ldots,\sigma_n),\tau/k} \cap U_{(\sigma_1,\ldots,\sigma_n)}).$$

Write $\mathscr{R} = (1/2)(\tau/k)$. Now we shall define a transformation

$$\iota\colon \langle 0, 1\rangle \to \{\overline{W^u_{f,x_0,\mathscr{R}}}\colon (x_n) \in G_x(f)\} = \mathscr{W}.$$

Write

$$\mathscr{A} = \Big\{t \in \langle 0, 1\rangle\colon t = \sum_{i=1}^{n} \sigma_i/(\sigma-1)^i,\ 0 \leqslant \sigma_i < \sigma - 1,\ n = 1, 2, \ldots\Big\}.$$

Of course, $\mathscr{A}$ is dense in $\langle 0, 1\rangle$. We first define $\iota'\colon \mathscr{A} \to \mathscr{W}$ by

$$\iota'\Big(\sum_{i=1}^{n} \sigma_i/(\sigma-1)^i\Big) = \overline{W^u_{f,\pi_0((\sigma_1,\sigma_2,\ldots,\sigma_n,0,0,\ldots))}}.$$

It is obvious that ι' is an injective Lipschitz map and its inverse is Lipschitz, too. Furthermore, $\iota'(\mathscr{A})$ is dense in $\mathscr{W}$ and in view of Theorem 2.5 $\mathscr{W}$ is compact. Therefore one can extend ι' to a homeomorphism $\iota\colon \langle 0,1\rangle \to \mathscr{W}$.

2.16. Example. The endomorphism

$$\begin{bmatrix} n & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & n \end{bmatrix}$$

for large n satisfies the hypothesis of Theorem 2.15.

2.17. Remark. If $(\sigma - 1) > (\lambda_1^r/\mu_1)$, then in Theorem 2.15 one can claim that $f \in C^r(M, M)$.

Another property of the family $(W^u_{x_0,\mathscr{R}})_{(x_n) \in G_x}$ is described in the following

2.18. Theorem. *Let $f \in S\,\mathrm{End}\,\mathrm{An}^r(M) - \{\mathrm{Diff\,An} \cup \mathrm{Expandings}\}$, $1 \leqslant r \leqslant +\infty$. Then for every $\varepsilon > 0$ and $x \in M$ there exist $f'\colon M \to M$, $\mathscr{R} > 0$ and* *a set $G'_x \subset G_x(f')$ such that $\varrho_{C^r}(f,f') < \varepsilon$, G'_x is homeomorphic to the Cantor set and for any $\eta \leqslant \mathcal{R}$ the map $G'_x \ni (x)_n \mapsto \overline{W^u_{f',x_0,\eta}} - W^u_{f',x_0,\eta}$ is a homeomorphism of G'_x onto the family $\{\overline{W^u_{f',x_0,\eta}} - W^u_{f',x_0,\eta}\}_{(x_n)\in G'_x}$, equipped with the Hausdorff metric.*

Analogously, $G'_x \ni (x_n) \mapsto E^u_{x_0}$ is a homeomorphism.

Sketch of the proof. Let x be a non-periodic point. Let τ, σ be the numbers defined in Theorem 2.15. Let a_1, $0 < a_1 < \tau$, be such a number that $f(B(f(x), a_1)) \cap B(f(x), a_1) = \emptyset$.

Let a_2, $0 < a_2 < \tau$, be such a number that $f(B(x, a_2)) \subset B(f(x), a_1)$. As in the proof of Theorem 2.15 we denote by (0) some point contained in $f^{-1}(\{x\})$ such that $(0) \notin B(x, a_2) \cup B(f(x), a_1)$. Let us denote the points of $\bigcup_{n=0}^{+\infty} f^{-n}(\{x\})$ in the following way (defined by induction): if $(\sigma_1, \ldots, \sigma_n) \in f^{-n}(\{x\})$, $0 \leqslant \sigma_i \leqslant \sigma - 1$, then there exists a point, which will be denoted by $(\sigma_1, \ldots, \sigma_n, 0)$, contained in $f^{-n-1}(\{x\})$ such that $(\sigma_1, \ldots, \sigma_n, 0) \notin B(x, a_2) \cup B(f(x), a_1)$.

Now, using a method similar to the method applied in Theorem 2.15, one can perturb f in the union of disjoint neighbourhoods of the following points:

$$(\sigma_1, \ldots, \sigma_{2n}) \quad \text{with} \quad \sigma_{2i+1} = 0 \quad \text{for} \quad i\colon 0 \leqslant i \leqslant n-1.$$

The set G' can be defined as the set of all trajectories $(\sigma_1, \ldots)$ for which $\sigma_{2i+1} = 0$, $i = 0, 1, \ldots$

2.19. COROLLARY. *Any element of $\mathcal{S}$ End An(M) – (Diff An ∪ Expandings) is not a structurally stable map.*

Proof. If there exists a homeomorphism $\varphi\colon M \to M$ such that $\varphi \circ f' = f \circ \varphi$ (where f' is constructed in Theorem 2.18), then, for some δ, $\varrho(x, y) < \delta$ implies $\varrho(\varphi(x), \varphi(y)) < R$. Hence

$$\varphi(W^u_{f',\pi_0((x_n)),\delta}) \subset W^u_{f,\varphi(x),R}, \qquad \varphi(W^u_{f',\pi_0((x'_n)),\delta}) \subset W^u_{f,\varphi(x),R}$$

for some f'-trajectories (x_n), (x'_n) such that $(x_n) \neq (x'_n)$, $(x_n), (x'_n) \in G'_x$. This contradicts the assumption that φ is a homeomorphism.

§ 3. Axiom A endomorphisms.

3.1. DEFINITION. Let $f\colon M \to M$ be a continuous map. The set $\Omega(f)$ of nonwandering points is defined by

$$\Omega(f) = \{x \in M\colon (\forall U \text{ a neighbourhood of } x)\ (\exists n > 0)\ f^n(U) \cap U \neq \emptyset\}.$$

Corollaries 1.14, 1.15 imply the following

3.2. PROPOSITION. *Let $f\colon M \to M$ be an Anosov endomorphism. Then $\overline{\mathrm{Per} f} = \Omega(f)$.*

Hence, as in the case of a diffeomorphism we can generalize the notion of an Anosov endomorphism:

3.3. DEFINITION. We call a map $f \in C^1(M, M)$ an *Axiom A endomorphism* iff:

(a) each point of $\Omega(f)$ is regular,

(b) $\Omega(f)$ is a hyperbolic set,

(c) $\overline{\mathrm{Per} f} = \Omega(f)$.

The following proposition can be verified directly:

3.4. PROPOSITION. $\Omega(\tilde f) = \widetilde{\Omega(f)} = \{(x_n) \in \tilde M\colon \text{for every } n,\ x_n \in \Omega(f)\}$.

(We recall that $\tilde f$ denotes the shift in the space $\tilde M = \underleftarrow{\lim}(M, f)$, $\widetilde{\Omega(f)} = \underleftarrow{\lim}(\Omega, f|\Omega)$).

3.5. PROPOSITION. *If f is an Axiom A endomorphism, then $\mathrm{Per}\tilde f$ is dense in $\Omega(\tilde f)$.*

We introduce the following metric on $\tilde M$:

$$\tilde\varrho((x_n), (y_n)) = \sup_{n\in Z} ((2+\lambda)/3)^{|n|} \varrho(x_n, y_n),$$

where λ is the hyperbolic coefficient of f. Let $(x_n) \in \Omega(\tilde f)$.

Remark. It is obvious that the condition (c) implies $f(\Omega(f)) = \Omega(f)$. The condition (a) implies f be an endomorphism in a neighbourhood of $\Omega(f)$ in the sense of Definition 1.1.

Define:

$$\tilde W^{s(u)}_{(x_n)} = \{(y_n) \in \tilde M\colon \tilde\varrho(\tilde f^k((y_n)), \tilde f^k((x_n))) \xrightarrow[k\to\infty(-\infty)]{} 0\},$$

$$\tilde W^{s(u)}_{(x_n),\delta} = \{(y_n) \in \tilde M\colon \tilde\varrho(\tilde f^k((y_n)), \tilde f^k((x_n))) < \delta \quad \text{for} \quad k \geqslant 0\ (k \leqslant 0)\}.$$

It is obvious that

$$\tilde W^{s(u)}_{(x_n)} = \{(y_n) \in \tilde M\colon \varrho(y_n, x_n) \xrightarrow[n\to\infty(-\infty)]{} 0\}.$$

Now we prove simple propositions:

3.6. PROPOSITION. *We have*

$$(\exists \delta^*)(\exists 0 < \mu < 1)(\forall k \geqslant 0)\ \tilde\varrho(\tilde f^k((x_n)), \tilde f^k((y_n))) \leqslant \mu^k \tilde\varrho((x_n), (y_n)) \quad \textit{for } (y_n) \in \tilde W^s_{(x_n),\delta^*},$$

$$(\forall k \leqslant 0)\ \tilde\varrho(\tilde f^k((x_n)), \tilde f^k((y_n))) \leqslant \mu^{|k|} \tilde\varrho((x_n), (y_n)) \quad \textit{for } (y_n) \in \tilde W^u_{(x_n),\delta^*}.$$

Proof. We put $\delta^* = R$ (R was defined in Theorem 2.1). If $(y_n) \in \tilde W^s_{(x_n),R}$, then

$$\tilde\varrho(\tilde f((x_n)), \tilde f((y_n))) = \sup_n ((\lambda+2)/3)^{|n|} \varrho(x_{n+1}, y_{n+1})$$

$$= \max\Big(\sup_{n\leqslant 0} ((\lambda+2)/3)^{|n-1|} \varrho(x_n, y_n), \sup_{n\geqslant 0} ((\lambda+2)/3)^{|n|} \varrho(x_{n+1}, y_{n+1})\Big)$$

$$\leqslant \max\Big(((2+\lambda)/3) \sup_{n\leqslant 0} ((\lambda+2)/3)^{|n|} \varrho(x_n, y_n), ((2+\lambda)/3) \sup_{n\geqslant 0} ((2+\lambda)/3)^{|n|} \varrho(x_n, y_n)\Big)$$

$$\leqslant ((2+\lambda)/3)\tilde\varrho((x_n), (y_n)).$$

It suffices to put $\mu = (2+\lambda)/3$.

A similar argument holds in the case of the unstable space.

3.7. PROPOSITION.

$$(\exists\, \delta^{**})(\forall\, 0 < \delta \leqslant \delta^{**})(\exists\, \varepsilon > 0)\big(\forall\, (x_n), (y_n) \in \Omega(f)\big) \quad \tilde{\varrho}\big((x_n), (y_n)\big) < \varepsilon \Rightarrow$$
$$\Rightarrow \tilde{W}^s_{(x_n),\delta} \cap \tilde{W}^u_{(y_n),\delta} \text{ contains exactly one point } r\big((x_n), (y_n)\big).$$

Moreover, $r\big((x_n), (y_n)\big) \in \Omega(\tilde{f})$ (i.e. $\tilde{\Omega}$ has the local product structure).

Proof. Take $\delta^{**} < \Delta$ (Δ is defined in Proposition 2.3). For $\delta \leqslant \delta^{**}$ take $\varepsilon = \min\big(\nu(\delta/2), \delta/2\big)$ (ν is defined in 2.3). Let $\tilde{\varrho}\big((x_n), (y_n)\big) < \varepsilon$, $(x_n), (y_n) \in \tilde{\Omega}$.

Write $z_0 = r(x_0, y_0)$ (cf. 2.4). Of course, $z_0 \in W^u_{y_0,\delta/2}$. Let $(z_n) \in \tilde{M}$ be an f-trajectory such that $z_n \in W^u_{y_n,\delta/2}$ for $n \leqslant 0$. If $n \geqslant 0$, then $\varrho(z_n, x_n) < \mu^n \varrho(z_0, x_0)$. Since $\varrho(x_n, y_n) \leqslant 1/\mu^n$, we have $\varrho(z_n, y_n) \leqslant (1/\mu^n)\,\delta$.

If $n \leqslant 0$, then $\varrho(z_n, y_n) \leqslant \mu^{|n|}\varrho(z_0, y_0)$. Since $\varrho(x_n, y_n) \leqslant 1/\mu^{|n|}$, we have $\varrho(z_n, x_n) \leqslant (1/\mu^{|n|})\,\delta$. Hence $(z_n) \in \tilde{W}^s_{(x_n),\delta} \cap \tilde{W}^u_{(y_n),\delta}$.

One can prove that $(z_n) \in \tilde{\Omega}$ in a standard way. It may be done as in the proof of Theorem 6.1 of [4] (apply Theorem 1.13 and Theorem 2.5).

In a similar way as in [4] one can prove the following

3.8. PROPOSITION. (a) *$\tilde{\Omega}$ is a locally maximal set (i.e., there exists an open set U: $\tilde{\Omega} \subset U \subset \tilde{M}$ such that*

$$\{\tilde{f}^i((x_n))\}_i \subset U \Rightarrow \{\tilde{f}^i((x_n))\}_i \subset \tilde{\Omega}.$$

(b) $\big(\forall (x_n) \in \tilde{M}\big)\big(\exists (y_n) \in \tilde{\Omega}\big)$ $(x_n) \in \tilde{W}^u_{(y_n)}$ (if f maps M onto M).

3.9. PROPOSITION. (a) *Ω has a local product structure (i.e., if (x_n), $(y_n) \in \tilde{\Omega}$ and $\varrho(x_0, y_0) < \nu(\Delta)$, then $r(x_0, y_0) \in \Omega$).*

(b) *Ω is a locally maximal set.*

(c) $(\forall x \in M)\big(\exists (y_n) \in \tilde{\Omega}\big)$ $x \in W^u_{y_0}$ (if f maps M onto M).

Proof. (a) follows from Theorem 1.13, (b) and (c) follow immediately from Proposition 3.8.

One can easily generalize Corollary 2.6 as follows:

3.10. PROPOSITION. *The map*

$$r(\cdot,\cdot)\colon \big\{\big((x_n), (y_n)\big) \in \Omega(\tilde{f}) \times \Omega(\tilde{f})\colon\ \tilde{\varrho}\big((x_n), (y_n)\big) < \varepsilon(\delta^{**})\big\} \to \Omega(\tilde{f})$$

is continuous.

Write $\hat{W}^{u(s)}_{(x_n)} = \tilde{W}^{u(s)}_{(x_n)} \cap \Omega(\tilde{f})$.

Propositions 3.5, 3.6, 3.7 imply that $\tilde{f}|\Omega(\tilde{f})$ is a homeomorphism satisfying Axiom A*. For such homeomorphisms R. Bowen in [1] proved the following:

3.11. THEOREM (*spectral decomposition theorem*). *There exists the decomposition $\Omega(\tilde{f}) = \Omega^1(\tilde{f}) \cup \ldots \cup \Omega^k(\tilde{f})$ such that $\Omega^j(\tilde{f})$ are closed disjoin*

sets, $\tilde{f}(\Omega^j) = \Omega^{g(j)}$, where g is a permutation of $(1, 2, \ldots, k)$, and if $g^r(j) = j$, then $\tilde{f}^r\colon \Omega^j \to \Omega^j$ is C-dense (C-density is defined in [1]).

Remark. One can define the sets $\Omega^j(\tilde{f})$ in the theorem above as the closures of the equivalence classes $\mathrm{Per}^j(\tilde{f})$ of the equivalence relation $\sim$ defined by:

$$(x_n) \sim (y_n) \quad \text{iff} \quad \overline{\hat{W}^u_{(x_n)}} \cap \overline{\hat{W}^u_{(y_n)}} \neq \emptyset.$$

3.12. DEFINITION. We define an *equivalence relation* in the set $\mathrm{Per}(f)$ as follows:

$$x \sim y \Leftrightarrow [k_{f,x}(R^m) \cap W^s_{x,r} \neq \emptyset,\ k_{f,y}(R^m) \cap W^s_{x,R} \neq \emptyset \text{ and each of these two inter-sections is transversal at least at one point}].$$

3.13. PROPOSITION. *The sets $\Omega^j(f) = \pi_0\Omega^j(\tilde{f})$, $j = 1, 2, \ldots, k$, are precisely the closures of equivalence classes of the relation introduced in Definition* 3.12: $\mathrm{Per}^j(f)$.

3.14. Remark. It is not difficult to prove that for any $(x_n) \in \tilde{\Omega}$, $x_0 \in \Omega^j$, we have $\overline{W^u_{x_0}} \cap \Omega \supset \Omega^j$.

For diffeomorphisms one can replace the above inclusion by the equality. In the case of the endomorphism the inclusion may be proper. There is an example:

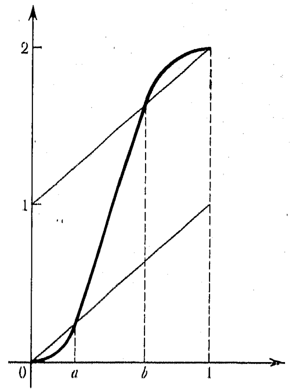


Fig. 2

3.15. EXAMPLE ([9]). Let $f\colon S^1 \to S^1$ be defined as shown in Fig. 2. We have $\Omega = \Omega^1 \cup \Omega^2$, $\Omega^1 = \{\exp(0)\}$, $\Omega^2 = S^1 - \bigcup_{n\geqslant 0} f^{-n}\big(\exp 2\pi i(\langle 0, a) \cup \cup \langle b, 1\rangle)\big)$.

It can be seen that for any $x \in \Omega^2$ if (x_n) is an f-trajectory such that $x_0 = x$, then the point $\exp(0)$ belongs to $W^u_{x_0}$.

The following two propositions will be used in § 4:

3.16. Proposition. *Let* $f\colon M \xrightarrow{\text{onto}} M$ *be an Axiom A endomorphism. Then for any points* $z^j \in \mathrm{Per}^j(f)$, $j = 1, 2, \ldots, k$, *we have* $\bigcup_{j=1}^{k} \overline{W^u_{z^j}} = M$.

Proof. This easily follows from Proposition 3.9 (c).

3.17. Proposition. *Let* $f\colon M \to M$ *be an Anosov endomorphism. Then there exists a neighbourhood* $\mathcal{O}_1 \subset \mathrm{End\,An}^1 M$ *of the endomorphism* f *such that if* $f' \in \mathcal{O}_1$ *then the conjugacy* $\theta_{f'f}\colon \mathrm{Per}(f) \to \mathrm{Per}(f')$ *preserves spectral decomposition.*

Proof. $h_{f'f}$ is a homeomorphism between $\varprojlim(M, f)$ and $\varprojlim(M, f')$. Moreover, $h_{f'f}$ conjugates $\tilde{f}'$ with $\tilde{f}$, therefore $h_{f'f}\hat{W}^u_{f,(x_n)} = \hat{W}^u_{f,h_{f'f}((x_n))}$. Hence $h_{f'f}$ preserves the relation $\sim$ defined in Remark 3.11. Consequently, our statement follows from the equality in Proposition 3.13.

We conclude this section with some results about measures invariant under f or $\tilde{f}$ if f is an endomorphism satisfying Axiom A. $\tilde{f}|\tilde{\Omega}$ is a homeomorphism which satisfies Axiom A* and has the property described in Proposition 3.10. Thus, for $\tilde{f}|\tilde{\Omega}$, Bowen's theorems proved in [1], [2] hold. So we have the following theorem:

3.18. Theorem. *Let* $g^{r_j}(j) = j$. *Denote* $\tilde{X}^j = \bigcup_{r=1}^{r_j} \Omega^{g^r(j)}(f)$. *There exists on* $\tilde{X}^j$ *exactly one probabilistic measure* $\tilde{\mu}$ *invariant under* $\tilde{f}$ *with entropy* $h_{\tilde{\mu}}(\tilde{f}|\tilde{X}^j) = h_{\mathrm{top}}(\tilde{f}|\tilde{X}^j)$. $\tilde{\mu}$ *can be obtained as the weak limit of the sequence* $(\tilde{\mu}_n)_1^\infty$, *where* $\tilde{\mu}_n(E) = \dfrac{N_{r_j\cdot n}(\tilde{f}|E)}{N_{r_j\cdot n}(\tilde{f}|X^j)}$ *for* E *a Borel subset of* $\tilde{X}^j$. *There exists a Markov partition for* $\tilde{f}|\tilde{X}^j$. *Therefore the system* $(\tilde{X}^j, \tilde{f}|\tilde{X}^j, \mu)$ *is isomorphic to a Markov chain.*

3.19. Note. Let $\tilde{\nu}$ be an invariant measure for $\tilde{f}|\Omega(\tilde{f})$. Write $\nu = \pi_0^*(\tilde{\nu})$ (i.e., for every E $\nu(E) = \tilde{\nu}\big(\pi_0^{-1}(E)\big)$). Then the system $\big(\Omega(\tilde{f}), \tilde{f}|\Omega(\tilde{f}), \tilde{\nu}\big)$ is a natural extention of $\big(\Omega(f), f|\Omega(f), \nu\big)$.

Theorem 3.18 and Note 3.19 imply:

3.20. *There exists on* X^j *exactly one probabilistic measure* μ *invariant under* f *with entropy* $h_\mu(f|X^j) = h_{\mathrm{top}}(f|X^j)$. μ *can be obtained as the weak limit of the sequence* $(\mu_n)_1^\infty$, *where*

$$\mu_n(E) = \frac{N_{r_j\cdot n}(f|E)}{N_{r_j\cdot n}(f|X^j)}$$

for E *a Borel subset of* X^j.

3.21. Remark. If $g(j) = j$, then $\tilde{f}|\tilde{X}^j$ is C-dense, hence the system $(\tilde{X}^j, \tilde{f}|\tilde{X}^j, \tilde{\mu})$, which is a natural extention of $(X^j, f|X^j, \mu)$, is isomorphic to a mixing Markov chain, hence it is isomorphic to a Bernoulli shift.

§ 4. Nonstability of Anosov endomorphisms.

4.1. Definition. We say that an Anosov endomorphism f has the *property* $(*)$ iff for every $x, y \in \mathrm{Per}(f)$ we have $x \notin k_y(R^m - \{0\})$ (for the definition of k_y see Theorem 2.8).

Notice that for $x = y$ this means just that $k_x^{-1}(\{x\}) = \{0\}$.

4.2. Examples. (a) The endomorphism from Example 1.3 has the property $(*)$. Indeed, if $x \in T^2$ is periodic, then its coordinates are rational. But $k_x(t) = x + t\cdot(a, b)$ and a/b is irrational.

(b) Expanding maps have not the property $(*)$. More precisely, if f is an expanding map of M, then for every $x, y \in M$ we have $x \in k_y(R^m - \{0\})$.

Denote by $\mathrm{End\,An}'^r(M)$ the set $\mathrm{End\,An}^r(M) - \{\text{Expandings}\}$.

4.3. Theorem. *The set of all elements of* $\mathrm{End\,An}'^r(M)$ *satisfying the property* $(*)$ *is a dense* G_δ *(residual) subset of* $\mathrm{End\,An}'^r(M)$.

Proof. Fix any Riemannian metric $\langle\cdot,\cdot\rangle$ on M (independent of f unlike the metrics adapted to f used in previous sections). Let ϱ be the metric induced by $\langle\cdot,\cdot\rangle$.

(i) Let n be an arbitrarily fixed integer. Write

$$\mathrm{Per}_n f = \{x_f^1, x_f^2, \ldots, x_f^{i(f)}\}, \qquad a_f = \min_{j_1 \neq j_2} \varrho(x_f^{j_1}, x_f^{j_2}).$$

Let Γ be the set of all noncontractible piecewise-C^1 loops in M. $l_\varrho(\gamma)$ denotes the length of γ. Finally, $\mathrm{diam}_\varrho(M)$ denotes the diameter of M in the metric ϱ. Put

$$\varepsilon_f = a_f/3\cdot\frac{\min(C, \mathrm{diam}_\varrho M)}{\mathrm{diam}_\varrho M}, \qquad \text{where} \qquad C = \inf_{\gamma\in\Gamma} l_\varrho(\gamma).$$

Of course, $\varepsilon_f < C/2$, hence for every f-trajectory (x_n) $k_{x_i}|W^u_{x_i,\varepsilon_f,\varrho^u}$ is an embedding (for the definition of $W^u_{x_i,\varepsilon_f,\varrho^u}$ see 2.10). Indeed, if on the contrary $k_{x_i}(a) = k_{x_i}(b)$ for $a \neq b$, then there exists a C^1-path in $W^u_{x_i,\varepsilon_f,\varrho^u}$ joining a with b, such that $l_{\varrho^u}(\gamma) < C$. For p sufficiently large $k_{x_{i-p}}|(g_{x_{i-1}x_i}\circ\ldots\circ g_{x_{i-p}x_{i-p+1}})^{-1}(W^u_{x_i,\varepsilon_f,\varrho^u})$ is an embedding, therefore the path $\gamma_1 = k_{x_{i-p}}\circ\circ(g\circ\ldots\circ g)^{-1}(\gamma)$ is not closed and $k_{x_i}(\gamma) = f^p(\gamma_1)$. But $l_\varrho(k_{x_i}(\gamma)) = l_{\varrho^u}(\gamma) < C$, thus $k_{x_i}(\gamma)$ is contractible. This contradicts the fact that f^p is a covering map.

In the sequel we shall identify $W^u_{x,{}_i\varepsilon,{}_f\varrho^u}$ with its image under k_{x_i}.

Now define

$$A^k = \{f \in \mathrm{End\,An}'^r(M)\colon (\forall y \in M)\ (f^k(y) \in \mathrm{Per}_n(f)) \Rightarrow y \notin \bigcup_{j=1}^{i(f)} (W^u_{f,x_f^j,\varepsilon_f,\varrho^u} - \{x_f^j\})\}.$$

We prove that for every k, A^k is dense in $\mathrm{End\,An}'^r(M)$. It is evident (in view of $\varepsilon_f \leqslant (a_f/3)$) that $A^0 = \mathrm{End\,An}'^r(M)$. It suffices to prove that

A^{k+1} is dense in A^k. Let $f \in A^k$; as long as f is the only transformation considered, we may omit the subscript f. Denote by $Y = \{y_1, \ldots, y_s\}$ the set of all points such that

$$f^{k+1}(y_t) \in \mathrm{Per}_n(f),$$
$$y_t \in \left(W^u_{x^{j(t)},\varepsilon,\varrho^u}\right) - \{x^{j(t)}\} \quad \text{for some } j(t),\ 1 \leqslant j(t) \leqslant i(f).$$

Write

$$Z = \left\{z \in M:\ f^{k+1}(z) \in \left(\mathrm{Per}_n(f)\right)\right\} - Y.$$

Let $\eta_1 < \varepsilon$ be a number satisfying the following conditions

(1) for $t = 1, \ldots, s$ $B_\varrho(y_t, \eta_1) \cap Z = \emptyset$

(in particular, $B_\varrho(y_t, \eta_1) \cap \mathrm{Per}_n(f) = \emptyset$).

(2) $$B_\varrho(y_{t_1}, \eta_1) \cap B_\varrho(y_{t_2}, \eta_1) = \emptyset \text{ if } t_1 \neq t_2,\ 1 \leqslant t_1,\ t_2 \leqslant s.$$

From (1) and (2) it follows that

(3) $$f\left(B_\varrho(y_t, \eta_1)\right) \cap \left(\{z \in M:\ f^k(z) \in \mathrm{Per}_n(f)\} - \{f(y_t)\}\right) = \emptyset \quad \text{for} \quad t = 1, \ldots, s.$$

Let η_2 be such that for every $x \in W^u_{x^{j(t)},\eta_2,\varrho^u}$ there exists an f-trajectory (x_τ) such that $x_0 = x$ and for every $\tau \geqslant 0$

$$\varrho\left(x_{-\tau}, f^{r\cdot \mathrm{ord}\, x^{j(t)}-\tau}(x^{j(t)})\right) < \eta_1/2.$$

From the definition of $W^u_{x^{j(t)}}$ it follows that there exists such p that for any $t = 1, \ldots, s$ we have

(4) $$W^u_{x^{j(t)},\varepsilon,\varrho^u} \subset f^p(W^u_{x^{j(t)},\eta_2,\varrho^u}).$$

Put

(5) $$\eta_3 = \mathrm{dist}_\varrho\Big(Y,\ \bigcup_{t=1}^{s} \bigcup_{q=1}^{p} K_{t,q}\Big),$$

where

$$K_{t,q} = k_{x^{j(t)}_{-q}} \circ g^{-1}_{x^{j(t)}_{-q},x^{j(t)}_{-q+1}} \circ \ldots \circ g^{-1}_{x^{j(t)}_{-1},x^{j(t)}}(W^u_{x^{j(t)},\varepsilon,\varrho^u})$$

and $x^{j(t)}_{-q}, x^{j(t)}_{-q+1}, \ldots, x^{j(t)}_{-1}, x^{j(t)}$ are the points of the periodic f-trajectory of $x^{j(t)}$.

Since $q \geqslant 1$, $f \in A^k$ and $f^q|K_{t,q}$ is injective, we have $Y \cap K_{t,q} = \emptyset$. But every $K_{t,q}$ is compact and therefore $\eta_3 > 0$.

Take $\eta \leqslant \min(\eta_1/2, \eta_3)$ such that $f|V_t$, where $V_t = B_\varrho(y_t, \eta)$ is a one-one map for $t = 1, \ldots, s$.

It is easy to see that there exists an f', arbitrarily close to f in C^r-topology, such that the following conditions are satisfied:

(6) $$f'|\Big(M - \bigcup_{t=1}^{s} V_t\Big) = f|\Big(M - \bigcup_{t=1}^{s} V_t\Big),$$

(7) $$\mathrm{Per}_n(f') = \{x^1, \ldots, x^{i(f)}\},$$

(8) $$f'(V_t) = f(V_t),$$

(9) $$f'|V_t \text{ is a one-one map},$$

(10) $y'_t \notin W^u_{f,x^{j(t)},\varepsilon_f,\varrho^u}$, where y'_t satisfy conditions $y'_t \in V_t, f'(y'_t) = f(y_t)$.

It is obvious that $y'_t \notin W^u_{f,x^{j(\bar t)},\varepsilon_f,\varrho^u}$ for each $\bar t$; $j(\bar t) \neq j(t)$, because $\eta < \eta_1 < \varepsilon_f < a_f/3$. Hence $V_t \cap W^u_{f,x^{j(\bar t)},\varepsilon_f,\varrho^u} = \emptyset$.

From this construction it follows that for any $y \in M$

(11) $$f'^{k+1}(y) \in \mathrm{Per}_n(f') \ \Rightarrow\ y \notin \bigcup_{j=1}^{i(f)} W^u_{f,x^j,\varepsilon_f,\varrho^u}.$$

Of course, $\varepsilon_f = \varepsilon_{f'}$. By the definition of η_2 we have

$$W^u_{f,x^j,\eta_2,\varrho^u} = W^u_{f',x^j,\eta_2,\varrho^u}.$$

Consequently, in view of (4) and (5), we get

$$W^u_{f,x^j,\varepsilon_f,\varrho^u} \subset W^u_{f',x^j,\varepsilon_{f'},\varrho^u}.$$

Hence and by the definition of the metric ϱ^u we obtain

$$W^u_{f,x^j,\varepsilon_f,\varrho^u} = W^u_{f',x^j,\varepsilon_{f'},\varrho^u}.$$

Thus $f' \in A^{k+1}$.

A^k is an open subset of $\mathrm{End\,An}'^r(M)$. Indeed, if f' is close to f in C^r-topology, then $W^u_{f',\theta_{f'f}(x),\varepsilon_f,\varrho^u}$ is close to $W^u_{f,x,\varepsilon_f,\varrho^u}$ in the Hausdorff metric. Now it suffices to observe that $\varepsilon_{(\cdot)}$: $\mathrm{End\,An}'^r(M) \to R$ is a continuous function.

(ii) In the above considerations we had fixed an integer n. Denote the set $\bigcap_{k \geqslant 0} A^k$ by A_n. Then $\bigcap_{n=1}^{\infty} A_n = \{f \in \mathrm{End\,An}'^r(M):\ f$ satisfies the property $(*)\}$ is G_δ.

In the above proof we perturbed f on the whole manifold M. By a local perturbation one can obtain the effect described below (we omit the proof).

4.4. Proposition. *For every $x \in \mathrm{Per}(f)$, U a neighbourhood of x and $\varepsilon > 0$ there exists f' such that $f|M - U = f'|M - U$, $\varrho_{C^r}(f, f') < \varepsilon$ and for every $y \in \mathrm{Per}(f)$ we have*

$$\theta_{f'f}(x) \notin k_{f',\theta_{f'f}(y)}(R^n - \{0\}) \quad \text{and} \quad \theta_{f'f}(y) \notin k_{f',\theta_{f'f}(x)}(R^n - \{0\}).$$

4.5. PROPOSITION. *The set of all Anosov endomorphisms of class C^r which do not satisfy the property* (*) *is dense in* $\operatorname{End An}^r(M) - \operatorname{Diff An}^r(M)$.

Proof. This is a simple corollary from the following lemma:

4.6. LEMMA. *Let* $f \in \operatorname{End An}^r(M) - \operatorname{Diff An}^r(M)$. *Take* $x \in M$ *such that* $f^s(x) \in \operatorname{Per}(f)$ *for some* $s > 0$ *and* $x \notin \operatorname{Per}(f)$. *For every neighbourhood* $U \ni x$, $\eta > 0$ *and any set* $\{z^1, z^2, \ldots, z^k\}$, $z^j \in \operatorname{Per}^j(f)$ *for* $j = 1, 2, \ldots, k$ (see Proposition 3.16), *there exist* f' *and* j_0 $(1 \leqslant j_0 \leqslant k)$ *such that* $\varrho_{C^r}(f, f') < \eta$, $f|M - U = f'|M - U$, $z^{j_0} \in \operatorname{Per}(f')$ *and* $f(x) \in k_{f', z^{j_0}}(R^m - \{0\})$.

Proof. In the proof of Theorem 4.3 we have removed all counterimages of periodic points from the unstable manifolds of periodic points. Here, on the contrary, we include these counterimages into the unstable manifolds of $z^1, \ldots, z^k$.

Write $X = \{f^a(x) \colon a \geqslant 0\}$. One can easily check that there exist an f-trajectory (x_n) and $\Delta > 0$ such that $x_0 = x$, $\operatorname{dist}_\varrho(x, f(X)) \geqslant \Delta$ and $\operatorname{dist}_\varrho(x_{-p}, X) \geqslant \Delta$ for every positive integer p.

Let $\delta \leqslant \min(\Delta, \Delta/2)$ (Δ was defined in Proposition 2.3) and let $\bar{a}$ be such that for any f-trajectory (y_n) we have $W^u_{y_0,\delta} \subset W^u_{y_0,\bar{a},\varrho^u}$.

From Proposition 3.16 it follows that if we take $a \geqslant \bar{a}$ sufficiently large, then the compact set $\bigcup_{j=1}^{k} W^u_{z^j,a,\varrho^u}$ is $\nu(\delta)$-dense (i.e., for every $z \in M$ $\operatorname{dist}_\varrho(z, \bigcup_{j=1}^{k} W^u_{z^j,a,\varrho^u}) < \nu(\delta)$).

Put

$$\tau = \min\Big(R, 1/2 \operatorname{dist}_\varrho\Big(x, \bigcup_{j=1}^{k} \bigcup_{n=0}^{+\infty} f^n(z^j)\Big)\Big)$$

(R was defined in Theorem 2.1). There exists a positive integer α such that for $j = 1, 2, \ldots, k$:

$$k_{z^j_{-\alpha}} \circ g^{-1}_{z^j_{-\alpha} z^j_{-\alpha+1}} \circ \ldots \circ g^{-1}_{z^j_{-1} z^j} (W^u_{z^j,2a,\varrho^u}) \subset W^u_{z^j,\tau},$$

where $z^j_{-\alpha}, z^j_{-\alpha+1}, \ldots, z^j_{-1}, z^j$ are successive points of the periodic f-trajectory of z^j.

Write

$$W^j_0 = W^u_{z^j,2a,\varrho^u} - \bigcup_{y \in X} B(y, \delta),$$

$$W^j_{-q} = k_{z^j_{-q}} \circ g^{-1}_{z^j_{-q} z^j_{-q+1}} \circ \ldots \circ g^{-1}_{z^j_{-1} z^j} (W^j_0) \quad \text{for } q = 0, 1, \ldots, \alpha.$$

From the above construction it follows that $x \notin W^j_{-q}$. Write

$$W = \bigcup_{j=1}^{k} \bigcup_{q=0}^{\alpha} W^j_{-q}, \qquad \sigma = \operatorname{dist}_\varrho(x, W), \qquad V = U \cap B(x, \min(\sigma, \delta, \tau)).$$

Fix $\varkappa > 0$ a number such that for every $\bar{x} \in M$ for which $\varrho(\bar{x}, x) < \varkappa$, there exists f' satisfying the following conditions:

(1) $$\varrho_{C^r}(f, f') < \eta, \quad f|M - V = f'|M - V, \quad f(x) = f'(\bar{x}).$$

Now fix a non-negative integer p such that $p+1$ is divisible by $\prod_{j=1}^{k} \operatorname{ord} z^j$ and $\delta \cdot ((2+\lambda)/3)^p < \varkappa$. There exist j_0 and $y \in W^u_{z^{j_0},a,\varrho^u}$ such that $\varrho(x_{-p}, y) < \nu(\delta)$. Hence $r(x_{-p}, y) \in W^s_{x_{-p},\delta} \cap W^u_{y,\delta}$ and $W^u_{y,\delta} \subset W^u_{z^j,2a,\varrho^u}$. Moreover, $W^u_{y,\delta} \subset W^j_0$ because $\operatorname{dist}_\varrho(x_{-p}, X) \geqslant 2\delta$.

We obtain $\varrho\big(f^p\big(r(x_{-p}, y)\big), x\big) < \varkappa$. Thus, the endomorphism f' satisfying (1) for $\bar{x} = f^p\big(r(x_{-p}, y)\big)$, is equal to f at each point $f^q\big(r(x_{-p}, y)\big)$, $0 \leqslant q < p$ and $f^\beta(x)$, $\beta \geqslant 1$.

Moreover, $f = f'$ at the points with negative indices belonging to any f-trajectory (y_n) such that $y_0 = r(x_{-p}, y)$ and $y_{-q} \in W^{j_0}_{-q}$ for $0 \leqslant q \leqslant \alpha$. Since $y_{-\alpha} \in W^u_{f,z^{j_0},\tau} = W^u_{f',z^{j_0},\tau}$, we obtain

$$f(x) = f'^{p+1+\alpha}(y_{-\alpha}) \in W^u_{f', f'^{p+1+\alpha}(z^{j_0})} = W^u_{f',z^{j_0}}.$$

Furthermore, $f'^p(y_0) \neq f'^{\operatorname{ord}(z^{j_0})-1}(z^{j_0})$, hence $f(x) \in k_{f',z^{j_0}}(R^m - \{0\})$.

Theorem 4.3 and Proposition 4.5 imply immediately the following

4.7. THEOREM. *Any Anosov endomorphism, except diffeomorphisms and Expandings, is not structurally stable.*

Now we shall give a classification of Anosov endomorphism in an arbitrarily small open set in $\operatorname{End An}^r(M)$ from the point of view of structural stability.

4.8. THEOREM. *Let* $f \in \operatorname{End An}'^r(M) - \operatorname{Diff An}^r(M)$. *Write*

$$\mathscr{Z} = \{Z = (Z_1, \ldots, Z_{k(f)}) \in \operatorname{Per}^1(f) \times \ldots \times \operatorname{Per}^{k(f)}(f)\}.$$

Let $\hat{f}\colon \mathscr{Z} \to \mathscr{Z}$ *be given by*

$$\hat{f}\big((Z_1, \ldots, Z_k)\big) = \big(f(Z_{g^{-1}(1)}), \ldots, f(Z_{g^{-1}(k)})\big),$$

where g *is defined in Theorem* 3.11. *Assume that for every point* $x \in \operatorname{Per}(f)$ *we have a set* $J_x \subset Z$ *such that* $J_{f(x)} = \hat{f}(J_x)$. $\mathcal{O}_1$ *is as in Proposition* 3.17.

Then, for every neighbourhood $\mathcal{O} \ni f$, $\mathcal{O} \subset \operatorname{End An}^r(M)$, *there exists* $f' \in \mathcal{O} \cap \mathcal{O}_1$ *with the following property*:

For every $x \in \operatorname{Per}(f)$ *and* $Z \in J_x$ *there is an integer* $s = s(x, Z)$, $1 \leqslant s \leqslant k$, *such that*

(1) $$\theta_{f'f}(x) \in k_{f', \theta_{f'f}(Z_{s(x,Z)})}(R^m - \{0\}),$$

(2) $$\theta_{f'f}(x) \notin \bigcup_{y \in S} k_{f', \theta_{f'f}(y)}(R^m - \{0\}).$$

where $S = \operatorname{Per}(f') - \bigcup_{Z \in J_x} \theta_{f'f}(Z_{s(x,Z)})$.

Proof. Write $\mathscr{Z}^t = \mathscr{Z} \cap (\mathrm{Per}_t^1(f) \times \ldots \times \mathrm{Per}_t^{k(f)}(f))$ for $t = 1, 2, \ldots$ If $f' \in \mathcal{O}_1$, then $k(f') = k(f)$ and we shall write simply k. By Proposition 3.17 we know that

$$\theta_{f'f}(Z) = (\theta_{f'f}(Z_1), \ldots, \theta_{f'f}(Z_k)) \in (\mathrm{Per}^1(f') \times \ldots \times \mathrm{Per}^k(f'))$$

for any $f' \in \mathcal{O}_1 \cap \mathcal{O}$ and $Z \in \mathscr{Z}$.

Let $\varepsilon > 0$ be a real number such that $\varrho_{C^r}(f', f) < \varepsilon \Rightarrow f' \in \mathcal{O} \cap \mathcal{O}_1$ and $\varepsilon \leqslant \mu$ (μ is as in Theorem 2.1).

Choose one point from each periodic orbit of f and denote the resulting set by $\widetilde{\mathrm{Per}}(f)$. For any $x \in \widetilde{\mathrm{Per}}(f)$ consider the partition of J_x into sets of the form $\{\hat{f}^{n\cdot \mathrm{ord}(x)}(Z)\}_{n=0}^{\infty}$, $Z \in J_x$. Choose one point from each element of this partition and denote the resulting set by $\tilde{J}_x$.

We shall construct f' by induction. Set $f_0 = f$. Assume that for each τ ($\tau = 1, 2, \ldots, t$) there are given: maps f_τ, numbers $\nu_\tau > 0$, numbers $a_\tau > 0$ such that for every τ

$$a_\tau \leqslant a < C/2. \tag{3}$$

(C is defined as at the beginning of the proof of Theorem 4.3, now for the metric ϱ_f. In the sequel we shall use the metric ϱ_f, but we shall omit the subscript f.) Assume that for any $x \in \widetilde{\mathrm{Per}}_t f$, $Z \in \tilde{J}_x \cap \mathscr{Z}^t$ there are given a sequence of points $(y_{(x,Z),n})_{n=-\infty}^{+\infty}$ and a number $s(x, Z) \in \{1, \ldots, k\}$.

Further, assume that the following conditions are satisfied for $\tau = 1, \ldots, t$:

$$\varrho_{C^r}(f_\tau, f_{\tau-1}) < \varepsilon/2^\tau, \tag{4}$$

(5) *if* $x \in \widetilde{\mathrm{Per}}_\tau(f)$, $Z \in \tilde{J}_x \cap \mathscr{Z}^\tau$ *then* $(y_{(x,Z),n})_{-\infty}^{+\infty}$ *is a non-periodic f_τ-trajectory*, $y_{(x,Z),0} = \theta_{f_\tau f}(x)$ *and* $\varrho(y_{(x,Z),-p\cdot \mathrm{ord}\, Z_{s(x,Z)}}, \theta_{f_\tau f}(Z_{s(x,Z)})) \xrightarrow[p\to+\infty]{} 0$,

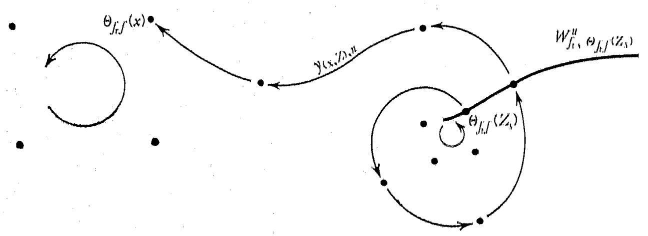


Fig. 3

$$f_{\tau+1}|\mathrm{Per}_{\tau+1} f_\tau = f_\tau|\mathrm{Per}_{\tau+1} f_\tau \quad \text{if} \quad \tau \leqslant t-1, \tag{6}$$

$$f_{\bar\tau}|f^{-\tau}(\mathrm{Per}_\tau f_\tau) = f_\tau|f^{-\tau}(\mathrm{Per}_\tau f_\tau) \quad \text{if} \quad t \geqslant \bar\tau \geqslant \tau, \tag{7}$$

(8) *for every* $\tau \leqslant \bar{\bar\tau} \leqslant \bar\tau \leqslant t$ *and* $y \in \mathrm{Per}_t f_t$ *such that* $\mathrm{ord}(y) = \tau$ *we have*

$$\mathrm{dist}_\varrho\Big(\Big(\Big(f_{\bar\tau}^{-\bar\tau}(\mathrm{Orb}\, y) - \bigcup_{\substack{x \in \widetilde{\mathrm{Per}}_{\bar\tau} f \\ Z \in J_x \cap \mathscr{Z}^{\bar\tau}}} \{y_{(x,Z),n}\}_{n=\infty}^{+\infty}\Big) - \mathrm{Per}_\tau f_\tau\Big), \bigcup_{w \in \mathrm{Per}_{\bar\tau}^{=} f_{\bar\tau}} W^u_{f_{\bar\tau},\, w, a_{\bar{\bar\tau}}}\Big) \geqslant \nu_{\bar\tau} > 0,$$

$$a_\tau \leqslant (1/2) \inf_{\substack{x_1 \neq x_2 \\ x_1, x_2 \in \mathrm{Per}_\tau f_{\tau-1}}} \varrho(x_1, x_2); \tag{9}$$

(10) *for every* $w \in \mathrm{Per}_\tau f_\tau$ *we have*

$$\varrho_{C^0}(W^u_{f_{\tau+1}, w, a}, W^u_{f_\tau, w, a}) < (1/2^{\tau+1}) \cdot \min(\nu_1, \ldots, \nu_\tau) \text{ for } \tau \leqslant t-1$$

(notice that for $t = 0$ there is no τ between 1 and t).

We have to define f_{t+1}, new sequences $(y_{(x,Z),n})$, numbers $s(x, Z)$ for $(x, Z) \in A = \{(x, Z)\colon x \in \widetilde{\mathrm{Per}}_t f, Z \in \tilde{J}_x \cap (\mathscr{Z}^{t+1} - \mathscr{Z}^t)$ or $x \in \widetilde{\mathrm{Per}}_{t+1} f, Z \in \tilde{J}_x \cap \cap \mathscr{Z}^{t+1}\}$ and numbers a_{t+1}, ν_{t+1}.

Denote the elements of A by $(x, Z)^1, \ldots, (x, Z)^\zeta$, $(x, Z)^\sigma = (x^\sigma, Z^\sigma)$ for $\sigma = 1, \ldots, \zeta$.

Now we shall construct transformations $f_{t,0}, \ldots, f_{t,\zeta}$, numbers $s((x, Z)^\sigma)$ and points $y_{(x,Z)^\sigma, n}$ by induction. Put $f_{t,0} = f_t$.

Assume that the following two conditions are satisfied:

$$f_t|\mathrm{Per}_{t+1}(f_t) = f_{t,\sigma}|\mathrm{Per}_{t+1}(f_t), \tag{11}$$

(12) condition (5) is fulfilled for $f_{t,\sigma}$ in place of f_t and for $(x, Z)^{\bar\sigma}$, $\bar\sigma = 1, \ldots, \sigma$.

Then for a so large that

$$(\deg f)^{a(t+1)!} > \sum_{x \in \widetilde{\mathrm{Per}}_t f} \mathrm{Card}(J_x \cap \mathscr{Z}^t) + \sigma$$

there exists an y and its neighbourhood U_y such that

$$f_{t,\sigma}^{a\cdot(t+1)!+1}(y) = \theta_{f_t f}(x^{\sigma+1}), \quad f_{t,\sigma}^{a\cdot(t+1)!}(y) \notin \mathrm{Per} f_{t,\sigma},$$

$$\Big(\mathrm{Per}_{t+1}(f_t) \cup \bigcup_{\substack{x \in \widetilde{\mathrm{Per}}_t f \\ Z \in \tilde{J} \cap \mathscr{Z}^t}} \{y_{(x,Z),n}\}_{-\infty}^{+\infty} \cup \bigcup_{\bar\sigma=1}^{\sigma} \{y_{(x,Z)^{\bar\sigma},n}\}_{n=-\infty}^{+\infty} \cup f_t^{-t}(\mathrm{Per}_t f_t)\Big) \cap U_y = \emptyset.$$

By Lemma 4.6 there exist $f_{t,\sigma+1}$ and $s((x, Z)^{\sigma+1})$ such that

$$\varrho_{C^r}(f_{t,\sigma}, f_{t,\sigma+1}) < \varepsilon/\zeta \cdot 2^{t+1}, \tag{13}$$

(14) *for every* $w \in \mathrm{Per}_t f_t$

$$\varrho_{C^0}(W^u_{f_{t,\sigma}, w, a}, W^u_{f_{t,\sigma+1}, w, a}) < 1/\zeta \cdot 2^{t+1} \min(\nu_1, \ldots, \nu_t),$$

$$f_{t,\sigma+1}|M - U_y = f_{t,\sigma}|M - U_y, \tag{15}$$

$$f_{t,\sigma}(y) \in k_{f_{t,\sigma+1}, Z_{s((x,Z)^{\sigma+1})}}(R^m - \{0\}). \tag{16}$$

The existence of a trajectory $(y_{(x,Z)^{\sigma+1},n})_{n=-\infty}^{+\infty}$ satisfying (5) for $f_{t,\sigma+1}$ in place of f_τ and $(x, Z)^{\sigma+1}$ follows from (16) and from the definition of the unstable manifold. Therefore, from the definition of U_y and (15) it follows that conditions (11) and (12) are satisfied for $\sigma+1$.

Hence, conditions (4), (5), (6), (7), (10) for $\tau = t+1$ are fulfilled if we put $f_{t,\zeta}$ instead of f_{t+1}.

Now we perturb $f_{t,\zeta}$ to an endomorphism f_{t+1} such that condition (8) will also be satisfied.

Denote the elements of the set

$$f_{t,\zeta}^{-(t+1)}\big(\mathrm{Per}_{t+1}(f_{t,\zeta})\big)-\Big(f_{t,\zeta}^{-t}(\mathrm{Per}_t f_{t,\zeta})\cup \bigcup_{\substack{x\in\widetilde{\mathrm{Per}}_{t+1}f\\ Z\in\tilde{J}_x\cap\mathscr{Z}^{t+1}}}\{y_{(x,Z),n}\}_{-\infty}^{+\infty}\Big)$$

by $\bar{x}^1, \ldots, \bar{x}^\Sigma$ in such a way that if, for some τ, $f_{t,\zeta}^\tau(\bar{x}^\beta)$ is periodic but $f_{t,\zeta}^\tau(\bar{x}^\gamma)$ is not, then $\beta<\gamma$. Take a_{t+1} satisfying (9) and (3) for $\tau = t+1$. If $\bar{x}^\beta\in\mathrm{Per}_{t+1}(f_{t,\zeta})-\mathrm{Per}_t(f_{t,\zeta})$, then, in view of (9) and (6), (7) for $f_{t,\zeta}$, we have: $\bar{x}^\beta\notin W^u_{f_{t,\zeta},w,a_{t+1}}$ for $w\in\mathrm{Per}_{t+1}(f_{t,\zeta})-\{\bar{x}^\beta\}$. This allows us to obtain the same result for every $\bar{x}^\beta$, $1\leqslant\beta\leqslant\Sigma$, where a_{t+1} is replaced by a_τ, $\tau = \mathrm{ord} f_{t,\zeta}^{t+1}(\bar{x}^\beta)$, using the method described in the proof of Theorem 4.3 and consisting taking successive perturbations of $f_{t,\zeta}$, removing $\bar{x}^\beta$ from unstable manifolds. If the perturbations are as in the proof of Theorem 4.3 and are small enough, then after the last step we obtain f_{t+1} satisfying (4), (5), (6), (7), (10) and we can take $\nu_{t+1}>0$ such that (8) is also satisfied.

Now set $f' = \lim\limits_{t\to\infty} f_t$.

If $x\in\widetilde{\mathrm{Per}}f$ and $Z\in J_x-\tilde{J}_x$, then $Z = f^{N\cdot\mathrm{ord}(x)}(\tilde{Z})$ for a unique $\tilde{Z}\in\tilde{J}_x$ and some positive integer N, and we can define $s(x, Z)$ as such a number that $\hat{f}^{N\cdot\mathrm{ord}(x)}(\tilde{Z}_{s(x,\tilde{Z})})\in\mathrm{Per}^{s(x,Z)}f$. Now if $x\in\mathrm{Per}(f)-\widetilde{\mathrm{Per}}(f)$ and $Z\in J_x$, then $x = f^N(\tilde{x})$ for a unique $\tilde{x}\in\widetilde{\mathrm{Per}}(f)$ and some positive integer N. From the assumptions of the theorem it follows that $Z = f^N(\tilde{Z})$, $\tilde{Z}\in J_{\tilde{x}}$. We can define $s(x, Z)$ as such a number that $f^N(\tilde{Z}_{s(\tilde{x},\tilde{Z})})\in\mathrm{Per}^{s(x,Z)}f$.

By (5) and (6), condition (5) is satisfied also for f' in place of f_τ and (1) follows. Condition (8) (with conditions (7) and (10) which allow to consider f' instead of f_τ) implies (2).

4.9. Proposition. *If $f\in\mathrm{End\,An}^1(M)$, then f has infinitely many periodic points.*

Proof. For Expanding the answer is positive and the proof is well-known. Assume that f is not an Expanding. Assume that the set $\mathrm{Per}(f)$ is finite. Hence $\Omega(f)$ is finite. This and the property that the dimension of each unstable manifold is less than the dimension of M, contradict Proposition 3.9 (c).

4.10. Proposition. *Let f be an Axiom A endomorphism and let $\mathrm{Per}^j f$ be an infinite set for some j. Then there exists an infinite set B^j of positive integers such that if $b_1, b_2\in B^j$ and $b_1\neq b_2$, then b_1 does not divide b_2 and for every $b\in B^j$ there exists $x\in\mathrm{Per}^j f$ such that $\mathrm{ord}(x) = b$.*

Proof. Let r be the smallest positive integer such that $g^r(j) = j$ (for definition of g see the spectral decomposition theorem 3.11). Then one can define B^j as the set of all numbers of the form $p\cdot r$, where p is a sufficiently large prime number. The existence of points with such minimal periods follows easily from Theorem 3.18.

4.11. Theorem. *For any open non-empty set $U\subset\mathrm{End\,An}'^r(M)-\mathrm{Diff\,An}(M)$ there exists an uncountable set $(f_t)_{t\in T}$ of Anosov endomorphisms contained in U such that if $t_1\neq t_2$, then there exists no surjective map $\varphi\in C^0(M, M)$ which makes the diagram*

$$\begin{array}{ccc} M & \xrightarrow{f_{t_1}} & M\\ \big\downarrow{\scriptstyle\varphi} & & \big\downarrow{\scriptstyle\varphi}\\ M & \xrightarrow[f_{t_2}]{} & M\end{array}$$

commute.

Proof. Let $f\in U$ and let $\mathscr{O}_1$ be as in Proposition 3.5. Fix j_0 such that $\mathrm{Per}^{j_0}(f)$ contains infinitely many elements. It is obvious that there exists an uncountable family $(G_t)_{t\in T}$ of subsets of the set B^{j_0} (see Proposition 4.10) such that $t_1\neq t_2$ implies $G_{t_1}\not\subset G_{t_2}$. For $t\in T$ let f_t be the endomorphism constructed in Theorem 4.8 for J_x defined as follows:

$$J_x = \begin{cases}\mathscr{Z} & \text{if } \mathrm{ord}\,(x)\in G_t,\\ \emptyset & \text{otherwise}.\end{cases}$$

If $t_1\neq t_2$, then there exists $x^{t_1}\in\mathrm{Per}^{j_0} f$ such that $\mathrm{ord}\,(x^{t_1})\in G_{t_1}-G_{t_2}$. Assume that there is a continuous map $\varphi\colon M\xrightarrow{\text{onto}}M$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{f_{t_1}} & M\\ \big\downarrow{\scriptstyle\varphi} & & \big\downarrow{\scriptstyle\varphi}\\ M & \xrightarrow[f_{t_2}]{} & M\end{array}$$

commutes.

It is obvious that if $x, y\in\mathrm{Per}^j(f_{t_1})$ for some $j\in\{1, \ldots, k\}$, then $\varphi(x)$, $\varphi(y)\in\mathrm{Per}^{\sigma(j)}(f_{t_2})$ for some $\sigma(j)\in\{1, \ldots, k\}$. For any point $x\in\mathrm{Per}(f_{t_2})$ there exists y such that $\varphi(y) = x$ and since $\mathrm{dist}_\varrho\big(f_{t_1}^n(y), \Omega(f_{t_1})\big)\xrightarrow[n\to\infty]{}0$, we have also $\mathrm{dist}_\varrho\big(\mathrm{Orb}_{f_{t_2}}(x), \varphi\big(\Omega(f_{t_1})\big)\big)\xrightarrow[n\to\infty]{}0$; therefore $x\in\varphi\big(\Omega(f_{t_1})\big)$. We have $\varphi\big(\Omega(f_{t_1})\big)\supset\mathrm{Per}(f_{t_2})$, so

(1) $$\varphi\big(\Omega(f_{t_1})\big)\supset\Omega(f_{t_2}).$$

Hence $\sigma\colon\{1, \ldots, k\}\to\{1, \ldots, k\}$ is a surjection, therefore it is also an injection.

For any $Z \in \mathscr{Z}$ there exists an s such that

$$\theta_{f_{t_1}}(x^{t_1}) \in W^u_{f_{t_1}, \theta_{f_{t_1}f}(Z_s)},$$

hence

$$\varphi\big(\theta_{f_{t_1}f}(x^{t_1})\big) \in W^u_{f_{t_2}, \theta_{f_{t_1}f}(Z_s)}.$$

But $\mathrm{ord}_{f_{t_2}}\Big(\varphi\big(\theta_{f_{t_1}f}(x^{t_1})\big)\Big) \notin G_{t_2}$, therefore

$$\varphi\big(\theta_{f_{t_1}f}(Z_s)\big) = \varphi\big(\theta_{f_{t_1}f}(x^{t_1})\big).$$

This implies $s = j_0$ (because σ is an injection). As $Z \in \mathscr{Z}$ was arbitrary, we obtain $\mathrm{Card}\,\varphi\big(\mathrm{Per}^{j_0}(f_{t_1})\big) = 1$. Note that by (1)

$$\varphi(\mathrm{Per}^j f_{t_1}) = \mathrm{Per}^{\sigma(j)} f_{t_2} \quad \text{for} \quad j = 1, \ldots, k.$$

Applying once more the injectivity of σ we see that the number of $j \in \{1, \ldots, k\}$ such that the set Per^j is infinite is greater for f_{t_1} than for f_{t_2}. But this is impossible.

Using the same arguments as above, one can easily prove:

4.12. THEOREM (a) *There exists a residual subset* $A^r \subset \mathrm{End\,An}''^r(M) - \mathrm{Diff\,An}(M)$ (*the set of all endomorphisms satisfying the property* (*)) *such that for every* $f \in A^r$ *and* U, *an open neighbourhood of* f *in* $\mathrm{End\,An}^r(M)$, *there is an* $f' \in U$ *such that there exists no continuous map* $\varphi\colon M \xrightarrow{\text{onto}} M$ *such that* $\varphi \circ f' = f \circ \varphi$.

(b) *There exists a dense subset* $B^r \subset \mathrm{End\,An}''^r(M) - \mathrm{Diff\,An}(M)$ *with the property*:

for every $f \in B^r$ *and* U, *an open neighbourhood of* f *in* $\mathrm{End\,An}^r$, *there exists* $f' \in U$ *such that there exists no continuous map* $\varphi\colon M \xrightarrow{\text{onto}} M$ *such that* $\varphi \circ f' = f \circ \varphi$.

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Received March 20, 1975 (977)